Lecture 3: Stability, Controllability & Observability

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This is our last set of notes where we briefly introduce some of the most basic concepts in the theory of linear systems: stability, controllability, and observability. In brief, a linear system is stable if its state does remains bounded with time, is controllable if the input can be designed to take the system from any initial state to any final state, and is observable if its state can be recovered from its outputs. We will get these definitions more accurate, and give simple conditions to check them out!

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3.1 Internal Stability

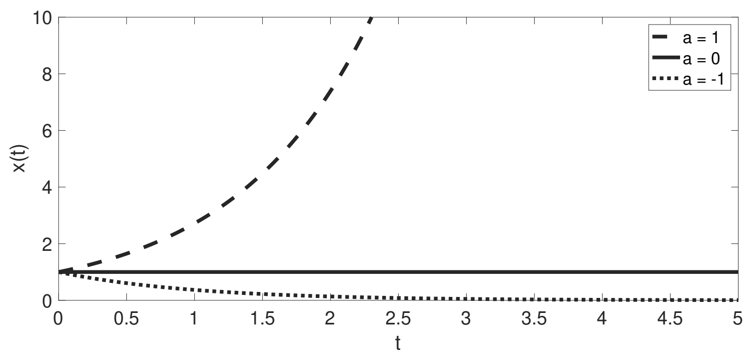
Consider the simple, scalar system without input

$$\dot{x}(t) = ax(t), \qquad x(0) = 1 \tag{3.1}$$

The solution, of course, is

$$x(t) = e^{at} \tag{3.2}$$

whose behavior depends critically on the sign of a:



When a is positive the solution blows up to infinity, when a is negative the solution dies down to zero, and when a is zero, neither happens – the solution remains at the same level it started. This is the core of what stability is all about!

Definition 3.1.1 (Marginal & asymptotic stability) The *zero-input* LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \qquad \text{or} \qquad \mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.3}$$

is

- "asymptotically stable" if $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ for every initial condition $\mathbf{x}_0$
- "marginally stable" if $\mathbf{x}(t) \not\to \mathbf{0}$ but remains bounded as $t \to \infty$ for every $\mathbf{x}_0$
- "stable" if it is either asymptotically or marginally stable
- "unstable" if it is not stable ($\|\mathbf{x}(t)\| \to \infty$ as $t \to \infty$ at least for some, if not all, $\mathbf{x}_0$)

□

If you are given a system of the forms in Eq. (3.3), you can compute its trajectories (either analytically or using numerical simulations) and check the stability of the system from the above definition. This is not a great idea, however, because it requires solving the differential/difference equation in Eq. (3.3), which is not always possible analytically. Numerical solutions always exist, but notice that the definition of asymptotic/marginal stability requires the solutions to go to zero/remain bounded for all initial conditions. It is never possible to numerically solve the dynamics for all possible initial conditions.

Therefore, we ideally want a simple test to determine stability of an LTI system, without a need to solve for the state trajectories explicitly. This is achieved, not surprisingly, using eigenvalues!

Recall the notion of *similarity transformations* from Section 2.9. We learned that for either of the LTI systems in Eq. (3.3), we can change the basis of the state vector from the standard basis $\mathbf{I}$ to a new basis $\mathbf{Q}$, which would change the representation of $\mathbf{x}(t)$ and $\mathbf{A}$ to

$$\hat{\mathbf{x}}(t) = \mathbf{Q}^{-1}\mathbf{x}(t), \qquad \hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$

Now, following our discussion of *diagonalization* in Section 1.7, let's use $\mathbf{Q} = \mathbf{V}$ where

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

is composed of the eigenvectors of $\mathbf{A}$ (and as before, we assume $\mathbf{A}$ is diagonalizable). You know from Section 1.7 that this choice of basis gives

$$\hat{\mathbf{A}} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbf{A}$. The great thing about the diagonal form of Λ is that it makes the computation of matrix exponential extremely easy:

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{3.4}$$

In other words, the matrix exponential of a diagonal matrix can be done element by element on its diagonal (can you show this using the definition of matrix exponential?). Even further, from Theorem 2.9.5 we know that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} \Rightarrow e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1} \tag{3.5}$$

so whether $e^{\mathbf{A}t}$ remains bounded or not is a direct consequence of whether $e^{\mathbf{\Lambda}t}$ remains bounded or not.

To assess the behavior of $e^{\mathbf{\Lambda}t}$ with time, recall that each λ_i is in general complex,

$$\lambda_i = \sigma_i + j\omega_i \qquad \begin{cases} \sigma_i = \mathrm{Re}\{\lambda_i\} \\ \omega_i = \mathrm{Im}\{\lambda_i\} \end{cases} \tag{3.6}$$

so

$$\begin{aligned} e^{\lambda_i t} &= e^{\sigma_i t + j\omega_i t} \\ &= e^{\sigma_i t}(\cos\omega_i t + j\sin\omega_i t) \end{aligned}$$

Notice that the factor $\cos\omega_i t + j\sin\omega_i t$ has always a unit modulus

$$|\cos\omega_i t + j\sin\omega_i t| = \sqrt{\cos^2\omega_i t + \sin^2\omega_i t} = 1$$

so

$$|e^{\lambda_i t}| = e^{\sigma_i t}$$

Therefore, whether $|e^{\lambda_i t}|$ converges to 0, diverges to infinity, or remains constant with time, depends only and only on the sign of $\sigma_i = \text{Re}\{\lambda_i\}$, as we saw in Eq. (3.2). This leads us to the following fundamental result about the stability of LTI systems:

Theorem 3.1.2 (Marginal & asymptotic stability) Similar, but different characterizations hold for the stability of continuous-time and discrete-time systems:

(i) The diagonalizable, continuous-time LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.7a}$$

is:

- asymptotically stable if $\text{Re}\{\lambda_i\} < 0$ for all i
- marginally stable if $\text{Re}\{\lambda_i\} \leq 0$ for all i, and, there exists at least one i for which $\text{Re}\{\lambda_i\} = 0$
- stable if $\text{Re}\{\lambda_i\} \leq 0$ for all i
- unstable if $\text{Re}\{\lambda_i\} > 0$ for at least one i

(ii) The diagonalizable, discrete-time LTI system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.7b}$$

is:

- asymptotically stable if $|\lambda_i| < 1$ for all i
- marginally stable if $|\lambda_i| \leq 1$ for all i, and, there exists at least one i for which $|\lambda_i| = 1$
- stable if $|\lambda_i| \leq 1$ for all i
- unstable if $|\lambda_i| > 1$ for at least one i

□

The second part of the above theorem, for discrete-time systems, may not be immediately clear to you, so let's dive deeper into it. Notice that the discrete-time conditions are a direct parallel of the continuous-time ones, except that the real part of λ_i has been replaced by its modulus, and comparison to 0 has been replaced by comparison to 1. This is why:

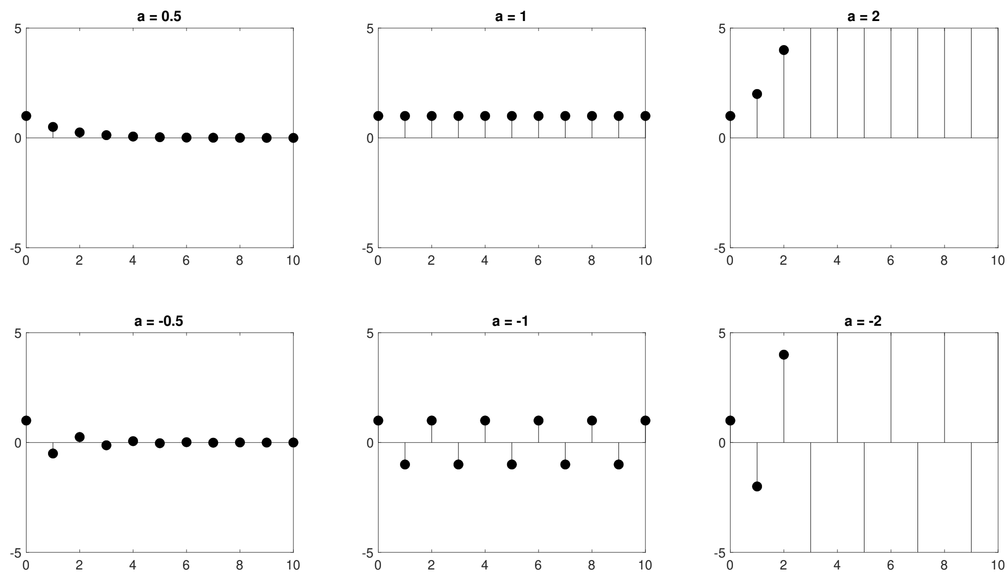
Consider the discrete-time counterpart of Eq. (3.1),

$$x(t+1) = ax(t), \qquad x(0) = 1$$

The solution, as we saw in Eq. (2.14), is

$$x(t) = a^t \tag{3.8}$$

whose convergence/divergence depends, not on the sign of a, but on the absolute value of a:



The solution dies down to zero if $|a| < 1$, blows up if $|a| > 1$, and neither dies down nor blows up if $|a| = 1$.

The extension of this discussion to the full, matrix version in Eq. (3.7b) is now straightforward. We perform the same diagonalization as in the continuous-time case, but instead of Eq. (3.5) use the fact that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} \Rightarrow \mathbf{A}^t = \mathbf{V}\mathbf{\Lambda}^t\mathbf{V}^{-1}$$

instead of Eq. (3.4) use the fact that

$$\mathbf{\Lambda}^t = \begin{bmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^t \end{bmatrix}$$

and instead of Eq. (3.6) use the polar representation of λ_i:

$$\lambda_i = r_i e^{j\theta_i}, \qquad \begin{cases} r_i = |\lambda_i| \\ \theta_i = \measuredangle \lambda_i \end{cases}$$

so

$$\lambda_i^t = r_i^t e^{jt\theta_i}$$

and therefore

$$|\lambda_i^t| = |r_i^t| \cdot \underbrace{|e^{jt\theta_i}|}_{1} = r_i^t$$

whose behavior is precisely that of Eq. (3.8). Just note that, unlike a in Eq. (3.8) which could be positive or negative, $r_i \geq 0$ since it is the modulus of λ_i.

Therefore, putting everything together, we see that whether $\mathbf{A}^t$ converges/diverges depends on whether the moduli of its eigenvalues are less than, equal to, or greater than 1, which is what we saw in Theorem 3.1.2.

Example 3.1.3 (Pendulum) Consider again the pendulum of Example 2.1.3, but without any external forces ($u = F_{\text{ext}} = 0$) and with linear (viscous) friction

$$F_{\text{fric}}(\dot{\theta}(t)) = b\dot{\theta}(t)$$

The state space equations then read

$$\dot{\mathbf{x}} = f(\mathbf{x}) = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin x_1 - \frac{b}{mL}x_2 \end{bmatrix}$$

which are nonlinear, so we have to first linearize them around an equilibrium point (recall Section 2.4). The equilibria are the solutions of

$$\begin{aligned} f(\mathbf{x}) = 0 &\Leftrightarrow \begin{cases} x_2 = 0 \\ -\frac{g}{L}\sin x_1 - \frac{b}{mL}x_2 = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} x_2 = 0 \\ \sin x_1 = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots \end{cases} \end{aligned}$$

so the system has infinitely many equilibrium points, corresponding to the downward position at rest $\mathbf{x}^*_{\text{down}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the upward position at rest $\mathbf{x}^*_{\text{up}} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$, and their rotations.

First, we linearize the system around $\mathbf{x}^*_{\text{down}}$

$$\begin{aligned} \frac{\partial f}{\partial \mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & -\frac{b}{mL} \end{bmatrix} \\ \mathbf{A} = \frac{\partial f}{\partial \mathbf{x}}\Big|_{\mathbf{x}^*_{\text{down}}} &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{b}{mL} \end{bmatrix} \end{aligned}$$

Recall that this linearization is only an approximation to the pendulum equations, and is only valid near $\mathbf{x}^*_{\text{down}}$. To assess its stability, we have to compute the eigenvalues of $\mathbf{A}$ (using MATLAB or by hand),

$$\begin{aligned} \lambda_1 &= \frac{-b - \sqrt{\Delta}}{2mL} \\ \lambda_2 &= \frac{-b + \sqrt{\Delta}}{2mL}, \qquad\qquad \Delta = b^2 - 4m^2gL \end{aligned}$$

and check their real parts. Clearly, two scenarios can happen:

- $\Delta < 0$: In this case, λ_1 and λ_2 are both complex, and their real parts are

  $$\text{Re}\{\lambda_1\} = \text{Re}\{\lambda_2\} = -\frac{b}{2mL} < 0$$

  so the linearized system is asymptotically stable (Theorem 3.1.2(i)).

- $\Delta \geq 0$: In this case, λ_1 and λ_2 are both real, so $\text{Re}\{\lambda_1\} = \lambda_1$ and $\text{Re}\{\lambda_2\} = \lambda_2$. Clearly, λ_1 is negative. λ_2 is also negative because

  $$\begin{aligned} \lambda_2 < 0 &\Leftrightarrow \frac{-b + \sqrt{b^2 - 4m^2gL}}{2mL} < 0 \\ &\Leftrightarrow b > \sqrt{b^2 - 4m^2gL} \\ &\Leftrightarrow b^2 > b^2 - 4m^2gL \end{aligned}$$

  which is clearly true, which means the linearized system is also asymptotically stable in this case.

Therefore, we see that $\mathbf{x}^*_{\text{down}}$ is asymptotically stable regardless of the values of the parameters, which makes perfect sense! If you release the pendulum near $\mathbf{x}^*_{\text{down}}$, friction will eventually dissipate all of its initial energy and it will stop at $\mathbf{x}^*_{\text{down}}$. Again, note that I said if you release the pendulum *near* $\mathbf{x}^*_{\text{down}}$, because the linearization is only a valid approximation there.

Next, we do the same around $\mathbf{x}^*_{\text{up}}$. The linearization is very similar,

$$\mathbf{A} = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}^*_{\text{up}}} = \begin{bmatrix} 0 & 1 \\ +\frac{g}{L} & -\frac{b}{mL} \end{bmatrix}$$

and the only difference is in the sign of a_{21}. This changes the eigenvalues of $\mathbf{A}$,

$$\begin{aligned} \lambda_1 &= \frac{-b - \sqrt{\Delta}}{2mL} \\ \lambda_2 &= \frac{-b + \sqrt{\Delta}}{2mL}, \qquad\qquad \Delta = b^2 + 4m^2 gL \end{aligned}$$

where the only difference is in the value of Δ. Clearly, Δ is always positive, so both eigenvalues are real regardless of the values of the parameters. $\lambda_1 < 0$, but $\lambda_2 > 0$ (why?), so the linearization is this time *unstable*. This again makes perfect intuitive sense, because if you release the pendulum near the upward position, it will move away from it (but of course not to infinity, because the linearization is only valid near the equilibrium point!). □

3.2 Input-Output Stability

Our discussion so far has been about the *internal* stability of an LTI system, meaning that we discarded the inputs and outputs of the system and only studied the system's *state response* due to initial conditions. In this section, we do the opposite: we discard the initial conditions and only study the system's *output response* due to the inputs. Because of the superposition property, this provides us with a complete picture of system's stability.

Consider a continuous-time LTI system which is initially at rest

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{0} \tag{3.9a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3.9b}$$

This time, we are interested in whether the system's output remains bounded in response to the input, or not. And notice that our focus here on the *output* (vs. state) is only for generality: you can always choose $\mathbf{C} = \mathbf{I}$ and $\mathbf{D} = \mathbf{0}$ to get $\mathbf{y} = \mathbf{x}$.

A critical aspect of input-output stability is appreciating the fact that the output is the result of an interaction between the input and the system. And both of them can make the output explode.

(i) If the input explodes (for example $u(t) = e^t$), the output of even the most stable systems (for example $\dot{x} = -x + u, y = x$) may explode as well.

(ii) Similarly, if the system is explosive (for example $\dot{x} = x + u, y = x$), the output may explode even in response to the weakest inputs (for example $u = k \sin t$ for arbitrarily small k).

The first case is kind of trivial (you put in infinite energy into any system and it explodes), and therefore not interesting. What is interesting for us is the second case: whether the system is explosive or not. This motives the definition of BIBO stability, as follows:

Definition 3.2.1 (Bounded-input bounded-output (BIBO) stability) The LTI system in Eq. (3.9) is BIBO stable if its response to any bounded input is bounded. In other words, the output remains bounded

$$|\mathbf{y}(t)| \leq \mathbf{y}_{\max}, \qquad \text{for some upper bound } \mathbf{y}_{\max}$$

whenever the input is bounded

$$|\mathbf{u}(t)| \leq \mathbf{u}_{\max}, \qquad \text{for some upper bound } \mathbf{u}_{\max}$$

In these equations, $|\mathbf{y}(t)|$ is the element-wise absolute value of $\mathbf{y}(t)$, similarly for $|\mathbf{u}(t)|$. □

The input-output nature of BIBO stability motivates the use of the system's convolution property (Eq. (**??**))

$$\mathbf{y}(t) = \mathbf{H}(t) \star \mathbf{u}(t)$$

or its frequency-domain version

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$$

These essentially bypass the state and describe the output directly as a function of the input. In fact, it turns out that:

Theorem 3.2.2 (BIBO stability and impulse response) The system in Eq. (3.9) is BIBO stable if and only if all the poles of $\mathbf{H}(s)$ have negative real parts. □

A few important remarks about this theorem are worthwhile:

- In item (iii) of the theorem, when we say all of the poles of $\mathbf{H}(s)$ have negative real parts, we mean all of the poles of all of the entries of $\mathbf{H}(s)$.
- More importantly, recall from Theorem 2.8.5 that the poles of $\mathbf{H}(s)$ are the same as the eigenvalues of $\mathbf{A}$, except if a zero-pole cancellation occurs. This means that BIBO stability and asymptotic stability are the same, except in the rare instances of zero-pole cancellations.

Example 3.2.3 (Zero-pole cancellations – revisited) Consider again the system of Example 2.8.4. The system's $\mathbf{A}$ matrix has eigenvalues

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

which means that the system is internally unstable. However, the system's transfer function has only one pole at

$$p_1 = -1$$

which means that the system is BIBO stable! □

Cases like the above example are extremely rare, but very important: the system explodes internally ($\mathbf{x}(t)$ diverges to infinity), but we do not observe it at the output ($\mathbf{y}(t)$ remains nice and bounded). In practice, this is clearly a dangerous situation and highlights the importance of placing enough and appropriate sensors in a system so that any internal instabilities can be observed and mitigated.

3.3 Observability

Consider the continuous-time state space system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad t \in \mathbb{R} \tag{3.10a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.10b}$$

or the discrete-time version

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad t = 0, 1, 2, \ldots \tag{3.11a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.11b}$$

Example 3.2.3 makes a critical point about these state space systems clear: *the output may only provide partial information about the state*. For most real-world systems, the dimension of the output, p, is less than the dimension of the state, n. This means that even if

$$\text{we know } \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \text{ and } \mathbf{u}(t), \mathbf{y}(t) \text{ for all } t \tag{3.12}$$

directly solving the system of equations

$$\mathbf{C}\mathbf{x}(t_0) = \mathbf{y}(t_0) - \mathbf{D}\mathbf{u}(t_0)$$

to obtain $\mathbf{x}(t_0)$ (for any time t_0 of interest) is often not possible, because of having more unknowns than equations $\Rightarrow$ infinitely many solutions. However, we have the *advantage of time*. That is, we are not limited to solve for $\mathbf{x}(t_0)$ only using $\mathbf{u}(t_0)$ and $\mathbf{y}(t_0)$ at time t, we can use the entire history of the signals $\mathbf{u}(t), \mathbf{y}(t)$ for all t. But how?

Before we move forward and see how we can best use all of this information, note that as long as Eq. (3.12) holds,

$$\boxed{\text{knowledge of } \mathbf{x}(t) \text{ for all } t \;\equiv\; \text{knowledge of } \mathbf{x}(0)}$$

This is because, once we have $\mathbf{x}(0)$, we can solve the dynamics and obtain $\mathbf{x}(t)$ for all other times. This motivates the notion of *observability*:

Definition 3.3.1 (Observability) The LTI system in Eq. (3.10) or Eq. (3.11) is called "observable" if the knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ over some finite time interval $0 \le t \le t_f$ (together with $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$) is enough to uniquely determine $\mathbf{x}_0$. □

Our emphasis of "some finite time interval" in this definition is to make it more realistic – we never have the entire, infinitely long history of the signals $\mathbf{u}(t)$ and $\mathbf{y}(t)$.

Checking observability is slightly different for continuous-time vs. discrete-time systems. The latter is simpler, so we start there! Assume Eq. (3.12) holds and we want to find $\mathbf{x}_0$ for the system in Eq. (3.11). As we saw in Section 2.5.1,

$$\begin{aligned}
\mathbf{y}(0) &= \mathbf{C}\mathbf{x}_0 + \mathbf{D}\mathbf{u}(0) \\
\mathbf{y}(1) &= \mathbf{C}\mathbf{A}\mathbf{x}_0 + \mathbf{C}\mathbf{B}\mathbf{u}(0) + \mathbf{D}\mathbf{u}(1) \\
\mathbf{y}(2) &= \mathbf{C}\mathbf{A}^2\mathbf{x}_0 + \mathbf{C}\mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{C}\mathbf{B}\mathbf{u}(1) + \mathbf{D}\mathbf{u}(2) \\
&\;\;\vdots \\
\mathbf{y}(t_f) &= \mathbf{C}\mathbf{A}^{t_f}\mathbf{x}_0 + \mathbf{C}\mathbf{A}^{t_f-1}\mathbf{B}\mathbf{u}(0) + \cdots + \mathbf{C}\mathbf{B}\mathbf{u}(t_f-1) + \mathbf{D}\mathbf{u}(t_f)
\end{aligned}$$

Note that everything in these equations are known, except for $\mathbf{x}_0$. So we need to solve the linear system of equations

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{t_f} \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} \mathbf{y}(0) - \mathbf{Du}(0) \\ \mathbf{y}(1) - \mathbf{CBu}(0) - \mathbf{Du}(1) \\ \mathbf{y}(2) - \mathbf{CABu}(0) - \mathbf{CBu}(1) - \mathbf{Du}(2) \\ \vdots \\ \mathbf{y}(t_f) - \mathbf{CA}^{t_f-1}\mathbf{Bu}(0) - \cdots - \mathbf{CBu}(t_f - 1) - \mathbf{Du}(t_f) \end{bmatrix} \tag{3.13}$$

If our knowledge of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ and $\mathbf{u}(t), \mathbf{y}(t)$ are accurate, this system of equations always has a solution (right?). But the question is whether it has a *unique* solution. From Eq. (1.11), we know that this is the case if and only if the coefficient matrix

$$\mathcal{O}_{t_f} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{t_f} \end{bmatrix}$$

is full column rank. For reasons that we skip here (the Cayley-Hamilton theorem), this matrix is full column rank for all $t_f \geq n$ if and only if the observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{3.14}$$

is full column rank. This gives us the following theorem (trust me for the continuous-time version!)

Theorem 3.3.2 (Observability & observability matrix) Both of the systems in Eq. (3.10) and Eq. (3.11) are observable if and only if the observability matrix $\mathcal{O}$ in Eq. (3.14) is full rank. □

Notice that $\mathcal{O}$ is np-by-n, so being full rank and full column rank are the same for it.

Example 3.3.3 (Zero-pole cancellation – revisited) Consider again the system of Example 2.8.4. Its observability matrix is

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

which is clearly not full rank, and the system is therefore not observable! You can even move further and compute the null space of $\mathcal{O}$, which is $\langle \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rangle$. This means that changing $\mathbf{x}_0$ along the direction of $\mathbf{z} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ has absolutely no effect on the output. This null direction is indeed the direction along which the solutions of this system diverge to infinity, making it invisible from the output! (Can you show this?) □

Note that the observability matrix is independent of $\mathbf{B}$ or $\mathbf{D}$, showing that observability is all about the output – the input can even be set to zero when studying observability. This is why we even sometimes say the pair $(\mathbf{A}, \mathbf{C})$ (instead of the system) is observable or not.

Now let's assume that the system is observable, and go back to Eq. (3.13). Recall from Section 1.4.7 that the solution can be found using the pseudo-inverse,

$$\mathbf{x}_0 = (\mathcal{O}_{t_f}^T \mathcal{O}_{t_f})^{-1} \mathcal{O}_{t_f}^T \tilde{\mathbf{y}}_{t_f}$$

where $\tilde{\mathbf{y}}_{t_f}$ is the large vector on the right hand side of Eq. (3.13). In this solution, and in the study of observability in general, the matrix $\mathcal{O}_{t_f}^T \mathcal{O}_{t_f}$ plays a central role, and is therefore given a special name:

Definition 3.3.4 (Observability Gramian) The *symmetric* matrix

$$\mathcal{W}_o(t_f) = \mathcal{O}_{t_f}^T \mathcal{O}_{t_f} = \sum_{t=0}^{t_f-1} (\mathbf{A}^T)^t \mathbf{C}^T \mathbf{C} \mathbf{A}^t, \qquad t_f \geq n$$

is called the "observability Gramian" in discrete time. Similarly, the symmetric matrix

$$\mathcal{W}_o(t_f) = \int_0^{t_f} e^{\mathbf{A}^T t} \mathbf{C}^T \mathbf{C} e^{\mathbf{A} t} dt, \qquad t_f > 0$$

is called the "observability Gramian" in continuous time. □

It is not hard to show that all of the eigenvalues of $\mathcal{W}_o(t_f)$ (which are real, right?) are nonnegative, for all t_f. Therefore it is always positive semidefinite. However, if the system is observable ($\mathcal{O}$ is full rank), $\mathcal{W}_o(t_f)$ becomes full rank (nonsingular) as well, and therefore positive definite. This is why the positive-definiteness of $\mathcal{W}_o(t_f)$ is another test of observability, similar to the full-rankness of $\mathcal{O}$.

3.4 Controllability

The notion of observability (Definition 3.3.1), at its core, is essentially asking whether the output is *rich enough* to determine the state. The notion of controllability, is the dual to this: whether the *input* is rich enough to determine the state:

Definition 3.4.1 (Controllability) The LTI system in Eq. (3.10) or Eq. (3.11) is called "controllable" if for any initial state $\mathbf{x}_0$ and any final state $\mathbf{x}_f$, the input signal $\mathbf{u}(t)$ can be designed such that the system, starting from $\mathbf{x}(0) = \mathbf{x}_0$, reaches $\mathbf{x}(t_f) = \mathbf{x}_f$ in some finite time t_f. □

Interestingly, determining whether the system is controllable or not is also easier in discrete time, and boils down to a system of linear equations: from 2.5.1,

$$\mathbf{x}(t_f) = \mathbf{x}_f = \mathbf{A}^{t_f}\mathbf{x}_0 + \mathbf{A}^{t_f-1}\mathbf{B}\mathbf{u}(0) + \cdots + \mathbf{A}\mathbf{B}\mathbf{u}(t_f-2) + \mathbf{B}\mathbf{u}(t_f-1)$$

This time, everything is known, except for the inputs $\mathbf{u}(0), \ldots, \mathbf{u}(t_f-1)$ which we have to design. So we have to solve the linear system of equations

$$\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{t_f-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}(t_f-1) \\ \mathbf{u}(t_f-2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix} = \mathbf{x}_f - \mathbf{A}^{t_f}\mathbf{x}_0 \tag{3.15}$$

This time, we are not worried about the uniqueness of solutions (the more options we have, the better designs we can get), we are seeking to know if this equation has any solutions! But from Section 1.4.1, we know that this is the case *for all* $\mathbf{x}_0$ *and* $\mathbf{x}_f$ if and only if the coefficients matrix

$$\mathcal{C}_{t_f} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{t_f-1}\mathbf{B} \end{bmatrix}$$

is full row rank. Similar to above, the Cayley-Hamilton theorem lets us know that this is the case for any $t_f \geq n$ if and only if the controllability matrix

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{3.16}$$

is full row rank. Therefore, we have the next theorem! (and again, trust me for the continuous time case)

Theorem 3.4.2 (Controllability & controllability matrix) Both of the systems in Eq. (3.10) and Eq. (3.11) are controllable if and only if the controllability matrix $\mathcal{C}$ in Eq. (3.14) is full rank. □

As for observability, we see that controllability is really all about $\mathbf{A}$ and $\mathbf{B}$, without any role played by the output matrices $\mathbf{C}$ or $\mathbf{D}$. As such, instead of saying whether the system is controllable or not, we sometimes say whether the pair $(\mathbf{A}, \mathbf{B})$ is controllable or not.

MATLAB 3.4.3 (Controllability & observability matrices) In MATLAB, you can easily compute the controllability matrix $\mathcal{C}$ via `ctrb(A, B)` and the observability matrix $\mathcal{O}$ via `obsv(A, C)`. As always, you can check their rank using `rank()`. □

Similar to what we did for observability, assume now that the system is controllable, and revisit Eq. (3.15). $\mathcal{C}_{t_f}$ is hardly full column rank (because its number of columns, nt_f, is usually larger than its number of rows, n), and therefore the solution is hardly unique (though is not impossible). But we know from Section 1.4.7 that at least one solution can be found easily using the pseudo-inverse:

$$\begin{bmatrix} \mathbf{u}(t_f - 1) \\ \mathbf{u}(t_f - 2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix} = \mathcal{C}_{t_f}^T (\mathcal{C}_{t_f} \mathcal{C}_{t_f}^T)^{-1} (\mathbf{x}_f - \mathbf{A}^{t_f} \mathbf{x}_0)$$

Similar to the observability Gramian, the matrix $\mathcal{C}_{t_f} \mathcal{C}_{t_f}^T$ is called the controllability Gramian, and plays a central role in the study of controllability:

Definition 3.4.4 (Controllability Gramian) The *symmetric* matrix

$$\mathcal{W}_c(t_f) = \mathcal{C}_{t_f}^T \mathcal{C}_{t_f} = \sum_{t=0}^{t_f-1} \mathbf{A}^t \mathbf{B} \mathbf{B}^t (\mathbf{A}^T)^t, \qquad t_f \geq n$$

is called the "controllability Gramian" in discrete time. Similarly, the symmetric matrix

$$\mathcal{W}_c(t_f) = \int_0^{t_f} e^{\mathbf{A}t} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T t} dt, \qquad t_f > 0$$

is called the "controllability Gramian" in continuous time. □

The relationship between the controllability Gramian and controllability is the same as the relationship between observability Gramian and observability: the controllability Gramian is always positive semidefinite, for all t_f, and is positive definite/nonsingular for all t_f if and only if the system is controllable.

Throughout this section, I am sure you could have not helped but notice the strong similarity between controllability and observability! The following theorem makes it formal:

Theorem 3.4.5 (Duality between controllability and observability) The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the pair $(\mathbf{A}^T, \mathbf{B}^T)$ is observable. □

Example 3.4.6 (Canonical forms – revisited) Consider again the canonical forms in Example 2.9.2. For the controllable canonical form, we have

$$\operatorname{rank}(\mathcal{C}) = 6 \Rightarrow \text{full rank} \Rightarrow \text{controllable}$$
$$\operatorname{rank}(\mathcal{O}) = 3 \Rightarrow \text{rank deficient} \Rightarrow \text{not observable}$$

hence the name! Similarly, for the observable canonical form, we have

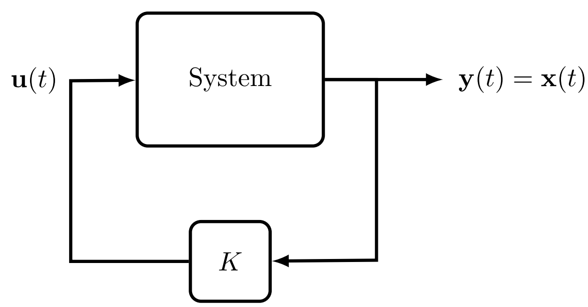
$$\begin{aligned}\operatorname{rank}(\mathcal{C}) &= 3 \Rightarrow \text{rank deficient} \Rightarrow \text{not controllable}\\ \operatorname{rank}(\mathcal{O}) &= 6 \Rightarrow \text{full rank} \Rightarrow \text{observable}\end{aligned}$$

Note that while the controllable canonical form is always controllable, it may or may not be observable. Similarly, the observable canonical form is always observable, but may or may not be controllable. □

3.5 Feedback Stabilization

The notion of controllability, but definition, ensures that we have full control over the state of the system using the control signal $\mathbf{u}(t)$, making it possible to take the system from any initial state to any final state in a finite amount of time. This is often called "point to point control". Even though point to point control is how controllability is defined, in many real-world scenarios we are interested in a different control problem: stabilization.

Assume that you have an internally unstable system. Stabilization (a.k.a. regulation) is the problem of designing the control input $\mathbf{u}(t)$ such that the controlled system becomes stable. This can take different forms, but the most common form is state feedback stabilization:



In this framework, the input is a constant (matrix) gain times the state,

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$$

and we want to choose the matrix $\mathbf{K}$ such that the close-loop system is stable.

By direct substitution, we readily see that

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{x}\\ &= (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}\end{aligned}$$

so the stability of the close-loop system depends on the eigenvalues of

$$\mathbf{A} + \mathbf{B}\mathbf{K}$$

Note that $\mathbf{K} \in \mathbb{R}^{m\times n}$ has mn elements, but $\mathbf{A} + \mathbf{B}\mathbf{K}$ has only n eigenvalues. So we have mn degrees of freedom (variables to freely choose) to determine n eigenvalues. Looks like a favorable situation, right?

Example 3.5.1 (Pole placement) Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} u$$

Without the effect of input (if $u \equiv 0$), the open-loop system has eigenvalues

$$\lambda_1 = 3, \qquad \lambda_{2,3} = \frac{3}{2} \pm j\frac{\sqrt{11}}{2}$$

and is therefore (internally) unstable. So by closing the loop and choosing u appropriately, we try to make the closed-loop system asymptotically stable. The linear state feedback gain is

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \in \mathbb{R}^{1\times 3}$$

and $u = \mathbf{kx}$. So the closed-loop coefficient matrix is

$$\mathbf{A} + \mathbf{BK} = \begin{bmatrix} 2+k_1 & -3+k_2 & 1+k_3 \\ 3-k_1 & 2-k_2 & 2-k_3 \\ 1+3k_1 & 3+3k_2 & 2+3k_3 \end{bmatrix}$$

and has the characteristic polynomial

$$\det(\lambda\mathbf{I} - (\mathbf{A}+\mathbf{BK})) = \lambda^3 + (-6 - k_1 + k_2 - 3k_3)\lambda^2 + (14 - 2k_1 - 13k_2 + 14k_3)\lambda + (-15 + 35k_1 + 10k_2 - 55k_3) \tag{3.17}$$

We want the closed-loop system to be asymptotically stable, so we want the roots of this polynomial to be in the left half plane. But otherwise, the roots can be anything. In other words, there are infinitely many ways to make the closed-loop system asymptotically stable, and just saying that we want the closed-loop system to be asymptotically stable does not uniquely determine $\mathbf{K}$. <u>To uniquely determine $\mathbf{K}$, we have to fully determine the closed-loop characteristic polynomial or, equivalently, fully determine the closed-loop eigenvalues.</u> This is of course a design choice. Assume we want

$$\text{closed-loop eigenvalues: } \lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3 \tag{3.18}$$

which gives the desired characteristic polynomial

$$(\lambda+1)(\lambda+2)(\lambda+3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

Comparing this with Eq. (3.17), we see that we need

$$\begin{cases} -6 - k_1 + k_2 - 3k_3 = 6 \\ 14 - 2k_1 - 13k_2 + 14k_3 = 11 \\ -15 + 35k_1 + 10k_2 - 55k_3 = 6 \end{cases}$$

This is a linear system of equations, with the unique solution

$$\mathbf{K} = \begin{bmatrix} -4.2 & -3 & -3.6 \end{bmatrix}$$

Now, assume everything is the same, except that we have

$$\mathbf{B} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

We still want to choose $\mathbf{K}$ such that the closed-loop system is asymptotically stable. This time, the closed-loop characteristic polynomial becomes

$$\det(\lambda\mathbf{I} - (\mathbf{A} + \mathbf{BK})) = \lambda^3 + (-6 - k_1 + k_2 + k_3)\lambda^2 + (14 + 2k_1 - 5k_2 - 2k_3)\lambda + (-15 + 3k_1 + 6k_2 - 3k_3) \tag{3.19}$$

and, if we still want the closed-loop eigenvalues to be as in Eq. (3.18), we get the linear system of equations

$$\begin{cases} -6 - k_1 + k_2 + k_3 = 6 \\ 14 + 2k_1 - 5k_2 - 2k_3 = 11 \\ -15 + 3k_1 + 6k_2 - 3k_3 = 6 \end{cases}$$

This system of equations, unlike last time, has no solutions (check it yourself)! In fact, it is straightforward to check that $\lambda = 3$ is always a root of Eq. (3.19), regardless of k_1, k_2, k_3 (*notice that* $\lambda = 3$ *was one of the open-loop eigenvalues*). So not only the desired eigenvalues in Eq. (3.18) are impossible to obtain, but in fact it is impossible to stabilize this system using state feedback! □

The difference between the two systems in Example 3.5.1 is of course their controllability:

Theorem 3.5.2 (Pole placement and controllability) The eigenvalues of $\mathbf{A}+\mathbf{BK}$ can be assigned arbitrarily (provided that complex eigenvalues are selected in conjugate pairs) if and only if $(\mathbf{A}, \mathbf{B})$ is controllable. □

You can easily check that the first system in Example 3.5.1 is controllable, while the second one is not.

When you have a controllable $(\mathbf{A}, \mathbf{B})$, as I said above, choosing the location of the closed-loop eigenvalues is a design choice. As a rule of thumb, do not choose the closed-loop eigenvalues too close to the imaginary axis, too far from it, or too far from the real axis. These result in, respectively, a very slow system, a noise-amplifying system, and a very rapidly oscillating system, neither of which are desirable in practice.

From Theorem 3.5.2, you can see that controllability may be too much to ask. If all we need is stabilization, we don't need to choose the closed-loop eigenvalues arbitrarily, we only need them to be in the left half plane. It turns out that sometimes, that is possible even if the system is not controllable:

Example 3.5.3 (Partial pole placement) Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -3 & -2 \\ 6 & -1 & 5 \\ 7 & 3 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} u$$

The open-loop eigenvalues are

$$\lambda_1 = 3, \qquad \lambda_{2,3} = -\frac{3}{2} \pm j\frac{\sqrt{11}}{2}$$

so the open-loop system ($u \equiv 0$) is unstable, and you can easily check that the system is not controllable. So by Theorem 3.5.2, we know there is no hope in trying to assign the closed-loop eigenvalues arbitrarily. But there is still a possibility that we can stabilize the system. Recall that in the second system of Example 3.5.1, stabilization was not possible because one of the unstable open-loop eigenvalues ($\lambda = 3$) was always a closed-loop eigenvalue, regardless of k_1, k_2, k_3. So let's check the open-loop eigenvalues for our system in this example. Here our closed-loop characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(\lambda\mathbf{I} - (\mathbf{A} + \mathbf{BK})) \\ &= \lambda^3 + (-k_1 + k_2 + 2k_3)\lambda^2 + (-4 - 3k_1 + 3k_2 + 6k_3)\lambda + (-15 - 5k_1 + 5k_2 + 10k_3) \end{aligned}$$

and we can check that

$$\begin{aligned} p(3) &= -23k_1 + 23k_2 + 46k_3 \\ p(-3/2 + j\sqrt{11}/2) &= 0 \\ p(-3/2 - j\sqrt{11}/2) &= 0 \end{aligned}$$

This means that two of the open-loop eigenvalues, namely, $\lambda_{2,3} = -\frac{3}{2} \pm j\frac{\sqrt{11}}{2}$ are always closed-loop eigenvalues as well (regardless of $\mathbf{K}$), but $\lambda_1 = 3$ is not necessarily. This is great news, because both of $\lambda_{2,3} = -\frac{3}{2} \pm j\frac{\sqrt{11}}{2}$ are stable, and we don't worry about them being closed-loop eigenvalues as well. All we need to do is change λ_1 with a stable eigenvalue. By factorizing $p(\lambda)$ (MATLAB `factor()`), we see that

$$p(\lambda) = (\lambda^2 + 3\lambda + 5)(\lambda - 3 - k_1 + k_2 + 2k_3)$$

clearly showing that the closed-loop system can be made stable if $\lambda_1 = 3 + k_1 - k_2 - 2k_3$ is chosen to be in the left half plane, which can be done in infinitely many ways. □

In general, if you want to stabilize a system that is uncontrollable, check the open loop eigenvalues one by one. At least one of them will be a root of the closed-loop characteristic polynomial $p(\lambda)$, no matter what $\mathbf{K}$ you choose. If that eigenvalue is unstable (like the second system of example 3.5.1), there is nothing you can do! But if only stable open-loop eigenvalues satisfy $p(\lambda) = 0$, then you can take their corresponding factors out of $p(\lambda)$ and then choose $\mathbf{K}$ such that the remaining eigenvalues are set as you wish! In this latter case, the system is called stabilizable:

Definition 3.5.4 (Stabilizability) The system

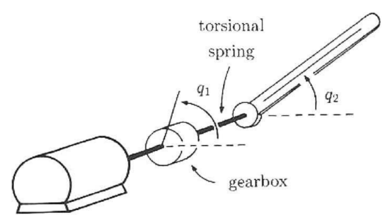
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

or, equivalently, the pair $(\mathbf{A}, \mathbf{B})$ is called "stabilizable" if $\mathbf{K}$ can be chosen such that all of the eigenvalues of $\mathbf{A} + \mathbf{B}\mathbf{K}$ belong to the left half of the complex plane. □

Note that stabilizability is a weaker condition than controllability (any controllable system is stabilizable, but not vice versa).

Exercise 3.5.5 (One link robotic arm) Consider a one link robotic arm described by the system of ODEs

$$\begin{aligned} &J_1\ddot{q}_1 + F_1\dot{q}_1 + \frac{K}{N}(q_2 - \frac{q_1}{N}) = T \\ &J_2\ddot{q}_2 + F_2\dot{q}_2 + K(q_2 - \frac{q_1}{N}) + mgd\cos q_2 = 0 \end{aligned}$$



For simplicity, assume $J_1 = J_2 = F_1 = F_2 = K = N = m = g = d = 1$, and take the angle of the arm as the system's output.

(i) Determine the system's state, input, and output.

(ii) Find the system's equilibria corresponding to $T^* = 0$.

(iii) Linearize the system around each of the above equilibrium points. Put the resulting linear system in the standard LTI form.

(iv) Compute and plot the impulse responses of each of the linearized systems. Do your results match your intuition?

(v) Determine the internal and input-output stability of each linearization. Does your answers match your intuition?

(vi) Are the linearized systems controllable? Observable?

(vii) Discretize each of the linearized systems using Euler discretization and a sample time of T_s. Are the discretized systems internally stable? Controllable? Observable? How does your responses depend on T_s?

(viii) Find a control signal that takes each of discretized systems from the initial state of downward rest to the final state of upward rest in $t_f = 1000T_s$.

(ix) Assume $T_s = 1$ms. Simulate each of the above designed control signals in MATLAB in the original nonlinear, continuous-time system using MATLAB `ode45`. Does the system go from downward rest to upward rest, as desired? Why or why not? (*Hint:* note that the control signal that you have designed is in discrete time, but `ode45` simulates the continuous-time system. So you need to transform your discrete-time input signal to a continuous-time signal. To do so, let your continuous-time signal be piecewise constant, taking the constant value of $u(k)$ for the interval $t \in [kT_s, (k+1)T_s)$.)

(x) Now stabilize the continuous-time linearization around the upward equilibrium using pole placement.

(xi) Simulate your controller using MATLAB `ode45`. Is the system stabilized towards the upward equilibrium? Does that depend on what initial condition you run the system from? Explain your results.

□