Event-Triggered Stabilization of Nonlinear Systems with Time-Varying Sensing and Actuation Delay

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Abstract

This paper studies the problem of stabilization of a nonlinear system with time-varying delays in both sensing and actuation using event-triggered control. Our proposed strategy seeks to opportunistically minimize the number of control updates while guaranteeing stabilization and builds on predictor feedback to compensate for arbitrarily large known time-varying delays. We establish, using a Lyapunov approach, the global asymptotic stability of the closed-loop system as long as the open-loop system is globally input-to-state stabilizable in the absence of time delays and sampling. We further prove that the proposed event-triggered law has inter-event times that are uniformly lower bounded and hence does not exhibit Zeno behavior. For the particular case of a stabilizable linear system, we show global exponential stability of the closed-loop system and analyze the trade-off between the rate of exponential convergence and a bound on the sampling frequency. We illustrate these results in simulation and also examine the properties of the proposed event-triggered strategy beyond the class of systems for which stabilization can be guaranteed.

1 Introduction

Event- and self-triggered approaches have recently gained popularity for controlling cyberphysical systems. The basic premise is that of abandoning the assumption of continuous or periodic updating of the control signal and instead adopt an opportunistic perspective that leads to deliberate, aperiodic updates. The challenge resides in determining precisely when control signals should be updated to improve efficiency while still guaranteeing convergence. This paper expands the state-of-the-art in resource-aware control by designing predictor-based event-triggered control strategies that stabilize nonlinear systems with *known* delays in both sensing and actuation that can be *arbitrarily large* and *time-varying*.

Literature review: There exists a vast literature on both event-triggered control and the control of time-delay systems. Here, we review the works most closely related to our treatment. Originating from event-based and discrete-event systems [Cassandras and Lafortune, 2007, Zou et al., 2017], the concept of event-triggered control (i.e., updating the control signal in an opportunistic fashion) was proposed in [Kopetz, 1991, Åström and Bernhardsson., 2002] and has found its way into the efficient use of sensing, computing, actuation, and communication resources in networked control systems, see e.g., [Tabuada, 2007, Wang and Lemmon, 2011, Heemels et al., 2012, Abdelrahim et al., 2017] and references therein. On the other hand, the notion of predictor feedback is a powerful method in dealing with controlled systems subject to time delay [Smith, 1959, Mayne, 1968, Manitius and Olbrot, 1979, Nihtila, 1991, Krstic, 2009, Karafyllis and Krstic, 2012]. In essence, a predictor feedback controller anticipates the future evolution of the plant using its forward model and sends the control signal early enough to compensate for the delay. Here, we pursue a Lyapunov-based analysis of predictor feedback following [Bekiaris-Liberis and Krstic, 2013]. Given that numerical implementations of predictor feedback controllers are particularly challenging [Mirkin, 2004, Zhong, 2004], we further discuss several methods for the implementation of our proposed controller and show that a carefully designed "closed-loop" method is numerically stable and robust to errors in delay compensation.

The joint treatment of time delay and event-triggering is particularly challenging. By its opportunistic nature, an event-triggered controller keeps the control value unchanged until the plant is close to instability and then updates the control value according to the current state. Now, if time delays exist, the controller only has access to some past state of the plant (delayed sensing) and it takes some time for an updated control action to reach the plant (delayed actuation), jointly increasing the possibility of the updated control value being already obsolete when it is implemented in the plant, resulting in instability. Therefore, the controller needs to be sufficiently proactive and update the control value sufficiently ahead of time to maintain closed-loop stability. This makes the design problem challenging. Delays in actuation and sensing may be due to communication delays between controller-actuator and controller-sensor pairs, and in that sense, previous work on the event-

⋆ A preliminary version of this paper appeared at the IEEE Conference on Decision and Control as [Nozari et al., 2016].

triggered control literature that specifically considers delays in the communication channel deals with a similar problem setup as the one considered here. Several event-triggered designs consider scenarios where the system dynamics are linear, see, e.g. [Zhang et al., 2017, Chen et al., 2017, Selivanov and Fridman, 2016a,b, Ge and Han, 2015, Garcia and Antsaklis, 2013]. The inclusion of nonlinearity, however, makes the problem more challenging. When digital controllers are used and the delay is smaller than the sampling time, [Hetel et al., 2006, Wu et al., 2015] design event-triggered controllers for the resulting delay-free discretized system. Robust event-triggered stabilizing controllers are also designed for nonlinear systems with sensing delays in [Li et al., 2012] and with both sensing and actuation delays in [Tabuada, 2007, Dolk et al., 2017]. In all these works, however, a key assumption is that the (maximum) delay is smaller than the (minimum) inter-transmission time. This assumption (also called the small-delay case) allows for the *treatment of delay as a disturbance* and, by construction, can tolerate unknown delays. In reality, however, (minimum) inter-transmission times can be very small, making this assumption restrictive. Similar to our preliminary work [Nozari et al., 2016], we take a different perspective here and consider arbitrarily large delays, with the expected tradeoff in our treatment that the delay can no longer be unknown. The technical approach is based on using predictors that capture the effect of the delay on the system to compensate for it. We rigorously analyze the case when the delay is accurately known and show in simulation that our design is indeed robust to small variations when the delay is only approximately known. Unlike [Nozari et al., 2016], here we consider event-triggering and time-varying delay both in sensing and actuation. Further, given the well-known difficulties in the computation of predictor-feedback controllers, we here provide a detailed discussion of the numerical challenges that arise in the implementation of predictor feedback and effective solutions to resolve them. Finally, this paper provides a complete and thorough technical treatment, including the proofs of all results, which are not available in [Nozari et al., 2016].

Statement of contributions: Our contributions are threefold. First, we design an event-triggered controller for stabilization of nonlinear systems with arbitrarily large sensing and actuation delays. We employ the method of predictor feedback to compensate for the delay in both and then co-design the control law and triggering strategy to guarantee the monotonic decay of a Lyapunov-Krasovskii functional. Our second contribution involves the closed-loop analysis of the event-triggered law, proving that the closed-loop system is globally asymptotically stable and the inter-event times are uniformly lower bounded (and thus no Zeno behavior may exist). Due to the importance of linear systems in numerous applications, we briefly discuss the simplifications of the design and analysis in this case. Our final contribution pertains to the trade-off between convergence rate and sampling. Our analysis in this part is limited to linear systems, where closed-form solutions are derivable for (exponential) convergence rate and minimum inter-event times. We provide a quantitative account of the well-known trade-off between sampling and convergence in event-triggered designs and show how this trade-off can be biased in either direction by tuning a design parameter. Finally, we present simulations to illustrate the effectiveness of our design and address its numerical implementation.

2 Preliminaries

We introduce notational conventions and briefly review notions on input-to-state stability. We denote by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ the sets of reals and nonnegative reals, respectively. Given $t \in \mathbb{R}$ and a function f on $\mathbb{R}$, $t_+ \triangleq \max\{t, 0\}$ while $f(t^+) \triangleq \lim_{s\to t^+} f(s)$ and $f(t^-) \triangleq \lim_{s\to t^-} f(s)$ when these limits exist. Given a vector or matrix, we use $|\cdot|$ to denote the Euclidean norm. We denote by $\mathcal{K}$ the set of strictly increasing continuous functions $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$. α belongs to $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$. We denote by $\mathcal{KL}$ the set of continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that, for each $s \in [0,\infty)$, $r \mapsto \beta(r,s)$ belongs to class $\mathcal{K}$ and, for each $r \in [0,\infty)$, $s \mapsto \beta(r,s)$ is monotonically decreasing with $\beta(r,s) \to 0$ as $s \to \infty$. We use the notation $\mathcal{L}_f S = \nabla S \cdot f$ for the Lie derivative of a function $S : \mathbb{R}^n \to \mathbb{R}$ along the trajectories of a vector field f taking values in $\mathbb{R}^n$.

We follow [Sontag and Wang, 1995] to review the definition of input-to-state stability of nonlinear systems and its Lyapunov characterization. Consider a nonlinear system of the form

$$\dot{x}(t) = f(x(t), u(t)), \qquad \text{a.a. } t \geq 0, \qquad x(0) = x_0, \quad (1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable, $f(0,0) = 0$, and "a.a." (almost all) denotes the fact that x may not be differentiable on a set of Lebesgue measure zero. System (1) is (globally) input-to-state stable (ISS) if there exist $\alpha \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that for any measurable locally essentially bounded input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and any initial condition $x(0) \in \mathbb{R}^n$, its solution satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \alpha\big(\operatorname{ess\,sup}_{t\geq 0} |u(t)|\big),$$

for all $t \geq 0$. For this system, a continuously differentiable function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function if there exist $\alpha_1, \alpha_2, \gamma, \rho \in \mathcal{K}_\infty$ such that

$$\forall x \in \mathbb{R}^n \qquad \alpha_1(|x|) \leq S(x) \leq \alpha_2(|x|), \quad (2a)$$
$$\forall (x,u) \in \mathbb{R}^{n+m} \quad \mathcal{L}_f S(x,u) \leq -\gamma(|x|) + \rho(|u|). \quad (2b)$$

According to [Sontag and Wang, 1995, Theorem 1], the system (1) is ISS if and only if it admits an ISS-Lyapunov function.

3 Problem Statement

Consider the nonlinear system ("plant") with dynamics

$$\dot{x}(t) = f(x(t), u_p(t)), \qquad \text{a.a. } t \geq 0, \qquad x(0) = x_0, \quad (3)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Our goal is to provide a state-feedback controller ensuring global asymptotic stability under the following challenges:

(i) **Actuation delay:** Let $u(t)$ be the control signal generated by the controller. Actuation delay is modeled as

$$u_p(t) = u(\phi(t)), \quad t \geq 0, \quad (4)$$

where $t - \phi(t) > 0$ is the amount of time that it takes for a control action generated at time $\phi(t)$ to reach the plant/actuator. For instance, In the case of a constant actuation delay D, we have $\phi(t) = t - D$. This delay further requires an initial value $\{u(t) \mid \phi(0) \le t < 0\}$ on the control input for (3) to be well-defined.

(ii) **Sensing delay:** We allow the existence of a delay between the sensor and the controller such that at any time t, the controller may have access to $x(s), s \le \psi(t)$ (alternatively, $x(t)$ takes $\psi^{-1}(t) - t$ seconds to reach the controller) for some delay function $\psi(t) \le t$.

(iii) **Actuation event-triggering:** We seek to design a controller that updates $u(t)$ only at a sequence of discrete times $\{t_k\}_{k=0}^{\infty}$,

$$u(t) = u(t_k), \quad t \in [t_k, t_{k+1}), \quad k \ge 0. \tag{5}$$

(iv) **Sensing event-triggering:** We further allow for the possibility that the event-triggering mechanism does not have access to the plant state at all times $t \in \mathbb{R}_{\ge 0}$, but only at some time instants denoted $\tau_\ell, \ell \in \mathbb{Z}_{\ge 0}$.[1] In this case, we let for simplicity that $\tau_0 = 0$, $t_0 = \psi^{-1}(0)$, and $u(t)$ be arbitrarily set in $[0, t_0)$ as the controller has not received any state information yet.

In the sequel, we impose the following assumptions on the system dynamics.

Assumption 3.1 *(Standing assumptions):*

(i) f is continuously differentiable, $f(0,0) = 0$, and (3) is forward complete (does not exhibit finite escape time) for all initial conditions and bounded inputs;
(ii) the initial control $\{u(t) \mid \phi(0) \le t < 0\}$ is given and continuously differentiable;
(iii) the delay function ϕ is continuously differentiable;
(iv) the delay functions ϕ and ψ are monotonically increasing so the argument of $u(\phi(t))$ and $x(\psi(t))$ do not go back in time;
(v) the origin of (3) is robustly globally asymptotically stabilizable in the absence of delays and with continuous sensing and actuation. Formally, there exists a globally Lipschitz feedback law $K : \mathbb{R}^n \to \mathbb{R}^m$, $K(0) = 0$, that makes

$$\dot{x}(t) = f(x(t), K(x(t)) + w(t)), \tag{6}$$

ISS with respect to the additive input disturbance w;
(vi) the delay function ϕ is known to the controller; on the other hand, ψ need not be known a priori or for all times, but only a posteriori and at times when state is measured;
(vii) the delay function ϕ and its derivative are bounded, i.e., there exist $M_0 > 0$, $M_1 \ge 1$, and $0 < m_2 \le 1$ such that

$$t - \phi(t) \le M_0 \ \text{ and } \ m_2 \le \dot{\phi}(t) \le M_1, \ \ \forall t \ge 0; \tag{7}$$

(viii) the sensing triggering times $\{\tau_\ell\}_{\ell=0}^{\infty}$ are given (determined by the sensor independently of our design). In particular, the sensor ensures that $\{\tau_\ell\}_{\ell \ge 0} \cap [a, b]$ is finite for any $a, b < \infty$ (lack of Zeno behavior) while $\{\tau_\ell\}_{\ell=0}^{\infty}$ can be arbitrary otherwise. •

Assumption 3.1(i)-(iv) are standard in predictor-based control of delay systems. In the case of digital communications, Assumption 3.1(iv) requires the lack of packet reordering. Nevertheless, the nature of the control system is such that any $u(t_k)$ has become obsolete and can be safely discarded, should it arrive later than $u(t_i), i \ge k$. The same applies to $\{x(\tau_\ell)\}_{\ell=1}^{\infty}$. Thus, ϕ and ψ can be, without loss of generality, replaced by a monotonically increasing upper bound if they are not so originally. Assumption 3.1(i), together with the piecewise-constant form of u_p, further ensures existence and uniqueness of solutions for (3). Assumption 3.1(v) is also standard in event-triggered control, though not necessarily with the globally Lipschitz property assumed here. This allows us to focus on the challenges that arise by time delays and event-triggered control. Further, the a priori knowledge of ϕ in Assumption 3.1(vi) is most realistic in applications where the same control task is repeatedly executed and thus a data-driven estimate of future ϕ can be computed using its history. Moreover, note that Assumption 3.1(vii) is trivially satisfied for a constant delay ($\phi(t) = t - D$) with $M_0 = D$ and $M_1 = m_2 = 1$. Finally, Assumption 3.1(viii) is imposed for simplicity and to let us focus on the design of the actuation triggering times. In fact, the values of $\{\tau_\ell\}$ other than τ_0 are irrelevant theoretically but practically critical for stability, a point we discuss in detail in Sections 4.4 and 6.

The resulting networked control scheme is illustrated in Figure 1. Our considered problem is then as follows.

Problem 1 *(Event-Triggered Stabilization under Sensing and Actuation Delay): Design the sequence of actuation triggering times*[2] $\{t_k\}_{k=1}^{\infty}$ *and the corresponding control values* $\{u(t_k)\}_{k=0}^{\infty}$ *such that* $\{t_{k+1} - t_k\}_{k \ge 0}$ *is uniformly lower bounded by a strictly positive constant and the closed-loop system* (3) *is globally asymptotically stable using the piecewise constant control* (5) *and the delayed information* $\{x(\tau_\ell)\}_{\ell=0}^{\infty}$ *received, resp., at* $\{\psi^{-1}(\tau_\ell)\}_{\ell=0}^{\infty}$.[3] •

The requirement that $\{t_k\}_{k \ge 0} \cap [a, b]$ be finite for any $0 \le a \le b < \infty$ ensures the resulting design is implementable by avoiding finite accumulation points, i.e., Zeno behavior. We propose a solution to Problem 1 in the next section.

4 Event-Triggered Design and Analysis

In this section, we propose an event-triggered control policy to solve Problem 1. We start our analysis with the simpler case where the controller receives state feedback continuously (i.e., $\{x(t)\}_{t=0}^{\infty}$ instead of $\{x(\tau_\ell)\}_{\ell=0}^{\infty}$) without delays (i.e., $\psi(t) = t$), and later extend it to the general case.

[1] We note that this sampled sensing scheme is also called *periodic event-triggered control*, even when the sampling times are not equally spaced. Nevertheless, we do not adopt this terminology here to avoid the latter interpretation.

[2] Recall that $t_0 = \psi^{-1}(0)$ is fixed.

[3] We require that the control law is causal, i.e., t_k and $u(t_k)$ depend only on the states $\{x(\tau_\ell)\}$ that have reached the controller by the time t_k. While sampling may be modeled as a specific type of delay, we capture it with the prediction error $e(t)$ (defined later). The values $\phi(t)$ and $\psi(t)$ only capture the delays in actuation and sensing, resp.

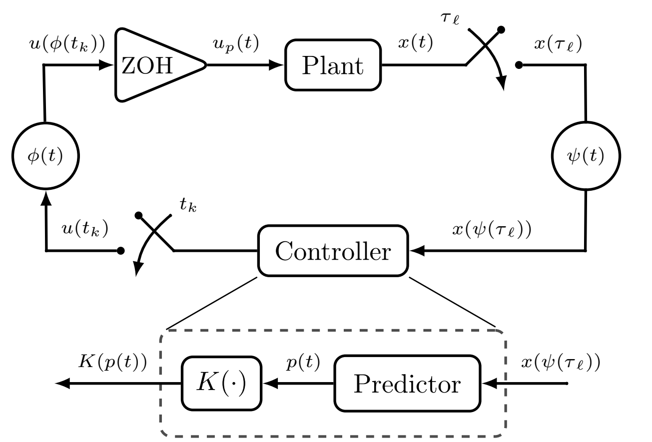

Fig. 1. The considered networked control scheme with sensing and actuation delays and event-triggering (top) and the proposed predictor-based controller (bottom).

4.1 Predictor Feedback Control for Time-Delay Systems

Here we review the continuous-time stabilization of the dynamics (3) by means of a predictor-based feedback control [Bekiaris-Liberis and Krstic, 2013]. For convenience, we denote the inverse of ϕ by $\sigma(t) = \phi^{-1}(t)$, for all $t \geq 0$. The inverse exists since ϕ is strictly monotonically increasing. From (7), for all $t \geq \phi(0)$,

$$\dot{\sigma}(t) \leq M_2 \triangleq m_2^{-1}.$$

To compensate for the delay, at any time $t \geq \phi(0)$, the controller makes the following prediction of the future state of the plant,

$$p(t) = x(\sigma(t)) = x(t_+) + \int_{\phi(t_+)}^{t} \dot{\sigma}(s) f(p(s), u(s)) ds. \quad (8)$$

This integral is computable by the controller since it only requires knowledge of the initial or current state of the plant and the history of $u(t)$ and $p(t)$, all of which are available to the controller. In the remainder, we thus assume that $p(t)$ can be computed exactly, but hint that numerical integration errors can lead to instability if not treated properly. We will give a detailed empirical discussion of this matter in Section 6 but its rigorous analysis remains open for future research.

As shown in Figure 1, the controller applies the control law K on the prediction p to compensate for the delay,

$$u(t) = K(p(t)), \qquad t \geq 0. \quad (9)$$

The next result shows convergence for the closed-loop system.

Proposition 4.1 *(Asymptotic Stabilization by Predictor Feedback [Bekiaris-Liberis and Krstic, 2013]): Under Assumption 3.1, the closed-loop system* (3) *under the controller* (9) *is globally asymptotically stable, i.e., there exists $\beta \in \mathcal{KL}$ such that for any $x(0) \in \mathbb{R}^n$ and bounded $\{u(t)\}_{t=\phi(0)}^{0}$, for all $t \geq 0$,*

$$|x(t)| + \sup_{\phi(t) \leq \tau \leq t} |u(\tau)| \leq \beta\Big(|x(0)| + \sup_{\phi(0) \leq \tau \leq 0} |u(\tau)|, t\Big).$$

4.2 Design of Event-triggered Control Law

Following Section 4.1, we let the controller make the prediction $p(t)$ according to (8) for all $t \geq \phi(0)$. Since the controller can only update $u(t)$ at discrete times $\{t_k\}_{k=0}^{\infty}$, it uses the piecewise-constant control (5) and assigns the control

$$u(t_k) = K(p(t_k)), \quad (10)$$

for all $k \geq 0$. In order to design the triggering times $\{t_k\}_{k=1}^{\infty}$, we use Lyapunov stability tools to determine when the controller has to update $u(t)$ to prevent instability. We define the triggering error for all $t \geq \phi(0)$ as

$$e(t) = \begin{cases} p(t_k) - p(t) & \text{if } t \in [t_k, t_{k+1}) \text{ for } k \geq 0, \\ 0 & \text{if } t \in [\phi(0), t_0), \end{cases} \quad (11)$$

so that $u(t) = K(p(t) + e(t))$, for $t \geq t_0$. Let

$$w(t) = u(t) - K(p(t) + e(t)), \qquad t \geq \phi(0), \quad (12)$$

where $w(t) = 0$ for $t \geq t_0$ but $w(t)$ is in general nonzero for $t \in [\phi(0), t_0)$. Computing $u(\phi(t))$ from (12) and substituting it in (3), the closed-loop system can be written

$$\dot{x}(t) = f\big(x(t), K\big(x(t) + e(\phi(t))\big) + w(\phi(t))\big), \quad (13)$$

for all $t \geq 0$. Notice that (13) simplifies to [Tabuada, 2007, Eq. (3)] in the absence of delay ($\phi(t) = t$). Let $g(x, w) = f(x, K(x) + w)$ for all x, w. By Assumption 3.1(v), there exists a continuously differentiable function $S : \mathbb{R}^n \to \mathbb{R}$ and $\alpha_1, \alpha_2, \gamma, \rho \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x(t)|) \leq S(x(t)) \leq \alpha_2(|x(t)|), \quad (14)$$

and $(\mathcal{L}_g S)(x, w) \leq -\gamma(|x|) + \rho(|w|)$. Therefore, we have

$$\begin{aligned} &(\mathcal{L}_f S)\big(x(t), K\big(x(t) + e(\phi(t))\big) + w(\phi(t))\big) \quad (15) \\ &= (\mathcal{L}_g S)\big(x(t), K\big(x(t){+}e(\phi(t))\big){+}w(\phi(t)){-}K(x(t))\big) \\ &\leq -\gamma(|x(t)|) + \rho\big(\big|K\big(x(t){+}e(\phi(t))\big){+}w(\phi(t)){-}K(x(t))\big|\big). \end{aligned}$$

As in [Bekiaris-Liberis and Krstic, 2013, eq. (8.47)], let [4]

$$V(t) = S(x(t)) + \frac{2}{b} \int_0^{2L(t)} \frac{\rho(r)}{r} dr, \quad (16a)$$

$$L(t) = \sup_{t \leq \tau \leq \sigma(t)} |e^{b(\tau - t)} w(\phi(\tau))|, \quad (16b)$$

where $b > 0$ is a design parameter. Note that the second term in (16a) may only be nonzero for $t \in [\phi(0), t_0)$ since the system is open-loop over this interval (cf. (11),(12)). The next result establishes an upper bound on dV/dt.

Proposition 4.2 *(Upper-bounding $\dot{V}(t)$): For the system* (3) *under the control defined by* (5) *and* (10) *and the*

[4] Note that ρ can always be chosen such that (16a) is well-defined, e.g., by choosing it such that $\rho(r)/r \in \mathcal{K}_\infty$ using [Sontag and Teel, 1995, Thm 1].

predictor (8)*, we have for any solution with maximal interval of existence* $[0, t_{\max})$*,*

$$\dot{V}(t) \leq -\gamma(|x(t)|) - \rho(2L(t)) + \rho(2L_K|e(\phi(t))|), \quad (17)$$

for all $t \in [0, t_{\max}) \setminus \{\bar{t}\}$ *and* $V(\bar{t}^-) \geq V(\bar{t}^+)$*, where* L_K *is the Lipschitz constant of* K *and* $\bar{t} \in [0, \sigma(0)]$ *is the greatest time such that* $w(t) = 0$ *for all* $t > \bar{t}$*.*

Proof. Using (15), we have

$$\begin{aligned}
&\mathcal{L}_f S(x(t)) \\
&\leq -\gamma(|x(t)|) + \rho\big(|w(\phi(t))| + |K(x(t)+e(\phi(t))) - K(x(t))|\big) \\
&\leq -\gamma(|x(t)|) + \rho\big(|w(\phi(t))| + L_K|e(\phi(t))|\big) \\
&\leq -\gamma(|x(t)|) + \rho(2|w(\phi(t))|) + \rho(2L_K|e(\phi(t))|). \quad (18)
\end{aligned}$$

In the following, we provide a rigorous proof of the fact $\dot{L}(t) = -bL(t)$ stated in [Bekiaris-Liberis and Krstic, 2013]. Similar to Lemma 8.9 therein, it holds that

$$L(t) = \lim_{n\to\infty} \left[\int_t^{\sigma(t)} e^{2nb(\tau - t)} w(\phi(\tau))^{2n} d\tau \right]^{\frac{1}{2n}} \triangleq \lim_{n\to\infty} L_n(t),$$

since $e^{-b(t-\tau)} w(\phi(\tau))$ is bounded for $\tau \in [t, \sigma(t)]$ and any $t \geq 0$ and $[t, \sigma(t)]$ has finite measure. In fact, it can be shown that this convergence is uniform over $[0, t_1]$ for any $t_1 < \bar{t}$. Therefore, since $\dot{L}_n(t) = -bL_n(t) - \frac{L_n}{2n}\left(\frac{w(\phi(t))}{L_n}\right)^{2n}$, $\frac{w(\phi(t))}{L_n} < 1$ for $t \in [0, t_1]$ and sufficiently large n and b, and $t_1 \in [0, \bar{t})$ is arbitrary, it follows from [Rudin, 1976, Thm 7.17] that $\dot{L}(t) = -bL(t)$ for $t \in (0, \infty) \setminus \{\bar{t}\}$. Combining this and (18), we get

$$\begin{aligned}
\dot{V}(t) &\leq -\gamma(|x(t)|) + \rho(2|w(\phi(t))|) + \rho(2L_K|e(\phi(t))|) \\
&\quad + \frac{2}{b} 2\dot{L}(t) \frac{\rho(2L(t))}{2L(t)} \\
&\leq -\gamma(|x(t)|) + \rho(2|w(\phi(t))|) + \rho(2L_K|e(\phi(t))|) \\
&\quad - 2\rho(2L(t)).
\end{aligned}$$

for $t \in (0, \infty) \setminus \{\bar{t}\}$. Equation (17) thus follows since $|w(\phi(t))| \leq L(t)$ (c.f. (16b)) and the fact that ρ is strictly increasing. Finally, since $S(x(t))$ is continuous, $L(\bar{t}^-) \geq 0$, and $L(\bar{t}^+) = 0$, we get $V(\bar{t}^-) \geq V(\bar{t}^+)$. ■

Proposition 4.2 is the basis for our event-trigger design. Formally, we select $\theta \in (0, 1)$ and require

$$\rho(2L_K|e(\phi(t))|) \leq \theta\gamma(|x(t)|), \qquad t \geq 0,$$

which can be equivalently written as

$$|e(t)| \leq \frac{\rho^{-1}(\theta\gamma(|p(t)|))}{2L_K}, \qquad t \geq \phi(0). \qquad (19)$$

Notice from (11) and the fact $t = 0$ that (19) holds on $[\phi(0), t_0]$. Equation (19) fully specifies the sequence of times $\{t_k\}_{k=1}^{\infty}$ and its dependence on the actuation delay. For each k, after each time t_k, the controller keeps evaluating (19) until it reaches equality. At this time, labeled t_{k+1}, the controller triggers the next event that sets $e(t_{k+1}) = 0$ and maintains (19). Notice that "larger" γ and "smaller" ρ (corresponding to "stronger" input-to-state stability in (2)) are then more desirable, as they are intuitively expected to let the controller update u less often. Our ensuing analysis shows global asymptotic stability of the closed-loop system and the existence of a uniform lower bound on the inter-event times.

4.3 Convergence Analysis under Event-triggered Law

In this section we show that our event triggered law (19) solves Problem 1 by showing, in the following result, that the inter-event times are uniformly lower bounded (so, in particular, there is no finite accumulation point in time) and the closed-loop system achieves global asymptotic stability.

Theorem 4.3 *(Uniform Lower Bound for the Inter-Event Times and Global Asymptotic Stability): Suppose that the class* $\mathcal{K}_\infty$ *function* $\mathcal{G} : r \mapsto \gamma^{-1}(\rho(r)/\theta)$ *is (locally) Lipschitz. For the system* (3) *under the control* (10) *and the triggering condition* (19)*, the following hold:*

(i) there exists $\delta = \delta(x(0), \{u(t)\}_{t=\phi(0)}^{0}) > 0$ *such that* $t_{k+1} - t_k \geq \delta$ *for all* $k \geq 1$*,*

(ii) there exists $\beta \in \mathcal{KL}$ *such that for any* $x(0) \in \mathbb{R}^n$ *and bounded* $\{u(t)\}_{t=\phi(0)}^{0}$*, we have for all* $t \geq 0$*,*

$$|x(t)| + \sup_{\phi(t)\leq\tau\leq t} |u(\tau)| \leq \beta\Big(|x(0)| + \sup_{\phi(0)\leq\tau\leq 0} |u(\tau)|, t\Big). \quad (20)$$

Proof. Let $[0, t_{\max})$ be the maximal interval of existence of the solutions of the closed-loop system. The proof involves three steps. First, we prove that (ii) holds for $t < t_{\max}$. Then, we show that (i) holds until $t_{\max}$, and finally that $t_{\max} = \infty$.

Step 1: From Proposition 4.2 and (19), we have

$$\begin{aligned}
\dot{V}(t) &\leq -(1-\theta)\gamma(|x(t)|) - \rho(2L(t)) \\
&\leq -\gamma_{\min}(|x(t)| + L(t)), \qquad t \in [0, t_{\max}) \setminus \{\bar{t}\},
\end{aligned}$$

where $\gamma_{\min}(r) = \min\{(1-\theta)\gamma(r), \rho(2r)$ for all $r \geq 0$, so $\gamma_{\min} \in \mathcal{K}$. Also, note that

$$V(t) \leq \alpha_2(|x(t)|) + \alpha_0(L(t)) \leq 2\alpha_{\max}(|x(t)| + L(t)),$$

where $\alpha_{\max}(r) = \max\{\alpha_2(r), \alpha_0(r)\}$ and $\alpha_0(r) = \frac{2}{b}\int_0^{2r} \frac{\rho(s)}{s} ds$ for all $r \geq 0$. Since $\alpha_0, \alpha_2 \in \mathcal{K}_\infty$, we have $\alpha_{\max} \in \mathcal{K}_\infty$, so $\alpha_{\max}^{-1} \in \mathcal{K}$. Hence,

$$\dot{V}(t) \leq -\alpha_{\min}(\alpha_{\max}^{-1}(V(t)/2)) \triangleq \overline{\alpha}(V(t)), \; t \in [0, t_{\max}) \setminus \{\bar{t}\},$$

where $\overline{\alpha} \in \mathcal{K}$. Therefore, using the Comparison Principle [Khalil, 2002, Lemma 3.4], [Khalil, 2002, Lemma 4.4], and $V(\bar{t}^-) \geq V(\bar{t}^+)$, there exists $\beta_1 \in \mathcal{KL}$ such that $V(t) \leq \beta_1(V(0), t)$, $t < t_{\max}$. Therefore,

$$|x(t)| + L(t) \leq \beta_2(|x(0)| + L(0), t), \qquad t < t_{\max},$$

where $\beta_2(r, s) = \alpha_{\min}^{-1}(\overline{\beta}(2\alpha_{\max}(r), s))$ for any $r, s \geq 0$. Note that $\beta_2 \in \mathcal{KL}$. Since we have

$$\sup_{\phi(t)\leq\tau\leq t} |w(\tau)| \leq L(t) \leq e^{bM_0} \sup_{\phi(t)\leq\tau\leq t} |w(\tau)|,$$

it then follows that

$$|x(t)| + \sup_{\phi(t)\le\tau\le t} |w(\tau)| \le \beta_3\Big(|x(0)| + \sup_{\phi(0)\le\tau\le 0} |w(\tau)|, t\Big), \quad (21)$$

for all $t < t_{\max}$, where $\beta_3(r,s) = \beta_2(e^{bM_0}r, s)$. This inequality leads to (20) using the same steps as in [Bekiaris-Liberis and Krstic, 2013, Lemmas 8.10, 8.11] (the only difference being the multiplicity of inputs).

Step 2: Equation (19) can be rewritten as

$$|p(t)| \ge \gamma^{-1}\Big(\frac{\rho(2L_K|e(t)|)}{\theta}\Big).$$

From step 1, the prediction $p(t) = x(\sigma(t))$ and its error $e(t) = p(t_k) - p(t)$ are bounded. Therefore, there exists $L_{\gamma^{-1}\rho/\theta} > 0$ such that for all $t \ge 0$,

$$\gamma^{-1}\Big(\frac{\rho(2L_K|e(t)|)}{\theta}\Big) \le 2L_{\gamma^{-1}\rho/\theta}L_K|e(t)|.$$

where $L_{\gamma^{-1}\rho/\theta}$ is the Lipschitz constant of $\mathcal{G}$ on the compact set that contains $\{e(t)\}_{t=0}^{t_{\max}}$. Hence, a sufficient (stronger) condition for (19) is

$$|p(t)| \ge 2L_{\gamma^{-1}\rho/\theta}L_K|e(t)|. \quad (22)$$

Note that (22) is only for the purpose of analysis and is *not* executed in place of (19). Clearly, if the inter-event times of (22) are lower bounded, so are the inter-event times of (19). Let $r(t) = \frac{|e(t)|}{|p(t)|}$ for any $t \ge 0$ (with $r(t) = 0$ if $p(t) = 0$). For any $k \ge 0$, we have $r(t_k) = 0$ and $t_{k+1} - t_k$ is greater than or equal to the time that it takes for $r(t)$ to go from 0 to $\frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K}$. Note that for any $t \ge 0$,

$$\begin{aligned}
\dot r &= \frac{d}{dt}\frac{|e|}{|p|} = \frac{d}{dt}\frac{(e^Te)^{1/2}}{(p^Tp)^{1/2}} \\
&= \frac{(e^Te)^{-1/2}e^T\dot e(p^Tp)^{1/2} - (p^Tp)^{-1/2}p^T\dot p(e^Te)^{1/2}}{p^Tp} \\
&= -\frac{e^T\dot p}{|e||p|} - \frac{|e|p^T\dot p}{|p|^3} \le \frac{|\dot p|}{|p|} + \frac{|e||\dot p|}{|p|^2} = (1+r)\frac{|\dot p|}{|p|},
\end{aligned}$$

where the time arguments are dropped for better readability. To upper bound the ratio $|\dot p(t)|/|p(t)|$, we have from (8) that $\dot p(t) = \dot\sigma(t)f(p(t), u(t))$ for all $t \ge \phi(0)$. By continuous differentiability of f (which implies Lipschitz continuity on compacts) and global asymptotic stability of the closed loop system, there exists $L_f > 0$ such that

$$\begin{aligned}
|\dot p(t)| &= |\dot\sigma(t)f(p(t), u(t))| \le M_2|f(p(t), K(p(t)+e(t)))| \\
&\le M_2L_f|(p(t), K(p(t)+e(t)))| \\
&\le M_2L_f(|p(t)| + |K(p(t)+e(t))|) \\
&\le M_2L_f(|p(t)| + L_K|p(t)+e(t)|) \\
&\le M_2L_f(1+L_K)|p(t)| + M_2L_fL_K|e(t)| \\
\Rightarrow \dot r(t) &\le M_2(1+r(t))(L_f(1+L_K) + L_fL_K|r(t)|).
\end{aligned}$$

Thus, using the Comparison Principle [Khalil, 2002, Lemma 3.4], we have $t_{k+1} - t_k \ge \delta, k \ge 0$ where δ is the time that it takes for the solution of

$$\dot r = M_2(1+r)(L_f(1+L_K) + L_fL_Kr), \quad (23)$$

to go from 0 to $\frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K}$.

Step 3: Since all system trajectories are bounded and $t_k \xrightarrow{k\to\infty} \infty$, we have $t_{\max} = \infty$, completing the proof. ■

A particular corollary of Theorem 4.3 is that the proposed event-triggered law does not suffer from Zeno behavior, i.e., t_k accumulating to a finite point $t_{\max}$. Also, note that the lower bound δ in general depends on the initial conditions $x(0)$ and $\{u(t)\}_{t=\phi(0)}^{0}$ through the Lipschitz constant $L_{\gamma^{-1}\rho/\theta}$.[5] Finally, while Theorem 4.3 explicitly bounds x and u, the simple time-shift relationship (8) between p and x ensures that any bound satisfied by $x(t), t \ge 0$, including that of Theorem 4.3, is also satisfied by $p(t), t \ge 0$.

4.4 *Delayed and Event-Triggered Sensing*

So far, we have not considered any delays in the availability of the sensing information about the plant state, which we consider next. Our treatment here shows that the above event-triggered controller with *the same triggering condition* (19)*, and with slight adjustments in the employed control and predictor signals,* globally asymptotically stabilizes the plant while maintaining the same lower bound on the inter-event times.

To address the general scenario in Problem 1, let

$$\bar\ell = \bar\ell(t) = \max\{\ell \ge 0 \,|\, \tau_\ell \le \psi(t)\},$$

be the index of the last plant state available at the controller at time t. Then, (8) is replaced with

$$p(t) = x(\tau_{\bar\ell}) + \int_{\phi(\tau_{\bar\ell})}^{t} \dot\sigma(s)f(p(s), u(s))ds, \; t \ge \psi^{-1}(0), \quad (24)$$

which is the best estimate of $x(\sigma(t))$ available to the controller[6]. Since $p(t)$ is not available before $\psi^{-1}(0)$, the control signal (5), (10) is updated as

$$u(t) = \begin{cases} K(p(t_k)) & \text{if} \quad t \in [t_k, t_{k+1}), \; k \ge 0, \\ 0 & \text{if} \quad t \in [0, t_0), \end{cases} \quad (25)$$

where the first event time is now $t_0 = \psi^{-1}(0)$. We next provide the same guarantees as Theorem 4.3.

Theorem 4.4 *Consider the plant dynamics* (3) *driven by the predictor-based event-triggered controller* (25) *with the predictor* (24) *and triggering condition* (19)*. Under Assumption 3.1, the closed-loop system is globally asymptotically stable, namely, there exists $\beta \in \mathcal{KL}$ such*

[5] However, for any given compact set of $|x(0)|$ and $|u(t)|, t < 0$, equations (21), (12), (11), and (8) ensure that $x(t)$ and therefore $p(t)$ are bounded for all t, and so $e(t)$ belongs to a compact set due to (19). Hence, $L_{\gamma^{-1}\rho/\theta}$ and thus δ can be chosen uniformly over this set.

[6] This only requires the controller to know $\psi(\tau_\ell)$ for every received state (not the full function ψ), which is realized by having a time-stamp for $x(\tau_\ell)$.

that (20) *holds for all* $x(0) \in \mathbb{R}^n$, *continuously differentiable* $\{u(t)\}_{t=\phi(0)}^{0}$, *and* $t \geq 0$. *Furthermore, there exists* $\delta = \delta(x(0), \{u(t)\}_{t=\phi(0)}^{0}) > 0$ *such that* $t_{k+1} - t_k \geq \delta$ *for all* $k \geq 0$.

Proof. For simplicity, let $U(t) = \sup_{\phi(t)\leq\tau\leq t} |u(t)|$. Since the open-loop system exhibits no finite escape time behavior, the state remains bounded during the initial period $[0, t_0]$. Hence, for any $x(0)$ and any $\{u(t)\}_{t=\phi(0)}^{0}$ there exists $\Xi > 0$ such that $|x(t)| \leq \Xi$ for $t \in [0, t_0]$. Without loss of generality, Ξ can be chosen to be a class $\mathcal{K}$ function of $|x(0)| + U(0)$. Thus,

$$
\begin{aligned}
|x(t)|+U(t) &\leq \Xi(|x(0)| + U(0)) + U(0) \qquad (26)\\
&\leq \big[\Xi(|x(0)| + U(0)) + U(0)\big]e^{-(t-t_0)}, \quad t \in [0, t_0].
\end{aligned}
$$

As soon as the controller receives $x(0)$ at t_0, it can estimate the state $x(t)$ by simulating the dynamics (3), i.e.,

$$
x(t) = x(0) + \int_0^t f(x(s), u(\phi(s)))ds. \tag{27}
$$

This estimation is updated whenever a new state $x(\tau_\ell)$ arrives and used to compute the predictor (8), which combined with (27) takes the form (24). Since the controller now has access to the same prediction signal $p(t)$ as before, the same Lyapunov analysis as above holds for $[t_0, \infty)$. Therefore, let $\hat{\beta} \in \mathcal{KL}$ be such that (20) holds for $t \geq t_0$. By (26),

$$
|x(t)| + U(t) \leq \hat{\beta}\big(\Xi(|x(0)| + U(0)) + U(0), t - t_0\big) \quad t \geq t_0.
$$

Therefore, (20) holds by choosing $\beta(r, t) = \max\big\{\hat{\beta}\big(\Xi(r)+ r, t - t_0\big), \big[\Xi(r) + r\big]e^{-(t-t_0)}\big\}$. Finally, since the triggering condition (19) has not changed, $t_{k+1} - t_k \geq \delta, k \geq 0$ for the same $\delta > 0$ as in Theorem 4.3. ■

Remark 4.5 *(Separation of sensing and actuation delays):* It is a standard practice in the literature to combine the sensing and actuation delays into a single quantity, i.e., "networked induced delays". This is in fact the basis of the predictor design in equation (23). However, in our treatment, it is beneficial to keep the two delays distinct since their sources are often physically distinct and the assumptions on the sensing delay ψ are significantly weaker than on the actuator delay ϕ (cf. Assumption 3.1). •

Remark 4.6 *(Practical importance of feedback):* While the controller can theoretically discard $\{x(\tau_\ell)\}_{\ell=1}^{\infty}$ and rely on $x(0)$ for estimating the state at all future times, closing the loop using the most recent state value $x(\tau_{\bar{\ell}})$ is in practice critical for preventing the estimator (27) from drifting due to noise and un-modeled dynamics, even when the system dynamics are perfectly known. This is apparent, for instance, in Example 6.2 shown later, where facing the errors caused by the numerical approximation of the prediction signal. •

5 The Linear Case

Here, we specialize the general treatment of Section 4 to the linear case

$$
\dot{x}(t) = Ax(t) + Bu(\phi(t)), \quad \text{a.a. } t \geq 0, \quad x(0) = x_0. \tag{28}
$$

For simplicity, we restrict our attention to the perfect sensing case, with similar generalizations to sampled and delayed sensing as in Section 4.4. Assuming that the pair (A, B) is stabilizable, we can use pole placement to find a linear feedback law $K : \mathbb{R}^n \to \mathbb{R}$ that satisfies Assumption 3.1(v). Moreover, $p(t)$ has the explicit form

$$
p(t) = e^{A(\sigma(t)-t_+)}x(t_+) + \int_{\phi(t_+)}^{t} \dot{\sigma}(s)e^{A(\sigma(t)-\sigma(s))}Bu(s)ds, \tag{29}
$$

for all $t \geq \phi(0)$ and the closed-loop system takes the form

$$
\dot{x}(t) = (A + BK)x(t) + Bw(\phi(t)) + BKe(\phi(t)).
$$

Furthermore, given an arbitrary $Q = Q^T > 0$, the continuously differentiable function $S : \mathbb{R}^n \to \mathbb{R}$ is $S(x) = x^T Px$, where $P = P^T > 0$ is the unique solution to the Lyapunov equation $(A + BK)^T P + P(A + BK) = -Q$. Clearly, (14) holds with $\alpha_1(r) = \lambda_{\min}(P)r^2$ and $\alpha_2(r) = \lambda_{\max}(P)r^2$. Also, using Young's inequality [Young, 1912],

$$
\mathcal{L}_f S = -x(t)^T Qx(t) + 2x(t)^T PB(w(\phi(t)) + Ke(\phi(t))),
$$

so (15) holds with $\gamma(r) = \frac{1}{2}\lambda_{\min}(Q)r^2$ and $\rho(r) = \frac{2|PB|^2}{\lambda_{\min}(Q)}r^2$. Thus, (19) also takes the simpler form

$$
|e(t)| \leq \frac{\lambda_{\min}(Q)\sqrt{\theta}}{4|PB||K|}|p(t)|. \tag{30}
$$

In addition to these simplifications, we show next that the closed-loop system is globally exponentially stable.

5.1 *Exponential Stability under Event-triggered Control*

We next show that, in the linear case, we obtain the stronger feature of global exponential stability using a slightly different Lyapunov-Krasovskii functional.

Theorem 5.1 *(Exponential Stabilization): The system* (28) *subject to the piecewise-constant closed-loop control* $u(t) = Kp(t_k)$, $t \in [t_k, t_{k+1})$, *with* $p(t)$ *given in* (29) *and* $\{t_k\}_{k=1}^{\infty}$ *determined according to* (30) *satisfies*

$$
|x(t)|^2 + \int_{\phi(t)}^{t} u(\tau)^2 d\tau \leq Ce^{-\mu t}\Big(|x(0)|^2 + \int_{\phi(0)}^{0} u(\tau)^2 d\tau\Big),
$$

for some $C > 0$, $\mu = \frac{(2-\theta)\lambda_{min}(Q)}{4\lambda_{max}(P)}$, *and all* $t \geq 0$.

Proof. For $t \geq 0$, let $L(t) = \int_t^{\sigma(t)} e^{b(\tau-t)}w(\phi(\tau))^2 d\tau$. One can see that $\dot{L}(t) = -w(\phi(t))^2 - bL(t)$, $t \geq 0$. Define $V(t) = x(t)^T Px(t) + \frac{4|PB|^2}{\lambda_{\min}(Q)}L(t)$. Therefore, using (30),

$$
\begin{aligned}
\dot{V}(t) &= -x(t)^T Qx(t) + 2x(t)^T PBw(\phi(t)) - \frac{4|PB|^2 b}{\lambda_{\min}(Q)}L(t)\\
&\quad + 2x(t)^T PBKe(\phi(t)) - \frac{4|PB|^2}{\lambda_{\min}(Q)}w(\phi(t))^2\\
&\leq -\frac{2-\theta}{4}\lambda_{\min}(Q)|x(t)|^2 - \frac{4|PB|^2 b}{\lambda_{\min}(Q)}L(t) \leq -\mu V(t),
\end{aligned}
$$

where $\mu = \min\left\{\frac{(2-\theta)\lambda_{\min}(Q)}{4\lambda_{\max}(P)}, b\right\} = \frac{(2-\theta)\lambda_{\min}(Q)}{4\lambda_{\max}(P)}$ if b is chosen sufficiently large. Hence, by the Comparison Principle [Khalil, 2002, Lemma 3.4], we have $V(t) \leq e^{-\mu t}V(0)$, $t \geq 0$. Let $W(t) = |x(t)|^2 + \int_{\phi(t)}^{t} u(\tau)^2 d\tau$. From [Bekiaris-Liberis and Krstic, 2013, Eq. (6-99)-(6-100)], $c_1 W(t) \leq V(t) \leq c_2 W(t)$, for some $c_1, c_2 > 0$ and all $t \geq 0$. Hence, the result follows with $C = c_2/c_1$. ■

From Theorem 5.1, the convergence rate μ depends both on the ratio $\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$ and the parameter θ. The former can be increased by placing the eigenvalues of $A+BK$ at larger negative values, though large eigenvalues result in noise amplification. Decreasing θ, however, comes at the cost of faster control updates, a trade-off we study next.

5.2 *Optimizing the Sampling-Convergence Trade-off*

Here, we analyze the trade-off between sampling frequency and convergence speed. In general, it is clear from the Lyapunov analysis of Section 4 that more updates (intuitively corresponding to smaller θ) hasten the decay of $V(t)$ and help convergence. Let δ be the time that it takes for the solution of (23) to go from 0 to $\frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K}$. As shown in Section 4.3, the inter-event times are lower bounded by δ, so it can be used to bound the sampling cost of implementing the controller. Let

$$a = M_2 L_f L_K, \quad c = M_2 L_f(1 + L_K), \quad R = \frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K},$$

where $L_f = \sqrt{2}(|A| + |B|)$, $L_K = |K|$, and $L_{\gamma^{-1}\rho/\theta} = \frac{2|PB|}{\lambda_{\min}(Q)\sqrt{\theta}}$. Then, the solution of (23) with initial condition $r(0) = 0$ is $r(t) = \frac{ce^{at} - ce^{ct}}{ae^{ct} - ce^{at}}$, so solving $r(\delta) = R$ for δ gives $\delta = \frac{\ln \frac{c+Ra}{c+Rc}}{a-c}$. The objective is to maximize δ and μ by tuning the optimization variables θ and Q. For simplicity, let $\theta = \nu^2$ and $Q = qI_n$ where $\nu, q > 0$. Then,

$$\delta(\nu) = \frac{1}{a-c}\ln\frac{c + \frac{\nu}{|P_1 B||K|}a}{c + \frac{\nu}{|P_1 B||K|}c}, \quad \mu(\nu) = \frac{2-\nu^2}{4\lambda_{\max}(P_1)},$$

where $P_1 = q^{-1}P$ is the solution of the Lyapunov equation $(A+BK)^T P_1 + P_1(A+BK) = -I_n$. Figure 2(a) depicts δ and μ as functions of ν and illustrates the sampling-convergence trade-off.

To balance these two objectives, we define the aggregate objective function as a convex combination of δ and μ,

$$J(\nu) = \lambda\delta(\nu) + (1-\lambda)\mu(\nu),$$

where $\lambda \in [0,1]$ determines the relative importance of convergence rate and sampling. The function J is strongly convex and its unique maximizer is the positive real solution of $c_3\nu^3 + c_2\nu^2 + c_1\nu + c_0 = 0$, where $c_3 = a(1-\lambda)$, $c_2 = (a+c)|P_1 B||K|(1-\lambda)$, $c_1 = c|P_1 B|^2|K|^2(1-\lambda)$, and $c_0 = -2\lambda_{\max}(P_1)|P_1 B||K|\lambda$. Figure 2(b) plots this maximizer for different values of λ, further illustrating the sampling-convergence trade-off.

6 Simulations

Here we illustrate the performance of our event-triggered predictor-based design. Example 6.2 is a two-dimensional

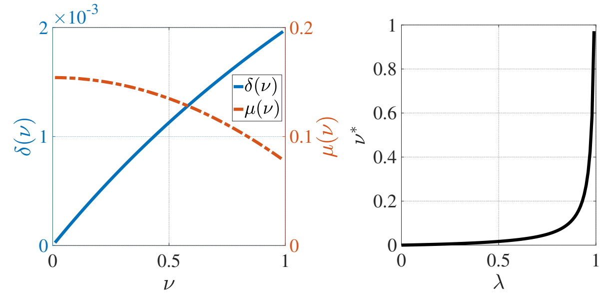

Fig. 2. Sampling-convergence trade-off for event-triggered control of linear systems. **(Left)**, values of the lower bound of the inter-event times (δ) and exponential rate of convergence (μ) for different values of the optimization parameter ν for a 3rd-order unstable linear system with $M_2 = 1$. **(Right)**, unique maximizer ν^* of the objective function $J(\nu)$ for different values of λ. As λ goes from 0 to 1, more weight is given to the maximization of δ, which increases ν^*.

nonlinear system that satisfies all the hypotheses required to ensure global asymptotic convergence of the closed-loop system. Example 6.3 is a different two-dimensional nonlinear system which, instead, does not but for which we observe convergence in simulation. We start by discussing some numerical challenges that arise because of the particular hybrid nature of our design, along with our approach to tackle them.

Remark 6.1 *(Numerical implementation of event-triggered control law):* The main challenge in the numerical simulation of the proposed event-trigger law is the computation of the prediction signal $p(t) = x(\sigma(t))$. To this end, at least three methods can be used, as follows:
(i) *Open-loop*: One can solve $\dot{p}(t) = \dot{\sigma}(t)f(p(t), u(t))$ directly starting from $p(\phi(0)) = x(0)$. The closed-loop system takes the form of a hybrid system (see, e.g., [Goebel et al., 2012] for an introduction to hybrid systems) with flow map

$$\dot{x}(t) = f(x(t), u(\phi(t))), \tag{31a}$$
$$\dot{p}(t) = \dot{\sigma}(t)f(p(t), u(t)), \tag{31b}$$
$$\dot{p}_{tk}(t) = 0, \tag{31c}$$
$$u(t) = K(p_{tk}(t)), \tag{31d}$$

jump map $p_{tk}((t_k)_+) = p((t_k)_+)$, jump set $D = \left\{(x, p, p_{tk}) \mid |p_{tk} - p| = \frac{\rho^{-1}(\theta\gamma(|p|))}{2L_K}\right\}$, and flow set $C = \overline{\mathbb{R}^{3n} \setminus D}$. Note that x and p do not change at jumps (i.e., identity maps). Here, the values of p (resp. p_{tk} and u) of any hybrid solution are arbitrary in the interval $[0, \phi(0))$ (resp. $[0, t_0)$). This formulation is computationally efficient but, if the original system is unstable, it is prone to numerical instabilities. The reason, suggesting the name "open-loop", is that the (p, p_{tk})-subsystem is completely decoupled from the x-subsystem. Therefore, as stated in Remark 4.6, if any mismatch occurs between $x(t)$ and $p(\phi(t))$ due to numerical errors, the x-subsystem tends to become unstable, and this is not "seen" by the (p, p_{tk})-subsystem.
(ii) *Semi-closed-loop*: One can add a feedback path from the x-subsystem to the (p, p_{tk}) subsystem by computing p directly from (24) every time a new state value arrives (i.e., at every $\psi^{-1}(\tau_\ell)$). This requires a numerical integration of $f(p(s), u(s))$ over the "history" of (p, u) from

$\phi(\tau_{\bar{\ell}})$ to t. This method is more computationally expensive but improves the numerical robustness. However, since we are still integrating over the history of p, any mismatch in the prediction takes more time to die out, which may not be tolerable for an unstable system.
(iii) *Closed-loop*: To further increase robustness, one can solve the differential form in (31b) rather than the integral form in (24) every time a new state value arrives (i.e., at every $\psi^{-1}(\tau_\ell)$) from $\phi(\tau_{\bar{\ell}})$ to t with "initial" condition $p(\phi(\tau_{\bar{\ell}})) = x(\tau_{\bar{\ell}})$. This method is as computationally expensive as (ii) but is considerably more robust. This is therefore the recommended method for the numerical implementation of the proposed predictor-based controller and used below in Examples 6.2 and 6.3. •

Example 6.2 *(Compliant Nonlinear System):* Consider the 2-dimensional system given by

$$f(x,u) = \begin{bmatrix} x_1 + x_2 \\ \tanh(x_1) + x_2 + u \end{bmatrix}, \quad \phi(t) = t - \frac{(t-5)^2 + 2}{2(t-5)^2 + 2},$$
$$\tau_\ell = \ell\Delta_\tau, \quad \ell \geq 0, \qquad \psi(t) = t - D_\psi,$$

where Δ_τ and D_ψ are constants. This system satisfies Assumption 3.1 with the feedback law $K(x) = -6x_1 - 5x_2 - \tanh(x_1)$, $S(x) = x^T P x$, and

$$L_f = \frac{\sqrt{2\sqrt{17}+10}}{2}, \quad L_K = \sqrt{74}, \quad (M_1, m_2) = 1 \pm \frac{3\sqrt{3}}{16},$$
$$M_0 = 1, \quad \gamma(r) = \frac{\lambda_{\min}(Q)}{2} r^2, \quad \rho(r) = \frac{2|PB|^2}{\lambda_{\min}(Q)} r^2,$$

where $P = P^T > 0$ is the solution of $(A+Bk)^T P + P(A+Bk) = -Q$ for $A = [1\ 1;\ 0\ 1]$, $B = [0;\ 1]$, $k = [-6\ \ -5]$, and arbitrary $Q = Q^T > 0$ (we use $Q = I$). A sample simulation result of this system is depicted in Figure 3(a). It is to be noted that for this example, (19) simplifies to $|e(t)| \leq \bar{\rho}|p(t)|$ with $\bar{\rho} = 0.022$, but the closed-loop system remains stable when increasing $\bar{\rho}$ about until 0.8 (Figure 3(b)).

While Theorem 4.3 guarantees the global asymptotic stability of the continuous-time system, discretization accuracy/error plays an important role in its digital implementation. It is with this in mind that one should interpret Figure 3(c), where depending on the discretization scheme and the stepsize employed, the numerical approximation errors in computing the prediction signal, cf. Remark 6.1, make the evolution of the Lyapunov function V not monotonically decreasing (whereas we know from Theorem 4.3 that it is monotonically decreasing for the continuous-time system). We see that, at least for this example, the effect on the evolution of V is sensitive to both the order of discretization and the stepsize (h), and benefits more from decreasing the latter.

Stability is also critically dependent on the sensing sampling rate $1/\Delta_\tau$, as noted in Remark 4.6. We can also see from Figure 3(c) that the decay of V clearly deteriorates for large Δ_τ (insufficient sampling) due to (in this example only discretization) noise but can be made monotonic for sufficiently small Δ_τ. To visualize this effect on stability more systematically, we varied Δ_τ and D_ψ and computed $|x(25)|$ as a measure of asymptotic stability. The average result is depicted in Figure 3(d) for 10 random initial conditions, showing that unlike our theoretical expectation, large Δ_τ and/or D_ψ result in instability even in the absence of noise because of the numerical error that degrades the estimation (27) over time (c.f. Remark 6.1). Nevertheless, taking the delays and sampling into account while designing the controller using the predictor-based scheme (10) significantly increases the robustness of the closed-loop system relative to a design that is oblivious to delays and sampling. As shown in [Mazenc et al., 2013], the asymptotic stability of the latter can only be guaranteed for this example *without actuation delays and event-triggering* if $\Delta_\tau + D_\psi \leq 7.1 \times 10^{-3}$ (given that, using the notation therein, we have $c_1 = 25, c_2 = 29/9, c_3 = 772$), which is more than two orders of magnitude more conservative than the empirical bound shown in Figure 3(d).

Finally, we have investigated the robustness of the closed-loop system to external disturbances (which are not theoretically included in our analysis but inevitably exist in practice). In an event-triggered system, disturbances may lead to instability and/or Zeno behavior, cf. [Dolk et al., 2017]. However, as shown in Figure 3(e-f), neither instability nor Zeno behavior occurs when adding (any strength of) the disturbance here, highlighting the practical relevance of the proposed event-triggered scheme. •

Example 6.3 *(Non-compliant Nonlinear System):* Here, we consider an example that violates several of our assumptions. Let

$$f(x,u) = (A + \Delta A)x + Bu + Ex_1^3, \quad E = [0\ \ 1]^T,$$
$$t - \phi(t) = D + a\sin(t),\ \tau_\ell = \ell\Delta_\tau,\ \psi(t) = t - \frac{1 - e^{-t}}{2},$$

where A and B are as in Example 6.2. The nominal delay D and nominal coefficient matrix A are known but their perturbations $a\sin(t)$ and ΔA are not (the controller *assumes* $\phi(t) = t - D$ and $f(x,u) = Ax + Bu + Ex_1^3$). We generate the elements of ΔA independently from $\mathcal{N}(0, \sigma_A^2)$. Furthermore, in our simulation, the actual time that it takes for a sensor message $x(\tau_\ell)$ to reach the controller is *not* the nominal delay $\psi^{-1}(\tau_\ell) - \tau_\ell$ but a random variable D_ℓ^ψ, where

$$E[D_\ell^\psi] = \psi^{-1}(\tau_\ell) - \tau_\ell, \quad \mathrm{var}(D_\ell^\psi) = \sigma_\psi > 0.$$

This serves to illustrate how the delay function ψ (and similarly ϕ), though being continuous and deterministic in our treatment, can be used to compensate for (in addition to physical sensor lag) computation and communication delays that are discrete and stochastic in nature [7].

Moreover, $K(x) = -6x_1 - 5x_2 - x_1^3$ makes the closed-loop system ISS but is not globally Lipschitz, and the zero-input system exhibits finite escape time. The simulation results of this example are illustrated in Figure 4. It can be seen that although V is significantly non-monotonic, the event-triggered controller is able to stabilize the system. While a thorough investigation of the stability of

[7] Since the triggering times τ_ℓ are themselves random and vary from execution to execution, the function ψ is defined for all t even though only the discrete sequence $\{\psi^{-1}(\tau_\ell)\}$ is relevant for each execution.

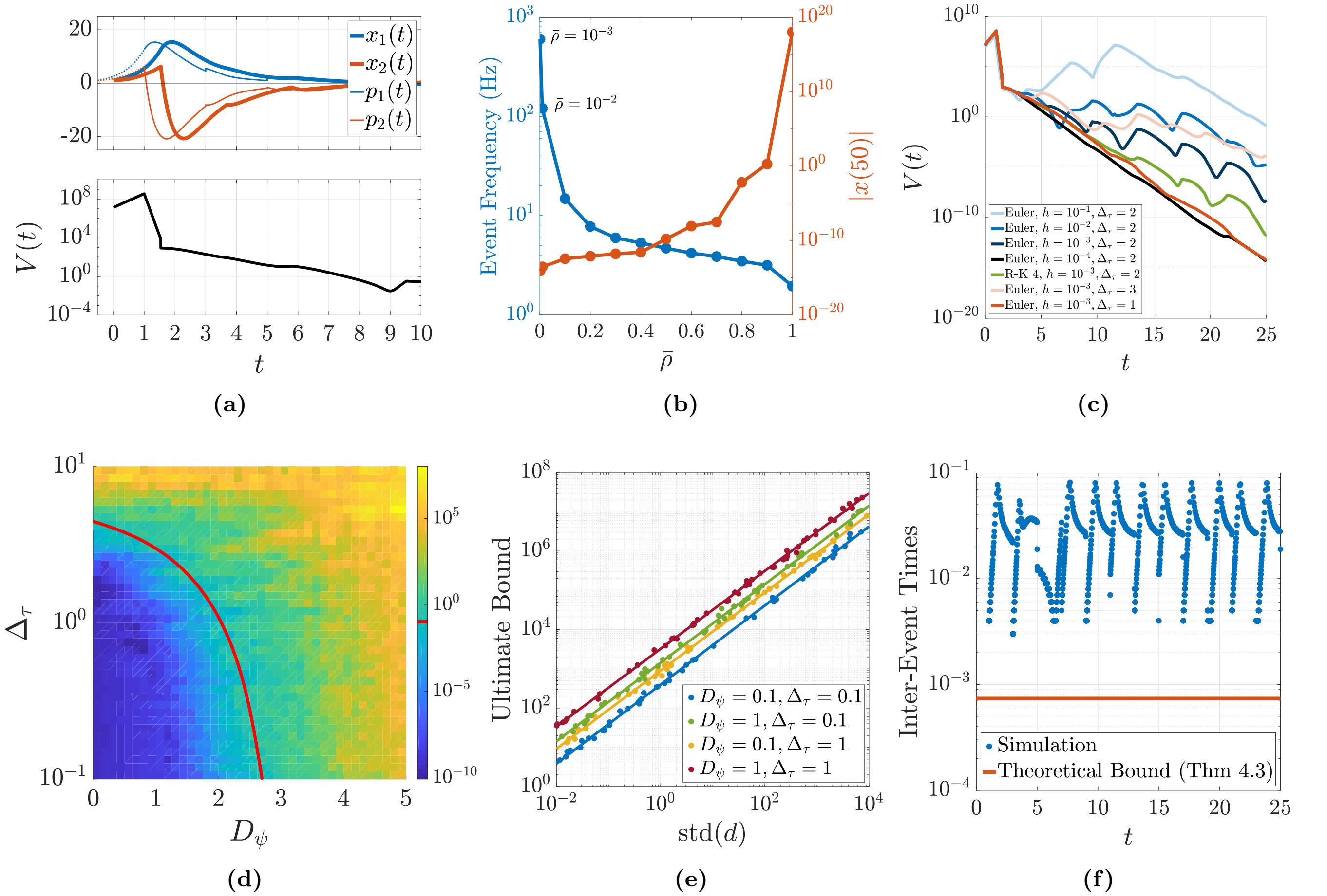

Fig. 3. Simulation results for Example 6.2. Unless otherwise stated, we use $x(0) = (1, 1)$, $\theta = 0.5$, $b = 10$, $\Delta_\tau = 2$, $D_\psi = 1$, and Euler discretization with $h = 10^{-3}$. **(a)** Sample trajectories. The dotted portion of $p(t)$ corresponds to the times $[\phi(0), \psi^{-1}(0))$ and is plotted only for illustration purposes (not used by the controller). **(b)** The event-frequency and average of $|x(50)|$ over 100 random initial conditions as a function of $\bar{\rho}$. **(c)** The effect of discretization and state sampling on stability. While stepsize h and sampling rate $1/\Delta_\tau$ have a strong impact on stability (blue and red curves, resp.), the effect of discretization order is less significant (green curve, 4th order Runge-Kutta). **(d)** Heat map of the average of $|x(25)|$ over 10 random initial conditions drawn from standard normal distribution. The red line shows an approximate border of stability. **(e-f)** Numerical verification of the robustness of the event-triggered controller to additive disturbances: we augment (3) as $\dot{x} = f(x, u_p) + d$, where d is zero-mean, white, and Gaussian. **(e)** The estimate of the ultimate bound of state ($\max_{i=1,2} \limsup_{t\to\infty} |x_i(t)|$) for varying standard deviation of d. The value of the ultimate bound depends on sampling delay and frequency, but the state always remains bounded for bounded disturbances and the best linear fit always has a slope $\simeq 1$, a behavior akin to globally input-to-state stable linear systems. **(f)** The inter-event times $\{t_{k+1} - t_k\}_{k\ge0}$ for $\mathrm{std}(d) = 1$. Unlike [Borgers and Heemels, 2014], the minimum inter-event time is lower bounded by δ in Theorem 4.3 irrespective of the existence or strength of disturbance (as long as $\Delta_\tau > \delta$) due to the fact that sensing only occurs at discrete-time instances $\{\tau_\ell\}$, making the controller oblivious to disturbance over each Δ_τ period. This may in principle lead to instability ($|x| \to \infty$) but we see from (e) that this is not the case.

the resulting stochastic dynamical system reaches far beyond our theoretical guarantees, this example suggests that the proposed controller is robust to small violations of its assumptions and is thus applicable to a wider class of systems than those satisfying Assumption 3.1. •

7 Conclusions and Future Work

We have proposed a prediction-based event-triggered control scheme for the stabilization of nonlinear systems with sensing and actuation delays. Assuming known time delay, globally-Lipschitz input-to-state stabilizability, and state feedback, we have shown that the closed-loop system is globally asymptotically stable and the inter-event times are uniformly lower bounded. We have specialized our results for linear systems, providing explicit expressions for our design and analysis steps, and further studied the sampling-convergence trade-off

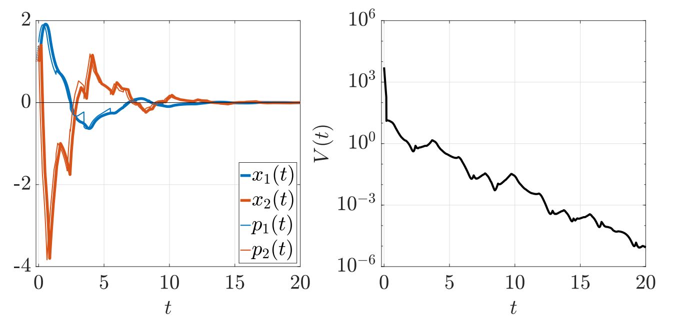

Fig. 4. Simulation of the non-compliant system in Example 6.3. We have used $x(0) = (1, 1)$, $\theta = 0.5$, $b = 10$, $a = 0.01$, $D = 0.2$, $\Delta_\tau = 1$, $\mu_\psi = 0.1$, $\sigma_\psi = \sigma_A = 0.02$, triggering condition $|e(t)| \le 0.5|p(t)|$, and Euler discretization of the continuous-time dynamics with $h = 10^{-2}$.

characteristic of event-triggered strategies. Finally, we have addressed the numerical challenges that arise in the computation of predictor feedback and demonstrated the effectiveness of our approach in simulation. Regarding future work, we highlight the extension of our results to systems with disturbances, unknown input delays, or output feedback, the characterization of the robustness properties resulting from incorporating the most recently available state information, the relaxation of the global Lipschitz requirement on the input-to-state stabilizer, and the study of the effect on performance of the numerical implementation of the event-triggered controller.

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