ESE 617/MEAM 613: Nonlinear Systems & Control (Fall 2019)

Homework #1

Due on 9/18/2019, 9 a.m., in class

1. **Qualitative behavior of second-order nonlinear systems.** As mentioned in class, the first step in the analysis of the behavior of any nonlinear system (i.e., before we get our hands dirty with the tools that we learn later in the course!) is to find its equilibria and see if linearization around each of them is conclusive (i.e., if the equilibrium points are hyperbolic). Computer simulations can also be very helpful in guiding the subsequent rigorous analysis. Here, we do these steps for the following toy systems:

   (i) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \frac{1}{6}x_1^3 - x_2 \end{cases}$

   (ii) $\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3 \end{cases}$

   (iii) $\begin{cases} \dot{x}_1 = -x_1 + x_2(1 + x_1) \\ \dot{x}_2 = -x_1(1 + x_1) \end{cases}$

   For each system, do the following (1 point/system each):

   1.1 Compute the equilibrium points

   1.2 Determine the type of each equilibrium point (stable node, unstable focus, etc.)

   1.3 Generate the phase portrait of the system using MATLAB.
   Hint: Consider a rectangle in the x_1-x_2 plane that encloses all the equilibrium points (with a reasonable space between the boundaries of the box and the equilibria). Pick 10 equally spaced points on each side of the box (giving a total of 36 points). Using a for loop, simulate the trajectories of the system starting from each of these 36 points as the initial point $x(0)$. Try to use a sufficiently long time horizon for stable trajectories to converge. In MATLAB, use `ode45`. Then, plot each of these trajectories in the x_1-x_2 plane (in MATLAB, you can use `plot(x(:, 1), x(:, 2))`, and don't forget to use `hold on`). Finally, set the bounding box of the figure to the original box that you chose (using `axis` in MATLAB) and mark the equilibria with large dots/... in the same plot.
   Hint: Use m-codes so that you don't need to re-code for each system!

2. **Chaos and the Lorenz system.** In class, we saw that ultra-sensitivity to initial conditions is a fundamental characteristic of chaotic systems. Here, we will see this for the Lorenz system,

   $$\begin{aligned} \dot{x}_1 &= \sigma(x_1 - x_1) \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2 \\ \dot{x}_3 &= x_1x_2 - \beta x_3 \end{aligned}$$

   which is the most well-known example of a chaotic system for some range of its parameters (σ, ρ, β). Set $\rho = 28$, $\sigma = 10$, and $\beta = 8/3$. Do the following (1 point each):

   2.1 Simulate (using `ode45` with the default options) and plot (using `plot3`) the trajectories from $x(0) = [1 \quad 1 \quad 1]^T$ until $t = 200$ in the x_1-x_2-x_3 space (not as a function of time). You should be able to see the so-called "butterfly attractor".

2.2 Now do the same, but for initial condition $y(0) = [1{+}\epsilon \quad 1{+}\epsilon \quad 1{+}\epsilon]^T$. Try $\epsilon = 10^{-6}, 10^{-15}, 10^{-16}$. For each value of ϵ, plot both x_1 and y_1 (i.e., `x(:, 1)` and `y(:, 1)`) as a function of time in the same plot. Do they lie on top of each other? Why or why not?

   Hint: recall that the machine precision (i.e., the smallest value of ϵ for which 1 and $1+\epsilon$ can be distinguished) for double precision arithmetic (the MATLAB standard) is given by the variable `eps` (just type `eps` in command window).

2.3 We know that even if we do not add the ϵ to the initial condition, numerical computations always have errors. To see the effect of these errors, this time simulate the system from the same initial condition $z(0) = [1 \quad 1 \quad 1]^T$ but change the precision of `ode45` using the command `options = odeset('AbsTol', 1e-10, 'RelTol', 1e-6);` (and include `options` as the last input to `ode45`). This makes `z` a *more accurate* estimate of the solutions of the Lorenz system starting from $[1 \quad 1 \quad 1]^T$ than `x`. Plot both x_1 and z_1 as a function of time in the same plot. Do they lie on top of each other? Why or why not?

2.4 Plot `x`, `y` (for $\epsilon = 10^{-6}$) and `z` in the phase space (using `plot3`). Do they (approximately) lie on top of each other? Why or why not?

2.5 Based on 2.2 - 2.4, do you think computer simulations can be useful in the study of chaotic systems?