Lecture 3: Stability, Controllability & State Feedback

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This is our last set of notes where we briefly introduce some of the most basic concepts in the theory of linear systems: stability, controllability, and state feedback control. In brief, a linear system is stable if its state does remains bounded with time, is controllable if the input can be designed to take the system from any initial state to any final state, and if so, then can be stabilized using state feedback.
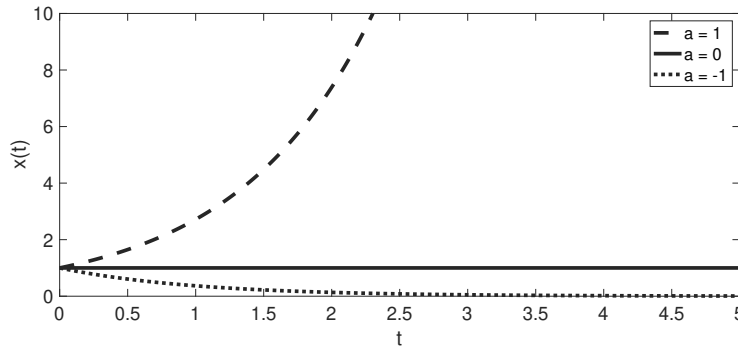
Contents

3.1   Internal Stability

Consider the simple, scalar system without input

$$\dot{x}(t) = ax(t), \qquad x(0) = 1 \tag{3.1}$$

The solution, of course, is

$$x(t) = e^{at} \tag{3.2}$$

whose behavior depends critically on the sign of a:

When a is positive the solution blows up to infinity, when a is negative the solution dies down to zero, and when a is zero, neither happens – the solution remains at the same level it started. This is the core of what stability is all about!

Definition 3.1.1 (Marginal & asymptotic stability) The *zero-input* LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.3}$$

is

- "asymptotically stable" if $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ for every initial condition $\mathbf{x}_0$

- "marginally stable" if $\mathbf{x}(t) \nrightarrow \mathbf{0}$ but remains bounded as $t \to \infty$ for every $\mathbf{x}_0$

- "stable" if it is either asymptotically or marginally stable

- "unstable" if it is not stable ($\|\mathbf{x}(t)\| \to \infty$ as $t \to \infty$ at least for some, if not all, $\mathbf{x}_0$)

$\square$

If you are given a system of the forms in Eq. (3.3), you can compute its trajectories (either analytically or using numerical simulations) and check the stability of the system from the above definition. This is not a great idea, however, because it requires solving the differential/difference equation in Eq. (3.3), which is not always possible analytically. Numerical solutions always exist, but notice that the definition of asymptotic/marginal stability requires the solutions to go to zero/remain bounded <u>for all initial conditions</u>. It is never possible to numerically solve the dynamics for all possible initial conditions.

Therefore, we ideally want a simple test to determine stability of an LTI system, without a need to solve for the state trajectories explicitly. This is achieved, not surprisingly, using eigenvalues!

Recall from our discussion of *diagonalization* in Section 1.6 that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} \Rightarrow e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1} \tag{3.4}$$

so whether $e^{\mathbf{A}t}$ remains bounded or not is a direct consequence of whether $e^{\mathbf{\Lambda}t}$ remains bounded or not. But

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{3.5}$$

2

and to assess the behavior of $e^{\mathbf{\Lambda} t}$ with time, recall that each λ_i is in general complex,

$$\lambda_i = \sigma_i + j\omega_i \qquad \begin{cases} \sigma_i = \mathrm{Re}\{\lambda_i\} \\ \omega_i = \mathrm{Im}\{\lambda_i\} \end{cases} \tag{3.6}$$

so

$$\begin{aligned} e^{\lambda_i t} &= e^{\sigma_i t + j\omega_i t} \\ &= e^{\sigma_i t}(\cos\omega_i t + j\sin\omega_i t) \end{aligned}$$

Notice that the factor $\cos\omega_i t + j\sin\omega_i t$ has always a unit modulus

$$|\cos\omega_i t + j\sin\omega_i t| = \sqrt{\cos^2\omega_i t + \sin^2\omega_i t} = 1$$

so

$$|e^{\lambda_i t}| = e^{\sigma_i t}$$

Therefore, whether $|e^{\lambda_i t}|$ converges to 0, diverges to infinity, or remains constant with time, depends only and only on the sign of $\sigma_i = \mathrm{Re}\{\lambda_i\}$, as we saw in Eq. (3.2). This leads us to the following fundamental result about the stability of LTI systems:

Theorem 3.1.2 (Marginal & asymptotic stability) A continuous-time diagonalizable LTI system is
- asymptotically stable if $\mathrm{Re}\{\lambda_i\} < 0$ for all i

- marginally stable if $\mathrm{Re}\{\lambda_i\} \leq 0$ for all i, and, there exists at least one i for which $\mathrm{Re}\{\lambda_i\} = 0$

- stable if $\mathrm{Re}\{\lambda_i\} \leq 0$ for all i

- unstable if $\mathrm{Re}\{\lambda_i\} > 0$ for at least one i

$\square$

3.2   Controllability

The notion of controllability, at the core, asks whether the input is *rich enough* to determine the state:

Definition 3.2.1 (Controllability) The LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad t \in [t_0, t_f] \tag{3.7a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.7b}$$

is called "controllable" if for any initial state $\mathbf{x}_0$ and any final state $\mathbf{x}_f$, the input signal $\mathbf{u}(t)$ can be designed such that the system, starting from $\mathbf{x}(0) = \mathbf{x}_0$, reaches $\mathbf{x}(t_f) = \mathbf{x}_f$ in some finite time t_f. $\square$

For reasons that we skip here, it's actually quite simple to check controllability for LTI system:

Theorem 3.2.2 (Controllability & controllability matrix) The systems in Eq. (3.7) is controllable if and only if the <u>controllability matrix</u>

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{3.8}$$

is full rank. $\square$

Notice that controllability is really all about $\mathbf{A}$ and $\mathbf{B}$, without any role played by the output matrices $\mathbf{C}$ or $\mathbf{D}$. As such, instead of saying whether the system is controllable or not, we sometimes say whether the pair $(\mathbf{A}, \mathbf{B})$ is controllable or not.

3.3   Feedback Stabilization

The notion of controllability, but definition, ensures that we have full control over the state of the system using the control signal $\mathbf{u}(t)$, making it possible to take the system from any initial state to any final state in a finite amount of time. This is often called "point to point control". Even though point to point control is how controllability is defined, in many real-world scenarios we are interested in a different control problem: stabilization.

Assume that you have an internally unstable system. Stabilization (a.k.a. regulation) is the problem of designing the control input $\mathbf{u}(t)$ such that the controlled system becomes stable. This can take different forms, but the most common form is <u>state feedback stabilization</u>:
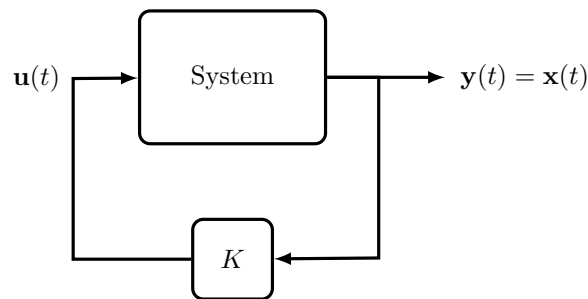
In this framework, the input is a constant (matrix) gain times the state,

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$$

and we want to choose the matrix $\mathbf{K}$ such that the close-loop system is stable.

By direct substitution, we readily see that

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
&= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{x} \\
&= (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}
\end{aligned}$$

so the stability of the close-loop system depends on the eigenvalues of

$$\mathbf{A} + \mathbf{B}\mathbf{K}$$

Note that $\mathbf{K} \in \mathbb{R}^{m \times n}$ has mn elements, but $\mathbf{A} + \mathbf{B}\mathbf{K}$ has only n eigenvalues. So we have mn degrees of freedom (variables to freely choose) to determine n eigenvalues. Looks like a favorable situation, right?

Example 3.3.1 (Pole placement) Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} u$$

Without the effect of input (if $u \equiv 0$), the open-loop system has eigenvalues

$$\lambda_1 = 3, \qquad \lambda_{2,3} = \frac{3}{2} \pm j\frac{\sqrt{11}}{2}$$

4

and is therefore (internally) unstable. So by closing the loop and choosing u appropriately, we try to make the closed-loop system asymptotically stable. The linear state feedback gain is

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \in \mathbb{R}^{1 \times 3}$$

and $u = \mathbf{kx}$. So the closed-loop coefficient matrix is

$$\mathbf{A} + \mathbf{BK} = \begin{bmatrix} 2 + k_1 & -3 + k_2 & 1 + k_3 \\ 3 - k_1 & 2 - k_2 & 2 - k_3 \\ 1 + 3k_1 & 3 + 3k_2 & 2 + 3k_3 \end{bmatrix}$$

and has the characteristic polynomial

$$\det(\lambda \mathbf{I} - (\mathbf{A} + \mathbf{BK})) = \lambda^3 + (-6 - k_1 + k_2 - 3k_3)\lambda^2 + (14 - 2k_1 - 13k_2 + 14k_3)\lambda + (-15 + 35k_1 + 10k_2 - 55k_3) \tag{3.9}$$

We want the closed-loop system to be asymptotically stable, so we want the roots of this polynomial to be in the left half plane. But otherwise, the roots can be anything. In other words, there are infinitely many ways to make the closed-loop system asymptotically stable, and just saying that we want the closed-loop system to be asymptotically stable does not uniquely determine $\mathbf{K}$. To uniquely determine $\mathbf{K}$, we have to fully determine the closed-loop characteristic polynomial or, equivalently, fully determine the closed-loop eigenvalues. This is of course a design choice. Assume we want

$$\text{closed-loop eigenvalues: } \lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3 \tag{3.10}$$

which gives the desired characteristic polynomial

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

Comparing this with Eq. (3.9), we see that we need

$$\begin{cases} -6 - k_1 + k_2 - 3k_3 = 6 \\ 14 - 2k_1 - 13k_2 + 14k_3 = 11 \\ -15 + 35k_1 + 10k_2 - 55k_3 = 6 \end{cases}$$

This is a linear system of equations, with the unique solution

$$\mathbf{K} = \begin{bmatrix} -4.2 & -3 & -3.6 \end{bmatrix}$$

Now, assume everything is the same, except that we have

$$\mathbf{B} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

We still want to choose $\mathbf{K}$ such that the closed-loop system is asymptotically stable. This time, the closed-loop characteristic polynomial becomes

$$\det(\lambda \mathbf{I} - (\mathbf{A} + \mathbf{BK})) = \lambda^3 + (-6 - k_1 + k_2 + k_3)\lambda^2 + (14 + 2k_1 - 5k_2 - 2k_3)\lambda + (-15 + 3k_1 + 6k_2 - 3k_3) \tag{3.11}$$

and, if we still want the closed-loop eigenvalues to be as in Eq. (3.10), we get the linear system of equations

$$\begin{cases} -6 - k_1 + k_2 + k_3 = 6 \\ 14 + 2k_1 - 5k_2 - 2k_3 = 11 \\ -15 + 3k_1 + 6k_2 - 3k_3 = 6 \end{cases}$$

5

This system of equations, unlike last time, has no solutions (check it yourself)! In fact, it is straightforward to check that $\lambda = 3$ is always a root of Eq. (3.11), regardless of k_1, k_2, k_3 (*notice that $\lambda = 3$ was one of the open-loop eigenvalues*). So not only the desired eigenvalues in Eq. (3.10) are impossible to obtain, but in fact it is impossible to stabilize this system using state feedback! $\qquad\square$

The difference between the two systems in Example 3.3.1 is of course their controllability:

Theorem 3.3.2 (Pole placement and controllability) The eigenvalues of $\mathbf{A}+\mathbf{BK}$ can be assigned arbitrarily (provided that complex eigenvalues are selected in conjugate pairs) if and only if $(\mathbf{A},\mathbf{B})$ is controllable. $\square$

You can easily check that the first system in Example 3.3.1 is controllable, while the second one is not.

When you have a controllable $(\mathbf{A},\mathbf{B})$, as I said above, choosing the location of the closed-loop eigenvalues is a design choice. As a rule of thumb, do not choose the closed-loop eigenvalues too close to the imaginary axis, too far from it, or too far from the real axis. These result in, respectively, a very slow system, a noise-amplifying system, and a very rapidly oscillating system, neither of which are desirable in practice.

From Theorem 3.3.2, you can see that controllability may be too much to ask. If all we need is stabilization, we don't need to choose the closed-loop eigenvalues arbitrarily, we only need them to be in the left half plane. It turns out that sometimes, that is possible even if the system is not controllable:

Example 3.3.3 (Partial pole placement) Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -3 & -2 \\ 6 & -1 & 5 \\ 7 & 3 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} u$$

The open-loop eigenvalues are

$$\lambda_1 = 3, \qquad \lambda_{2,3} = -\frac{3}{2} \pm j\frac{\sqrt{11}}{2}$$

so the open-loop system ($u \equiv 0$) is unstable, and you can easily check that the system is not controllable. So by Theorem 3.3.2, we know there is no hope in trying to assign the closed-loop eigenvalues arbitrarily. But there is still a possibility that we can stabilize the system. Recall that in the second system of Example 3.3.1, stabilization was not possible because one of the unstable open-loop eigenvalues ($\lambda = 3$) was always a closed-loop eigenvalue, regardless of k_1, k_2, k_3. So let's check the open-loop eigenvalues for our system in this example. Here our closed-loop characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(\lambda\mathbf{I} - (\mathbf{A}+\mathbf{BK})) \\ &= \lambda^3 + (-k_1 + k_2 + 2k_3)\lambda^2 + (-4 - 3k_1 + 3k_2 + 6k_3)\lambda + (-15 - 5k_1 + 5k_2 + 10k_3) \end{aligned}$$

and we can check that

$$p(3) = -23k_1 + 23k_2 + 46k_3$$
$$p(-3/2 + j\sqrt{11}/2) = 0$$
$$p(-3/2 - j\sqrt{11}/2) = 0$$

This means that two of the open-loop eigenvalues, namely, $\lambda_{2,3} = -\frac{3}{2} \pm j\frac{\sqrt{11}}{2}$ are always closed-loop eigenvalues as well (regardless of $\mathbf{K}$), but $\lambda_1 = 3$ is not necessarily. This is great news, because both of $\lambda_{2,3} = -\frac{3}{2} \pm j\frac{\sqrt{11}}{2}$ are stable, and we don't worry about them being closed-loop eigenvalues as well. All we need to do is change λ_1 with a stable eigenvalue. By factorizing $p(\lambda)$ (MATLAB `factor()`), we see that

$$p(\lambda) = (\lambda^2 + 3\lambda + 5)(\lambda - 3 - k_1 + k_2 + 2k_3)$$

6

clearly showing that the closed-loop system can be made stable if $\lambda_1 = 3 + k_1 - k_2 - 2k_3$ is chosen to be in the left half plane, which can be done in infinitely many ways. □

In general, if you want to stabilize a system that is uncontrollable, check the open loop eigenvalues one by one. At least one of them will be a root of the closed-loop characteristic polynomial $p(\lambda)$, no matter what **K** you choose. If that eigenvalue is unstable (like the second system of example 3.3.1), there is nothing you can do! But if only stable open-loop eigenvalues satisfy $p(\lambda) = 0$, then you can take their corresponding factors out of $p(\lambda)$ and then choose **K** such that the remaining eigenvalues are set as you wish! In this latter case, the system is called stabilizable:

Definition 3.3.4 (Stabilizability) The system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

or, equivalently, the pair $(\mathbf{A}, \mathbf{B})$ is called "stabilizable" if **K** can be chosen such that all of the eigenvalues of $\mathbf{A} + \mathbf{B}\mathbf{K}$ belong to the left half of the complex plane. □

Note that stabilizability is a weaker condition than controllability (any controllable system is stabilizable, but not vice versa).