Think Quantumly About Computing

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Computing on "Bits"

Conventional Digital Circuits Implement Boolean Algebra

- Bits — {0, 1}: The only two possible values in inputs/outputs
- Basic operators
  - AND (•) — a • b
    - returns 1 only if both a **and** b are 1s
    - otherwise returns 0
  - OR (+) — a + b
    - returns 1 if a **or** b is 1
    - returns 0 if none of them are 1s
  - NOT (‘) — a’
    - returns 0 if a is 1
    - returns 1 if a is 0

Truth tables

- A table sets out the functional values of logical expressions on each of their functional arguments, that is, for each combination of values taken by their logical variables

AND

Input		Output
A	B	
0	0	**0**
0	1	**0**
1	0	**0**
1	1	**1**

OR

Input		Output
A	B	
0	0	**0**
0	1	**1**
1	0	**1**
1	1	**1**

NOT

Input	Output
A	
0	1
1	0

Derived Boolean operators

- NAND — (a • b)′
- NOR — (a + b)′
- XOR — (a + b) • (a′ + b′) or ab′ + a′b
- XNOR — (a + b′) • (a′ + b) or ab + a′b′

NAND

Input		Output
A	B	
0	0	**1**
0	1	**1**
1	0	**1**
1	1	**0**

NOR

Input		Output
A	B	
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**0**

XOR

Input		Output
A	B	
0	0	**0**
0	1	**1**
1	0	**1**
1	1	**0**

XNOR

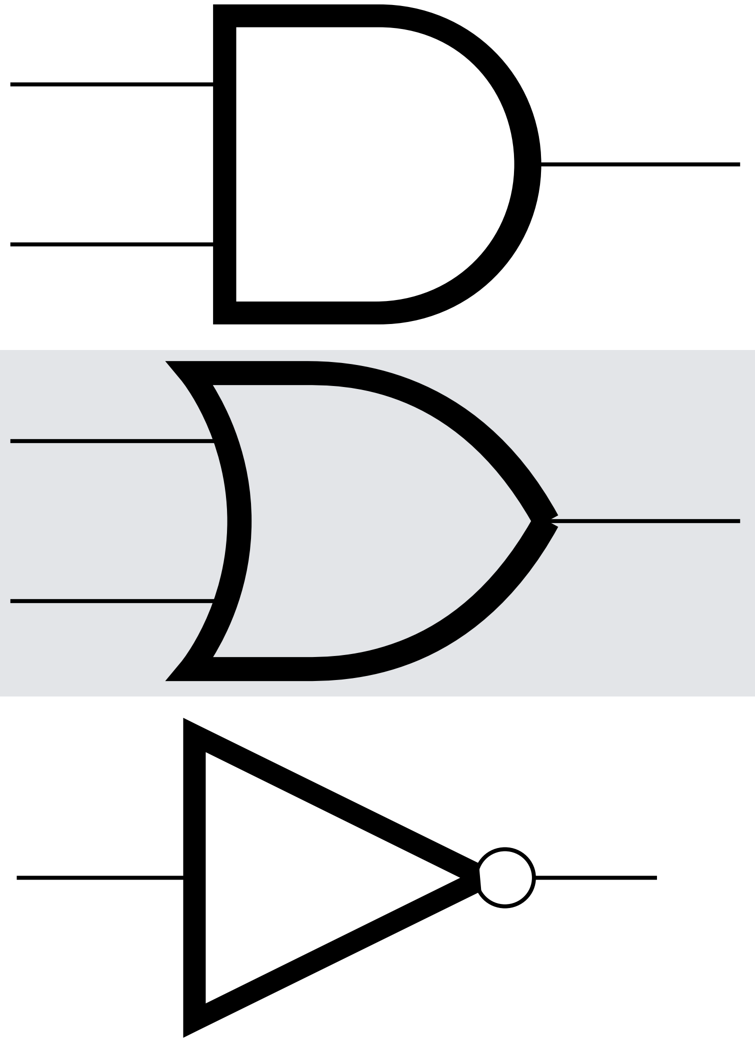
Input		Output
A	B	
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**1**

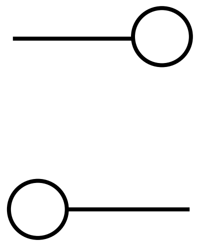
Boolean operators their circuit “gate” symbols

AND

OR

NOT





represents where we take a compliment value on an input

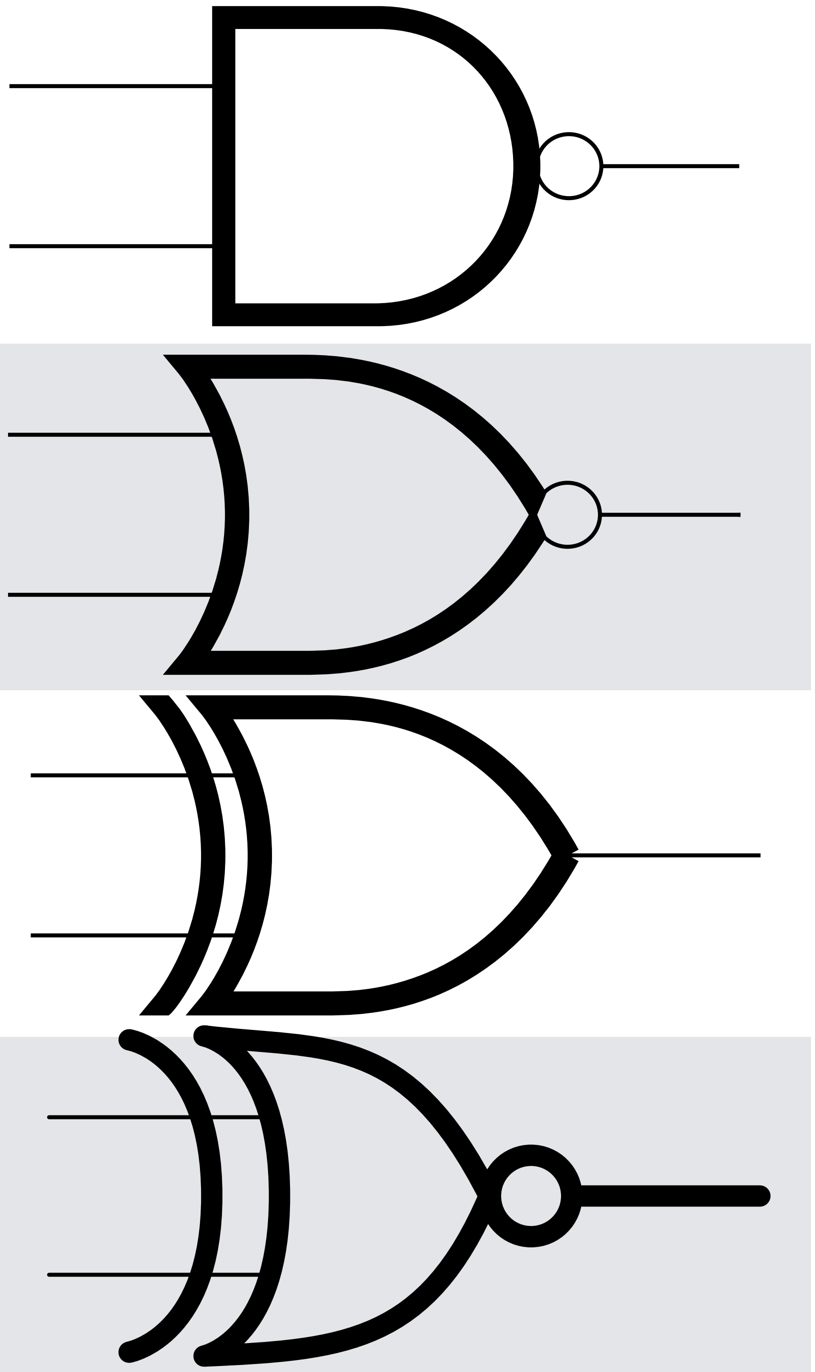
represents where we take a compliment value on an output

NAND

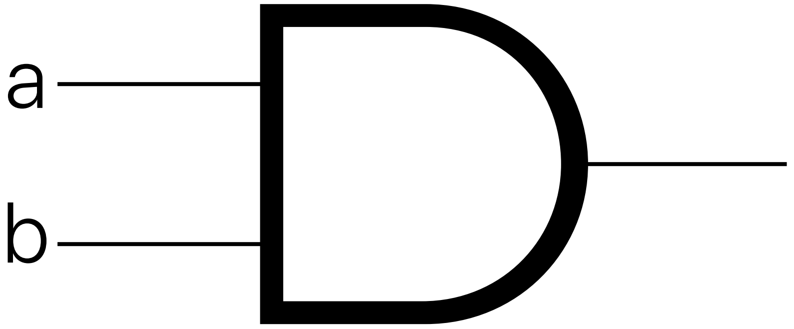
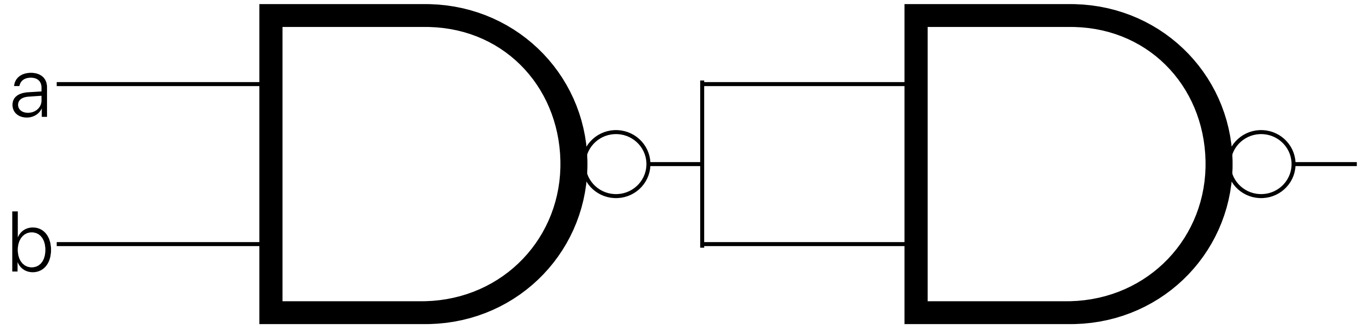
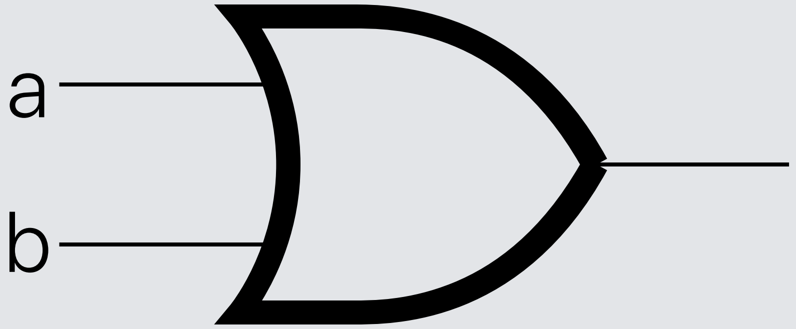
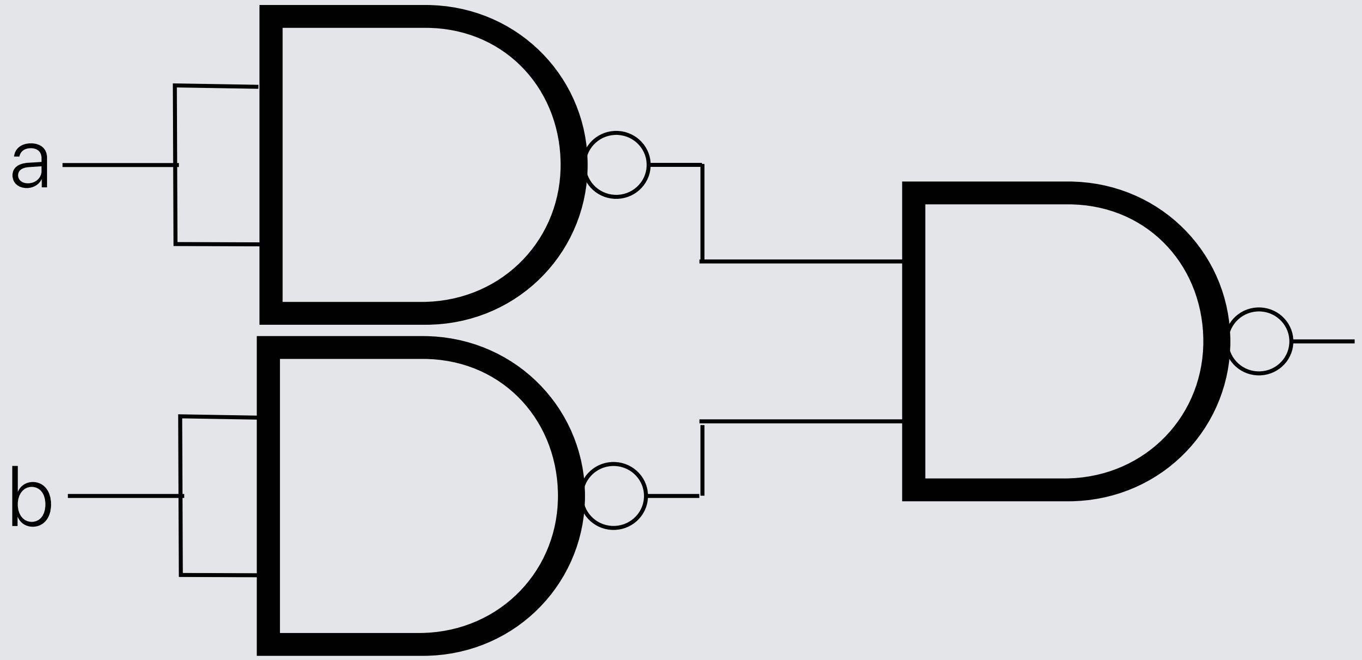
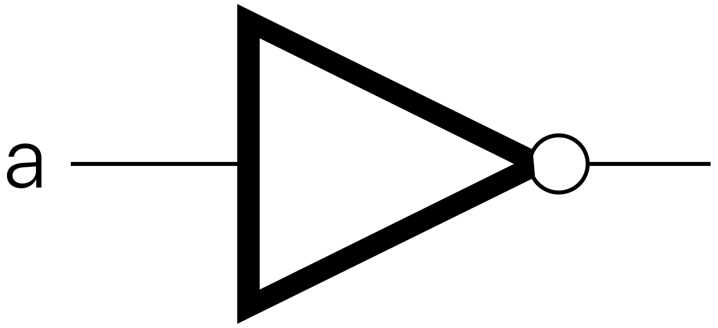
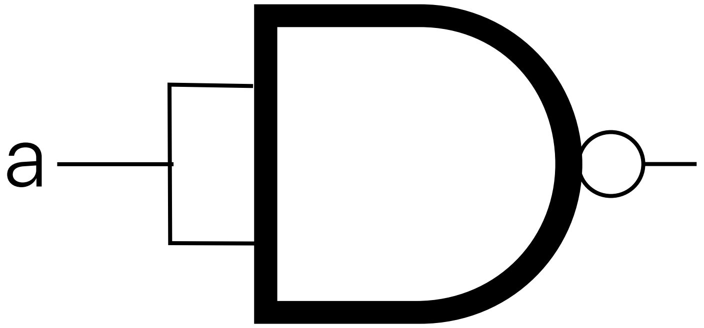
NOR

XOR

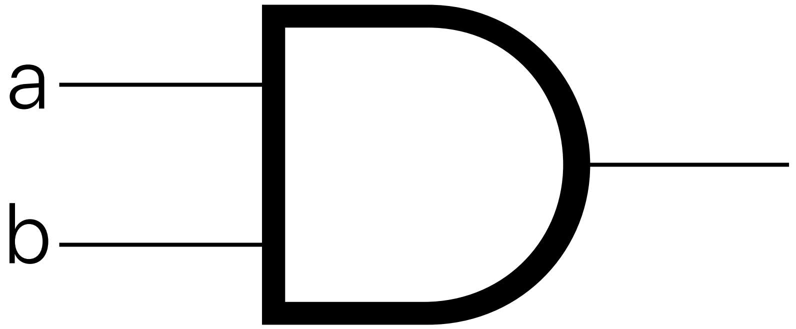
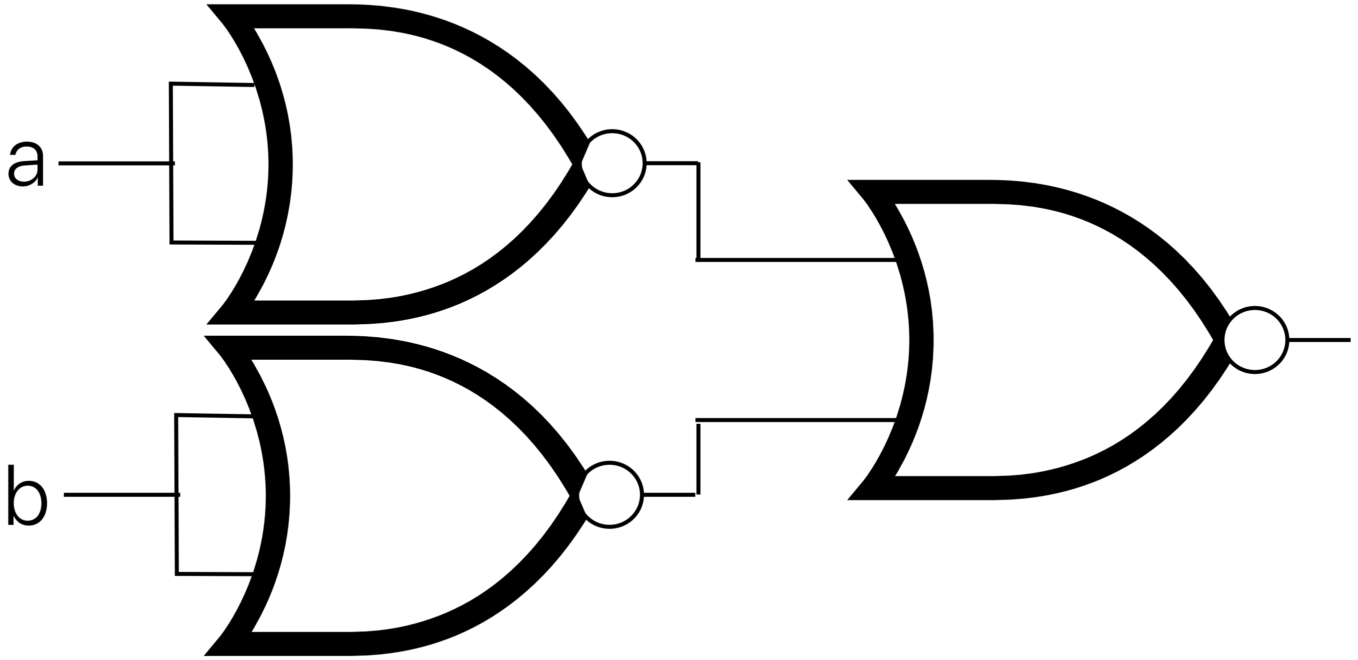
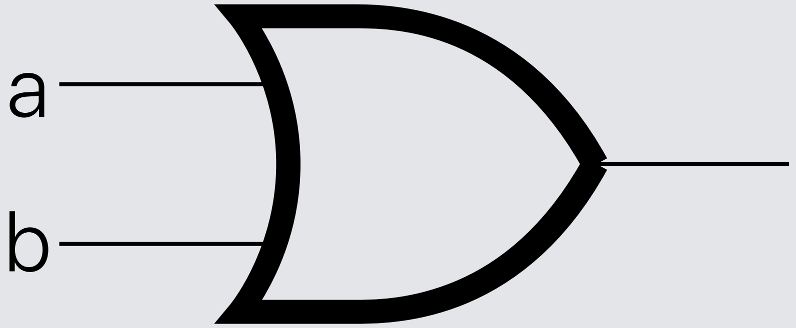
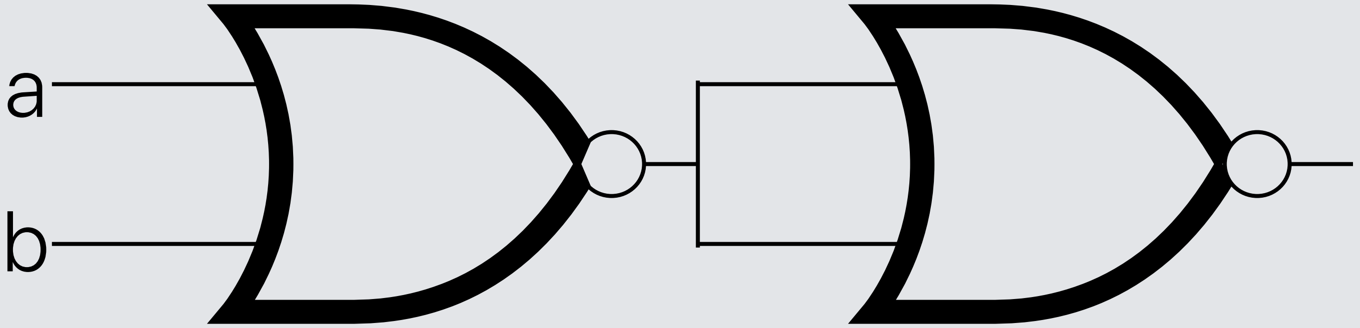
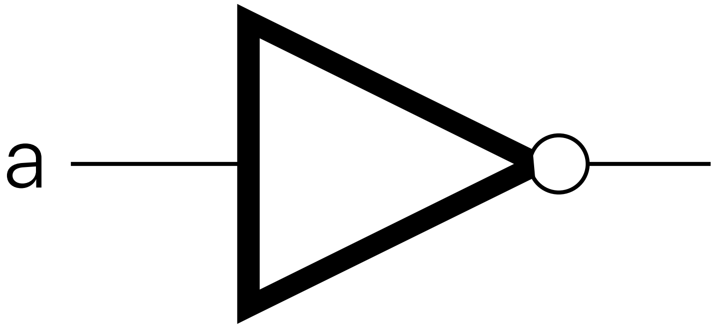
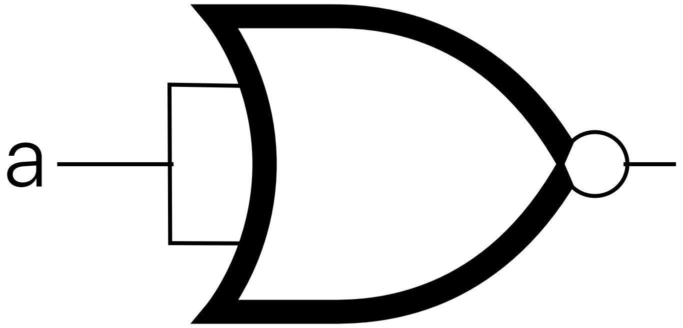
NXOR



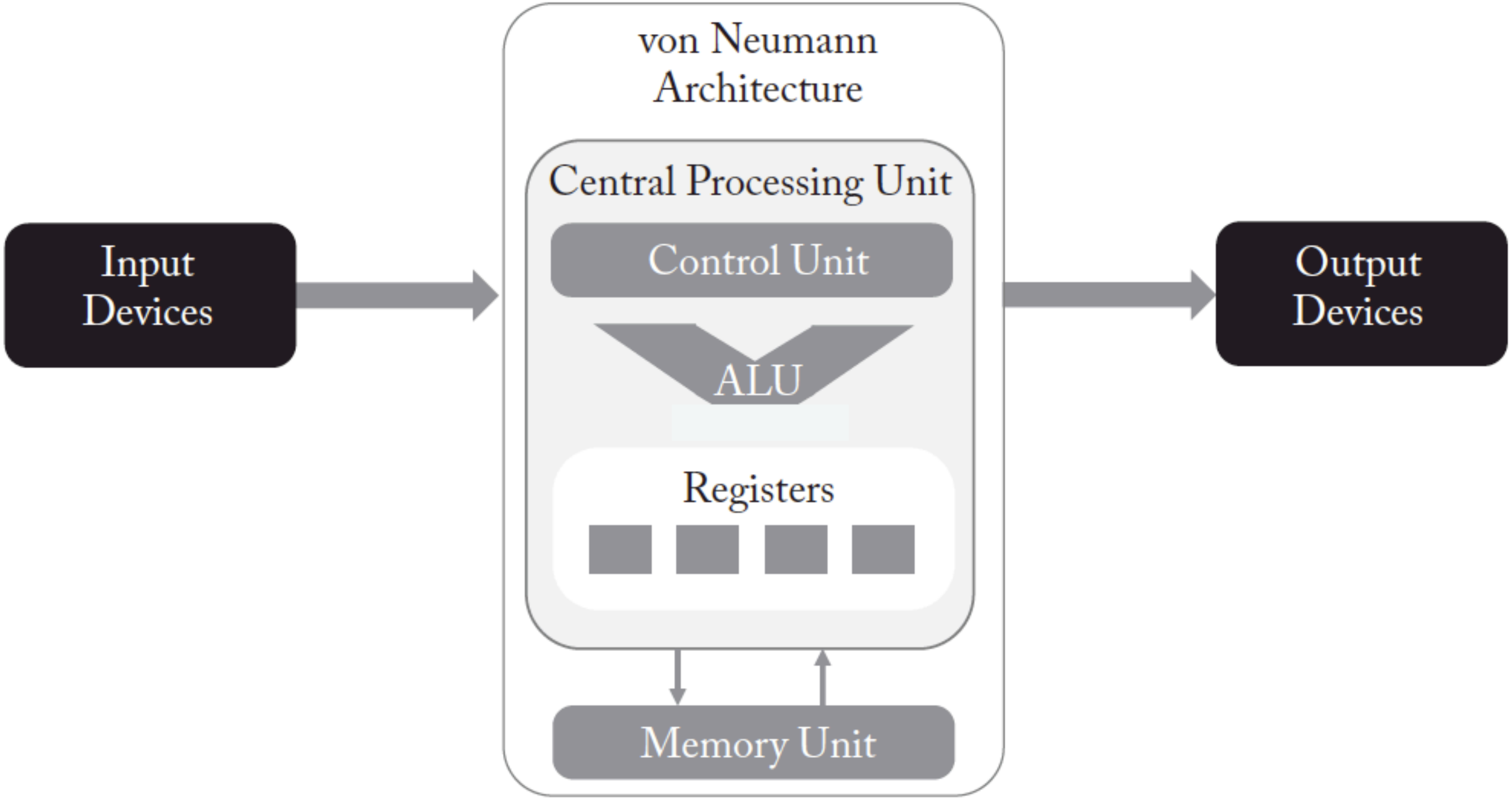
We can make everything NAND!

	Original	NAND
AND		
OR		
NOT		

We can also make everything NOR!

	Original	NAND
AND		
OR		
NOT		

von Neumann Architecture



Truth tables

- A table sets out the functional values of logical expressions on each of their functional arguments, that is, for each combination of values taken by their logical variables

AND

Input		Output
A	B	
0	0	**0**
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1	0	**0**
1	1	**1**

OR

Input		Output
A	B	
0	0	**0**
0	1	**1**
1	0	**1**
1	1	**1**

NOT

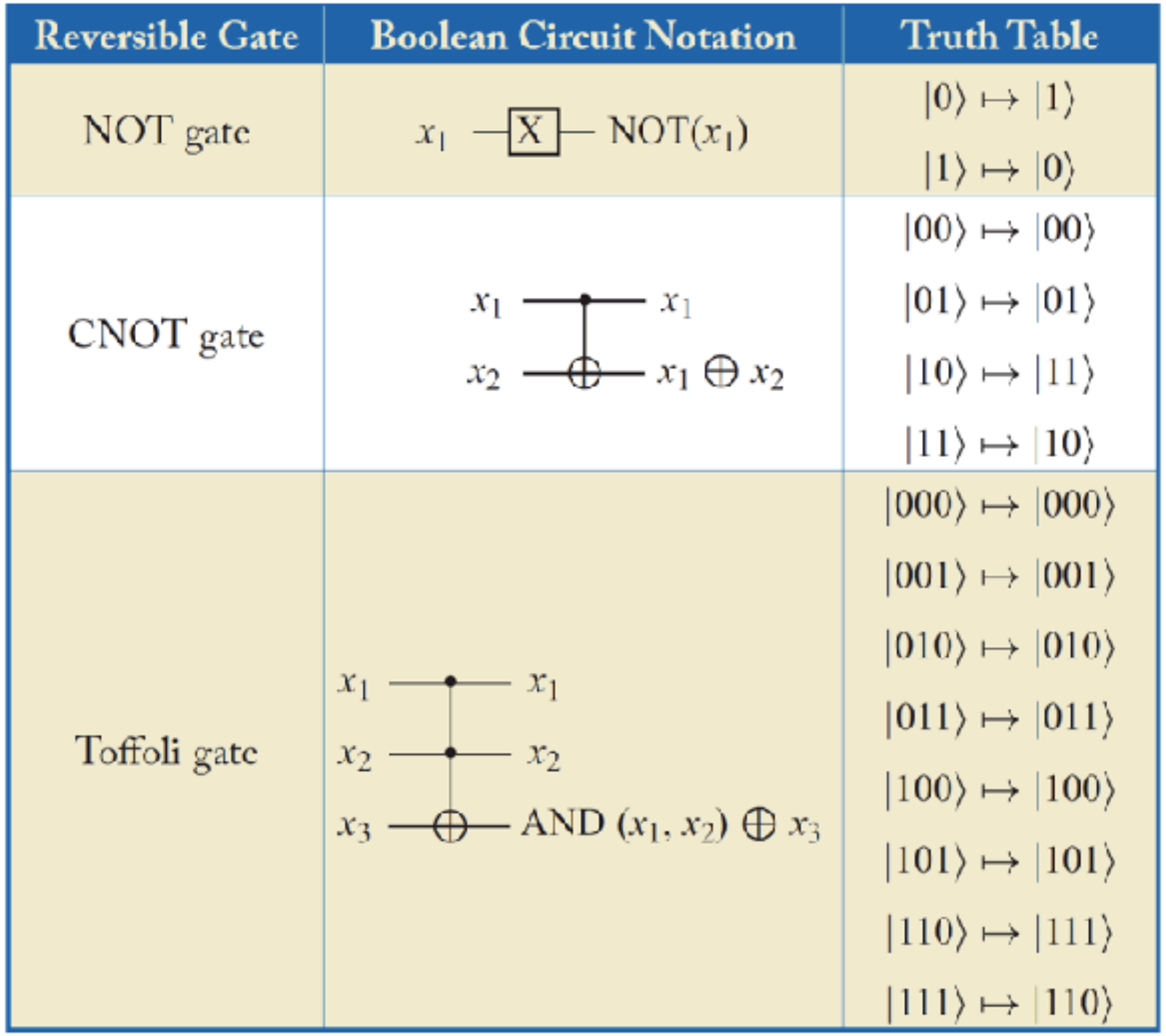
Input	Output
A	
0	1
1	0

Reversible

Two concepts before talking about "Qubits"

Reversible computation

- Second law of thermodynamics — Irreversible bit operation consumes energy
- Reversible gates



Reversible Gate	Boolean Circuit Notation	Truth Table
NOT gate	x_1 X NOT(x_1)	$\|0\rangle \mapsto \|1\rangle$ $\|1\rangle \mapsto \|0\rangle$
CNOT gate	x_1 x_1 x_2 $x_1 \oplus x_2$	$\|00\rangle \mapsto \|00\rangle$ $\|01\rangle \mapsto \|01\rangle$ $\|10\rangle \mapsto \|11\rangle$ $\|11\rangle \mapsto \|10\rangle$
Toffoli gate	x_1 x_1 x_2 x_2 x_3 AND $(x_1, x_2) \oplus x_3$	$\|000\rangle \mapsto \|000\rangle$ $\|001\rangle \mapsto \|001\rangle$ $\|010\rangle \mapsto \|010\rangle$ $\|011\rangle \mapsto \|011\rangle$ $\|100\rangle \mapsto \|100\rangle$ $\|101\rangle \mapsto \|101\rangle$ $\|110\rangle \mapsto \|111\rangle$ $\|111\rangle \mapsto \|110\rangle$

Randomized Computation

- Two forms
  - An algorithm takes random inputs
  - An algorithm make random choices
- Notation of random bits
  - $|x_1\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$ (1) x_1 has 50% probability to be 0, 50% to be 1
  - $|x_1x_2\rangle = \frac{1}{8}|00\rangle + \frac{1}{4}|01\rangle + \frac{1}{8}|10\rangle + 0|11\rangle$ (2)
- Conditional probability
  - For case (2), if we observed as x_1 0
  - $|x_1x_2\rangle(given\ x_1 = 0) = \frac{1/8}{3/8}|00\rangle + \frac{1/4}{3/8}|01\rangle = \frac{1}{3}|00\rangle + \frac{2}{3}|01\rangle$

Now, quantum computing

Qubit System

- $|\psi\rangle = \sum_{b \in \{0,1\}^n} \alpha_b |b\rangle$
- α_b—amplitude of the basis bit-string b
  - α_b can be any complex numbers
  - $\sum_{b \in \{0,1\}^n} |\alpha_b|^2 = 1$
  - amplitudes (as being possibly negative) can either accumulate (constructively) and cancel (destructively)
- Probability distribution across bs is called the **superposition** of all bit strings
- The correlation between bits is called **entanglement** of qubits

Measure the Qubit system

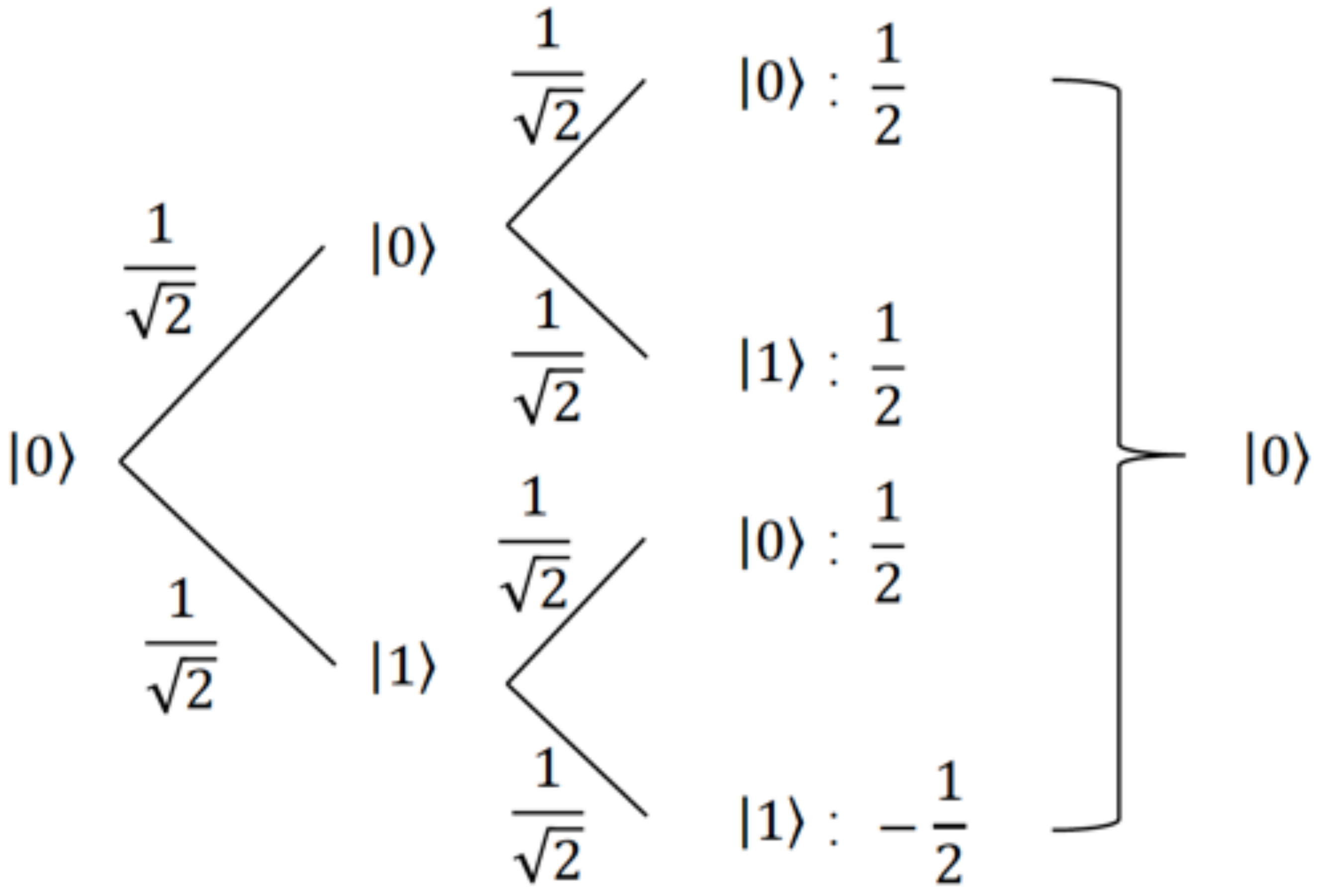
- Upon measurement, the state of the system "collapses" to the single classical definite value and can no longer revert to the superposition as it was before
- $Meas(|\psi\rangle) = |b\rangle$
- For $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
  $Pr[Meas(|\psi\rangle) = |0\rangle] = Pr[Meas(|\psi\rangle) = |1\rangle] = \frac{1}{2}$

Feynman's Sum-Over-Path Approach

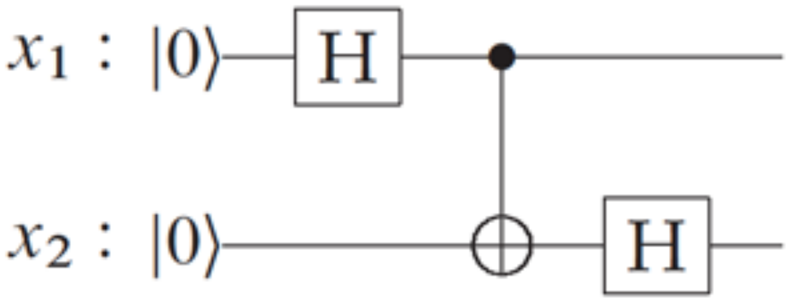
- The final amplitude is given by adding the contributions from all paths; and
- The contribution from a path is given by multiplying the coefficients along the path.

Hadamard transformation

$$H = \begin{cases} |0\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \\ |1\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle. \end{cases}$$



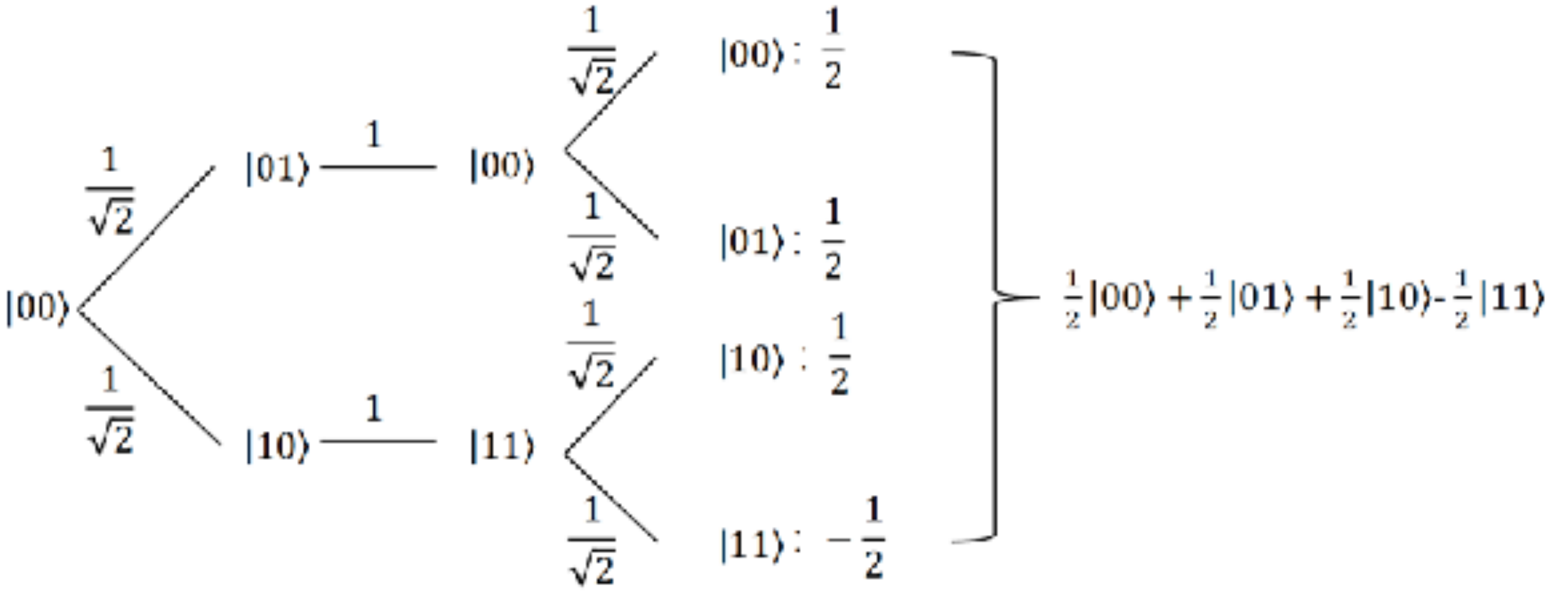
Interference prevention — Entangled Ancilla Qubit



- Bell state — After the first H gate and CNOT gate, we arrive at

$$|x_1x_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

  - The other qubit is guaranteed to be measured in the same state as the first
  - This "correlation" between the two qubits are called quantum entanglement.



Architecture constraints of Quantum Computers

- Probabilistic outcomes
    - a quantum program is intrinsically probabilistic
    - a good quantum program will make sure the desired bit-strings are observed with much higher probability compared to the undesired ones
    - a quantum program may need thousands of shots before meaningful statistics
- No copying of qubits
- Qubit-qubit interactions
- Analog noises

Principles of Quantum Computing

Superposition

- Amplitude distribution
  - $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1.$
  - $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
- A d-dimensional qubit system is defined as a superposition of d basis states
  - $|\psi\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle + \cdots + \alpha_{d-1}\,|d-1\rangle$

Composition

- The joint state of two separate quantum systems
- $|\psi\rangle = |\psi_0\rangle \otimes |\psi_1\rangle = \sum_j \sum_k \alpha_j \beta_k (|a_j\rangle \otimes |b_j\rangle),$ where $|a_j\rangle \otimes |b_j\rangle$ can often be shortened as $|a_j b_k\rangle$

Example 2.4 The four basis states of the two-qubit system are

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

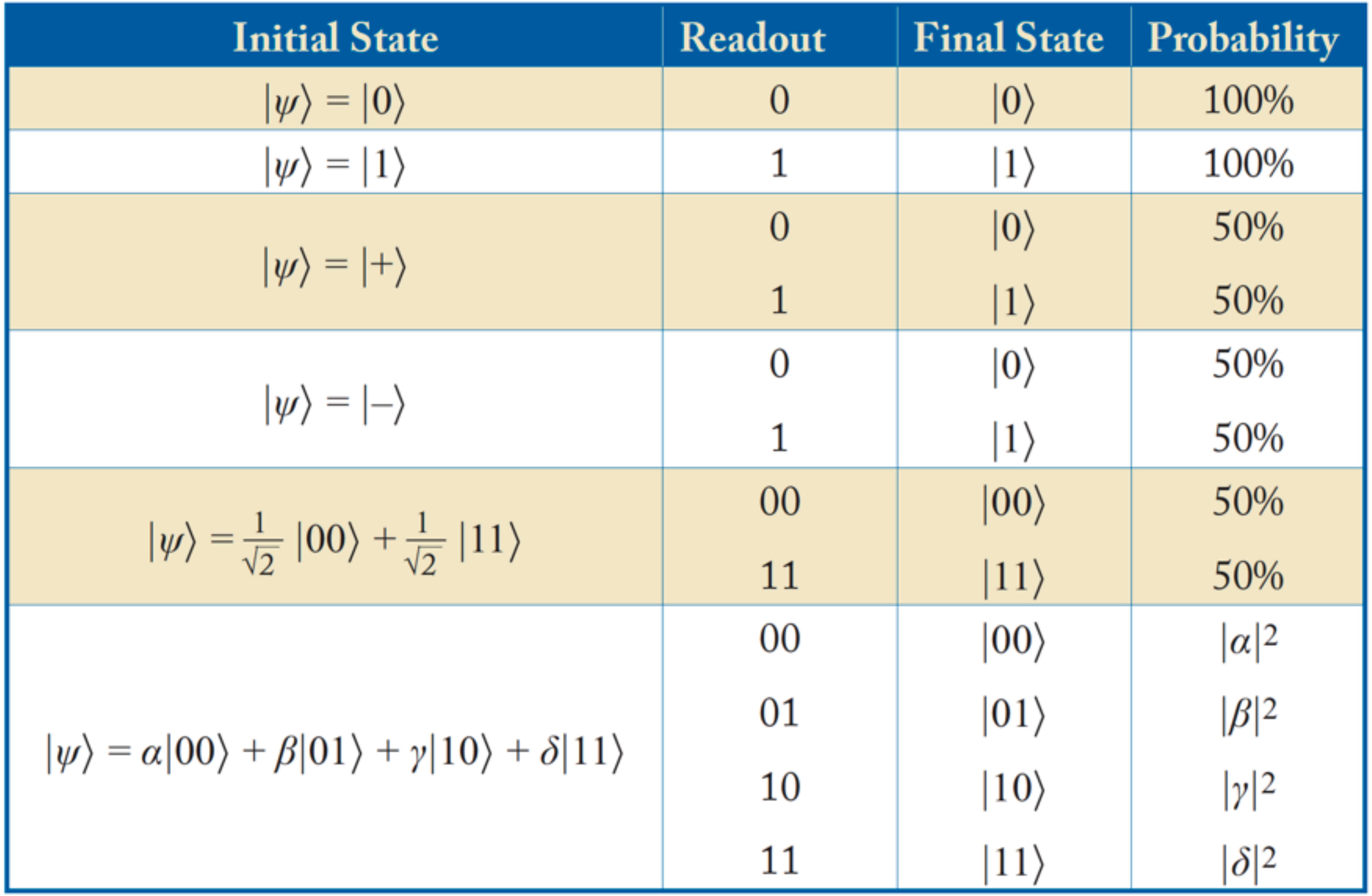
$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Example 2.5 Let's take a look at the example of two generic qubits. Suppose the first qubit is $|\psi_0\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ and the second qubit is $|\psi_1\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$, then their joint state is:

$$|\psi\rangle = |\psi_0\rangle \otimes |\psi_1\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}.$$

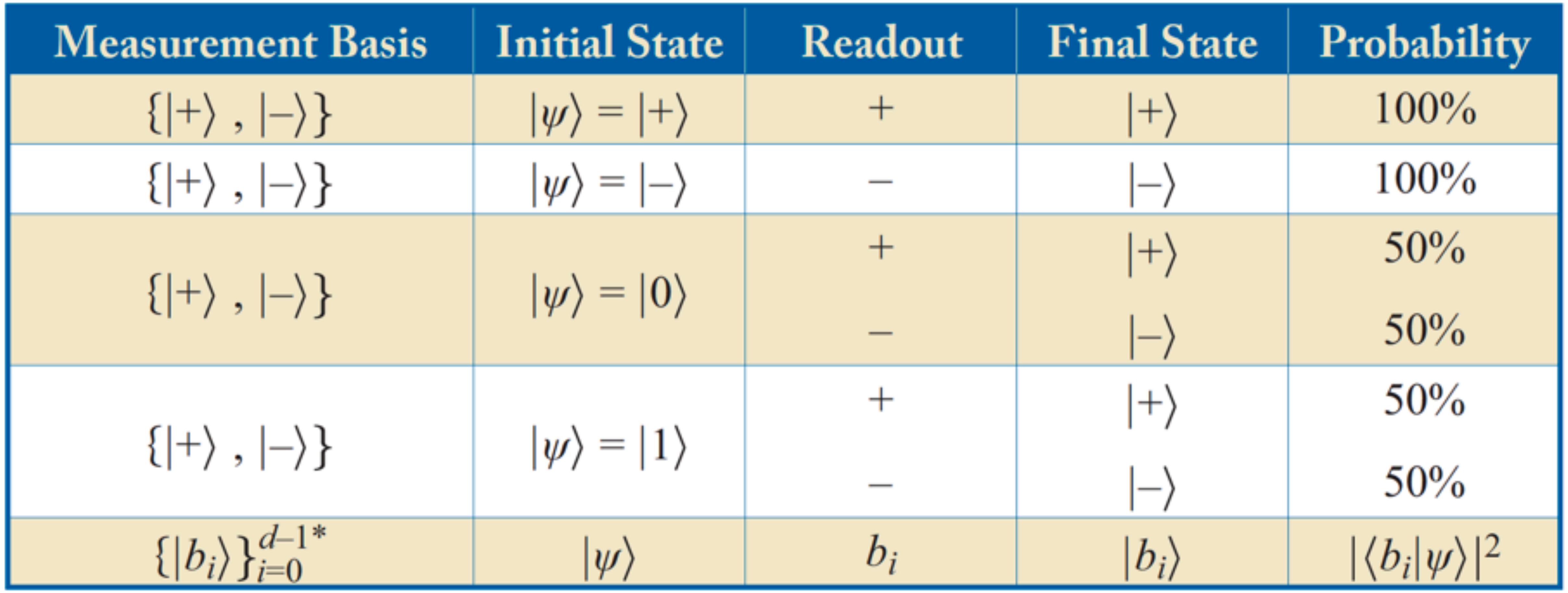
Measurement

- When we measure a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$we observe the basis state $0\rangle$ with probability $|\alpha|^2$and the basis state $1\rangle$ with probability $|\beta|^2$
- The process of measurement is irreversible and probabilistic
    - the state $|\psi\rangle$ collapses into one of the two basis states ($0\rangle$ or $1\rangle$)
    - the original quantum superposition cannot be recovered
    - outcome is related to the latitude of the quantum state— global phase (longitude) does not matter



Initial State	Readout	Final State	Probability
$\|\psi\rangle = \|0\rangle$	0	$\|0\rangle$	100%
$\|\psi\rangle = \|1\rangle$	1	$\|1\rangle$	100%
$\|\psi\rangle = \|+\rangle$	0	$\|0\rangle$	50%
	1	$\|1\rangle$	50%
$\|\psi\rangle = \|-\rangle$	0	$\|0\rangle$	50%
	1	$\|1\rangle$	50%
$\|\psi\rangle = \frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	00	$\|00\rangle$	50%
	11	$\|11\rangle$	50%
$\|\psi\rangle = \alpha\|00\rangle + \beta\|01\rangle + \gamma\|10\rangle + \delta\|11\rangle$	00	$\|00\rangle$	$\|\alpha\|^2$
	01	$\|01\rangle$	$\|\beta\|^2$
	10	$\|10\rangle$	$\|\gamma\|^2$
	11	$\|11\rangle$	$\|\delta\|^2$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle$$



Measurement Basis	Initial State	Readout	Final State	Probability
$\{\|+\rangle\ ,\ \|-\rangle\}$	$\|\psi\rangle = \|+\rangle$	$+$	$\|+\rangle$	100%
$\{\|+\rangle\ ,\ \|-\rangle\}$	$\|\psi\rangle = \|-\rangle$	$-$	$\|-\rangle$	100%
$\{\|+\rangle\ ,\ \|-\rangle\}$	$\|\psi\rangle = \|0\rangle$	$+$	$\|+\rangle$	50%
		$-$	$\|-\rangle$	50%
$\{\|+\rangle\ ,\ \|-\rangle\}$	$\|\psi\rangle = \|1\rangle$	$+$	$\|+\rangle$	50%
		$-$	$\|-\rangle$	50%
$\{\|b_i\rangle\}_{i=0}^{d-1*}$	$\|\psi\rangle$	b_i	$\|b_i\rangle$	$\|\langle b_i\|\psi\rangle\|^2$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

For instance, for the computational basis measurement, we take $M_1 = |0\rangle\langle 0|$ and $M_2 = |1\rangle\langle 1|$. Upon measurement, we obtain the outcome "i" with probability

$$\mathbf{Pr}[\text{observe } i] = |M_i\,|\psi\rangle\,|^2 = \langle\psi|M_i^\dagger M_i|\psi\rangle\,,$$

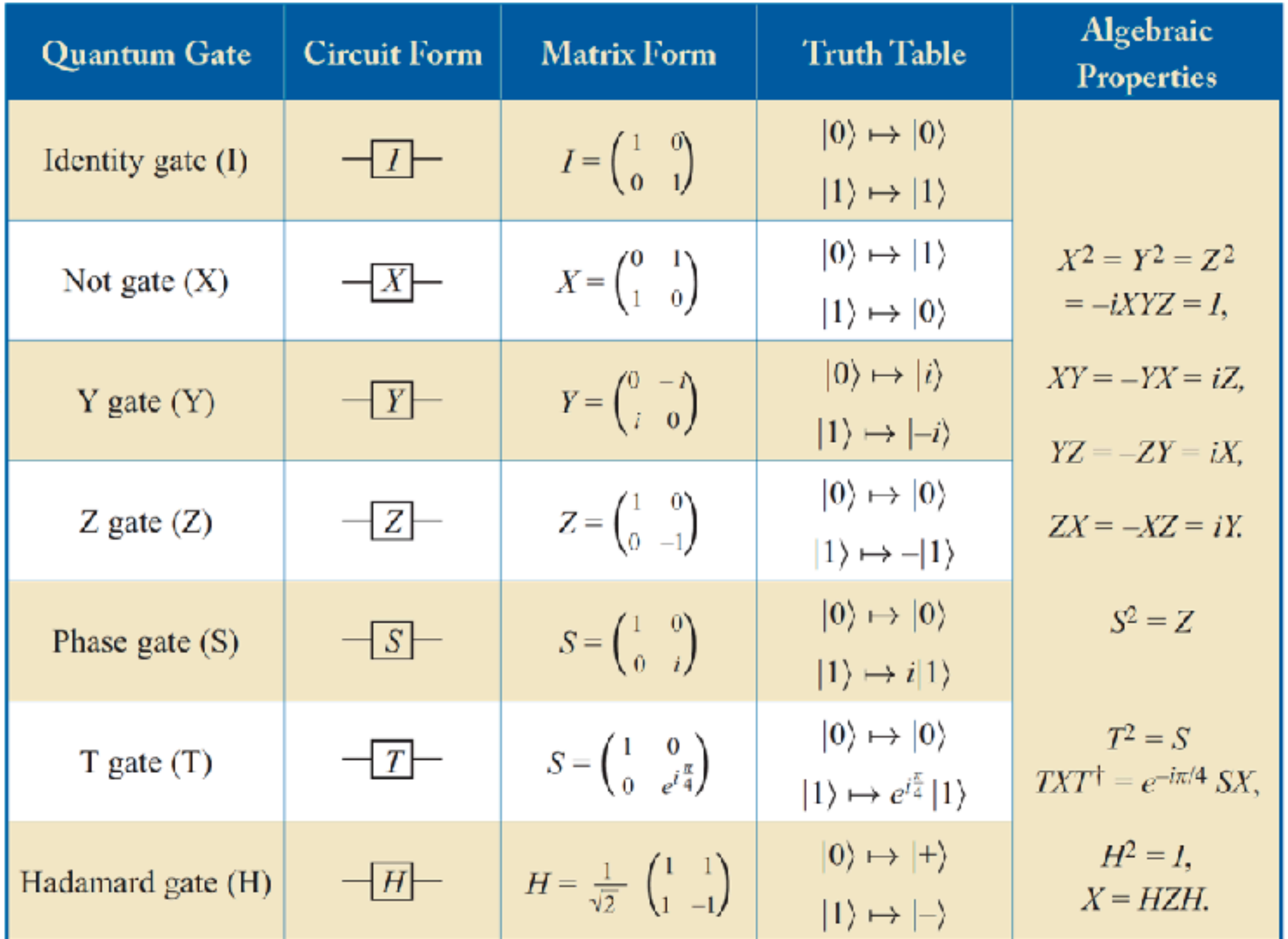
which results in a quantum state

$$|\psi'\rangle = \frac{M_i\,|\psi\rangle}{|M_i\,|\psi\rangle\,|} = \frac{M_i\,|\psi\rangle}{\sqrt{\langle\psi|M_i^\dagger M_i|\psi\rangle}}.$$

Quantum Gates

- Transformation through gates must be reversible and deterministic

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), \quad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$



Quantum Gate	Circuit Form	Matrix Form	Truth Table	Algebraic Properties
Identity gate (I)	$-\boxed{I}-$	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\|0\rangle \mapsto \|0\rangle$ $\|1\rangle \mapsto \|1\rangle$	
Not gate (X)	$-\boxed{X}-$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\|0\rangle \mapsto \|1\rangle$ $\|1\rangle \mapsto \|0\rangle$	$X^2 = Y^2 = Z^2$ $= -iXYZ = I,$
Y gate (Y)	$-\boxed{Y}-$	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\|0\rangle \mapsto \|i\rangle$ $\|1\rangle \rightarrow \|-i\rangle$	$XY = -YX = iZ,$ $YZ = -ZY = iX,$
Z gate (Z)	$-\boxed{Z}-$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\|0\rangle \mapsto \|0\rangle$ $\|1\rangle \mapsto -\|1\rangle$	$ZX = -XZ = iY.$
Phase gate (S)	$-\boxed{S}-$	$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$\|0\rangle \mapsto \|0\rangle$ $\|1\rangle \rightarrow i\|1\rangle$	$S^2 = Z$
T gate (T)	$-\boxed{T}-$	$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$	$\|0\rangle \mapsto \|0\rangle$ $\|1\rangle \mapsto e^{i\frac{\pi}{4}}\|1\rangle$	$T^2 = S$ $TXT^\dagger = e^{-i\pi/4}\,SX,$
Hadamard gate (H)	$-\boxed{H}-$	$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\|0\rangle \mapsto \|+\rangle$ $\|1\rangle \mapsto \|-\rangle$	$H^2 = I,$ $X = HZH.$

For example, when a qubit is in a superposition state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ then the operation applies to each of the basis states, e.g.,

$$H |\psi\rangle = \alpha(H |0\rangle) + \beta(H |1\rangle) = \alpha |+\rangle + \beta |-\rangle = \frac{\alpha + \beta}{\sqrt{2}} |0\rangle + \frac{\alpha - \beta}{\sqrt{2}} |1\rangle .$$

Example 2.12 It is usually convenient to include generic single-qubit rotation gates (e.g., R_x, R_y, R_z gates) along the Pauli axes in our gate set. We write $R_x(\theta)$ to indicate a rotation of θ angle about the x-axis. Several of the gates we've already discussed are just examples of the $R_z(\theta)$ gates, specifically the Z, S, and T gates which rotate by a π, $\frac{\pi}{2}$, and $\frac{\pi}{4}$ angle, respectively. Formally, the rotation gate can be written in their matrix forms as follows:

$$R_x(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_y(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_z(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}.$$

Example 2.13 **Two-qubit gates** take two qubits as inputs. They typically have an "entangling" effect—the operation applied to one qubit is dependent on the state of the other qubit, in other words, they are conditional gates. Among the most common two-qubit operations are the controlled-not gate (or CNOT gate), and the controlled-phase gate (or CZ gate), as shown in Table 2.5.

In the example, the CNOT gate is a two-input two-output gate which performs a NOT operation on the second (target) qubit only when the first (control) qubit is $|1\rangle$. Similarly for CZ gate, if the control qubit is $|1\rangle$, then we apply a Z gate to the target qubit. But looking at the truth table of the CZ gate, we notice that, in fact, it makes no distinction between the first and the second qubits—a phase is accumulated for the $|11\rangle$ basis. Hence, the CZ gate has a symmetric circuit symbol. One can in fact implement a CNOT gate with a CZ gate and vice versa. For example, CNOT is equivalent to a CZ gate with two Hadamard gates on both sides, since $HZH = X$:

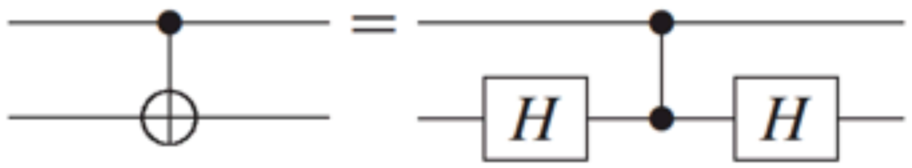


Table 2.5: Example measurement outcomes by $MeasZ$ on initial state $|\psi\rangle$.

Quantum Gate	Circuit Form	Matrix Form	Truth Table
CNOT gate		$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\|00\rangle \mapsto \|00\rangle$ $\|01\rangle \mapsto \|01\rangle$ $\|10\rangle \mapsto \|11\rangle$ $\|11\rangle \mapsto \|10\rangle$
CZ gate		$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\|00\rangle \mapsto \|00\rangle$ $\|01\rangle \mapsto \|01\rangle$ $\|10\rangle \mapsto \|10\rangle$ $\|11\rangle \mapsto -\|11\rangle$

Qubit Technologies

General Design Philosophy

- scalable system with well-characterized qubits
- ability to initialize qubits (e.g., prepare in computational basis)
- stability of qubits (i.e., long decoherence times)
- support for a universal instruction set (e.g., single qubit gates and CNOT gate) for arbitrary computation
- ability to measure qubits (e.g., readout in computational basis)

Trapped Ion Qubits

- Optical Qubits
- Hyperfine Qubits

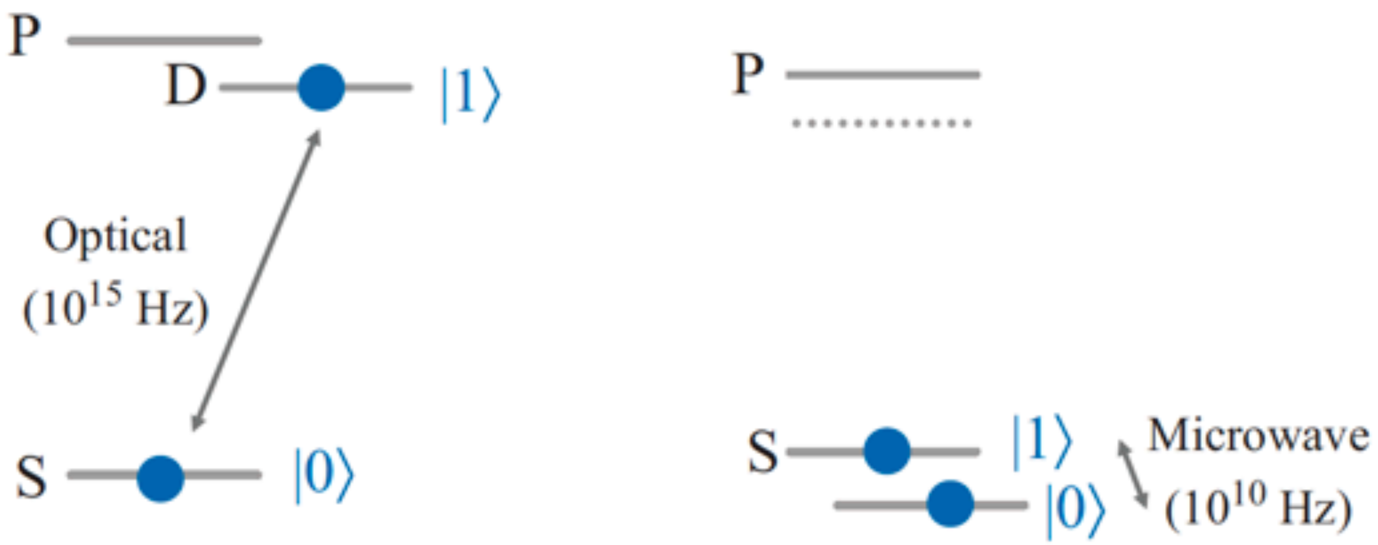


Figure 2.6: State transitions for two common types of trapped ion qubits: the optical qubit and the hyperfine qubit.

Measuring Qubits

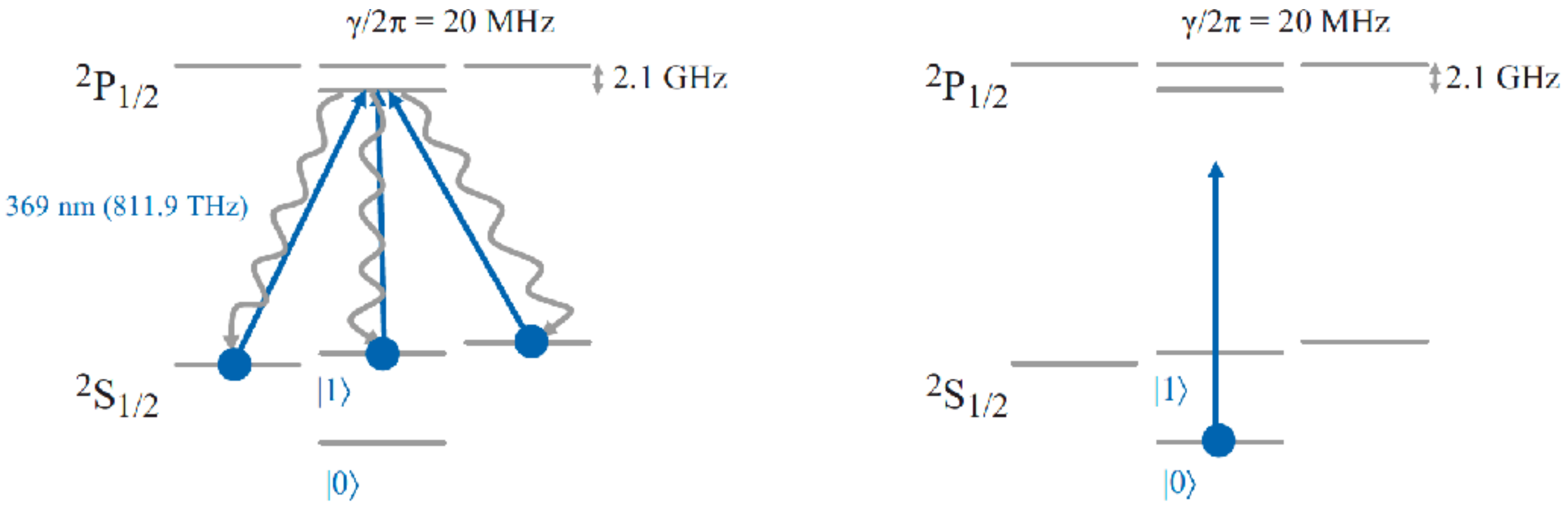


Figure 2.7: Measurement outcome is observed by state-dependent flourescence.

Single-Qubit Gate: Raman or Microwave Transition

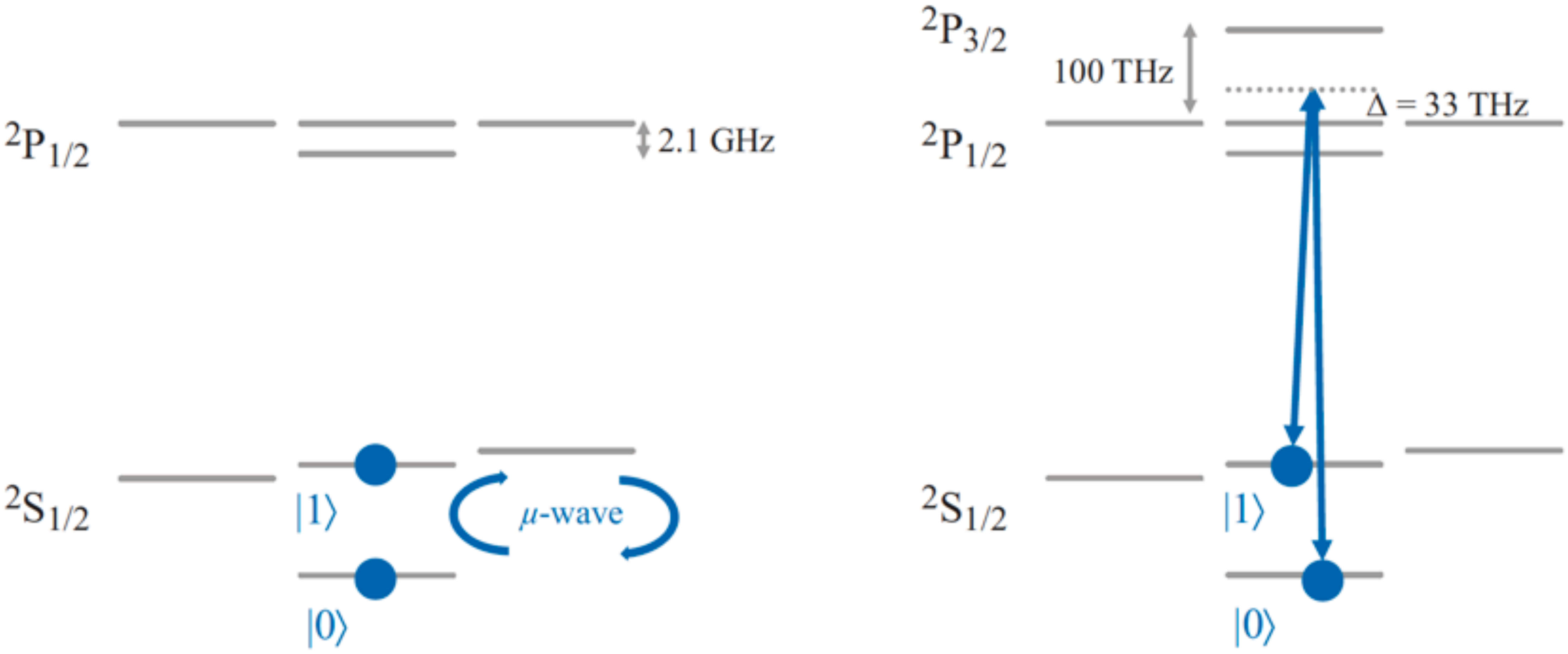


Figure 2.8: Single qubit gates via Raman transition or microwave transition.

Loading Qubits

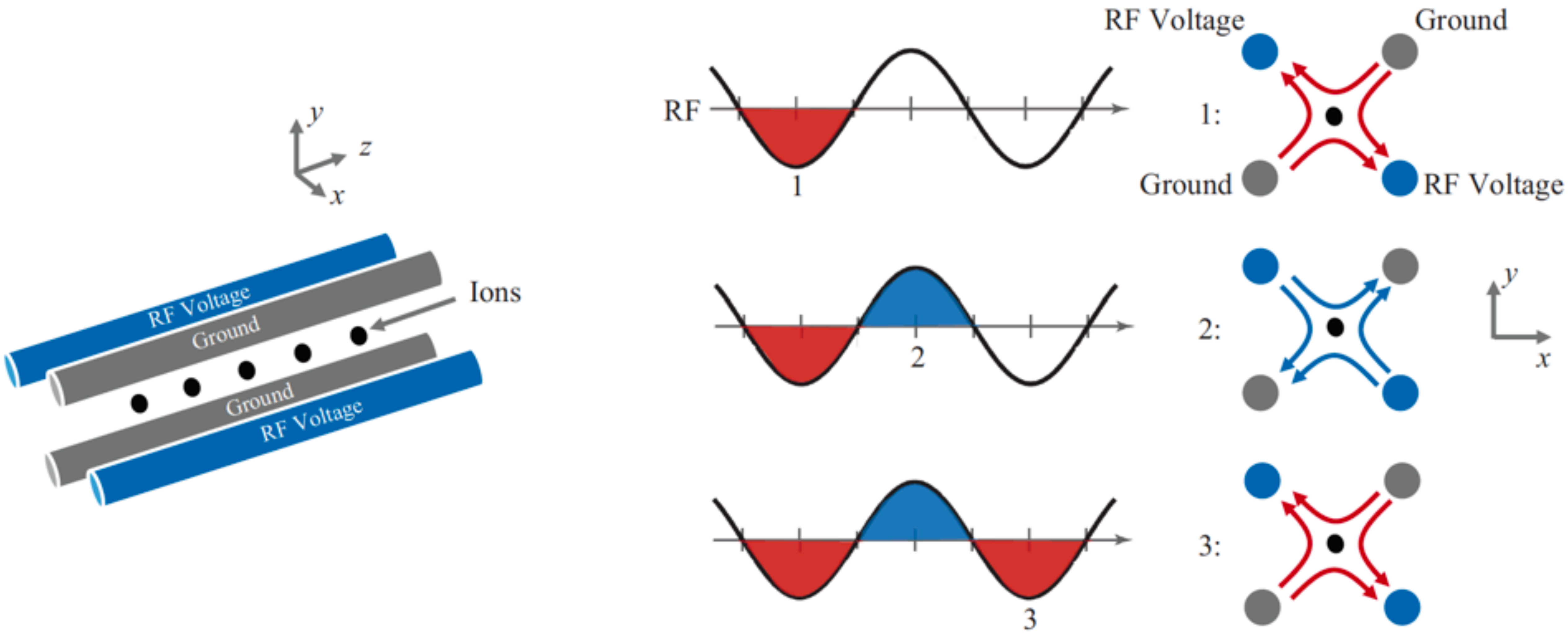


Figure 2.9: Schematics for a RF Paul trap.

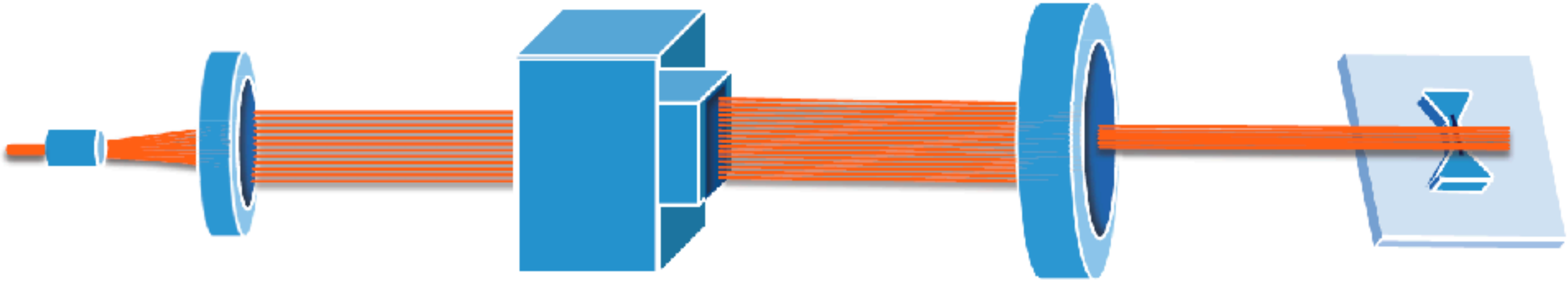


Figure 2.10: Schematics for a trapped ion QPU. After initialized from the optical source on the left, the laser are split into independently modulated beams, and then focused on the HOA trap on the right, providing individual controls over the array of ions in the trap.

Superconducting Qubits

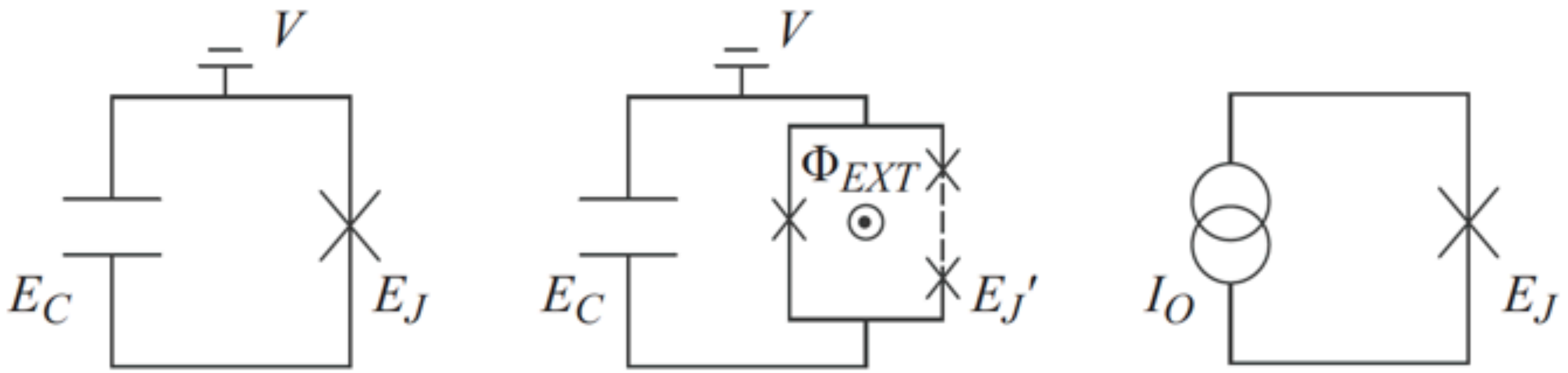
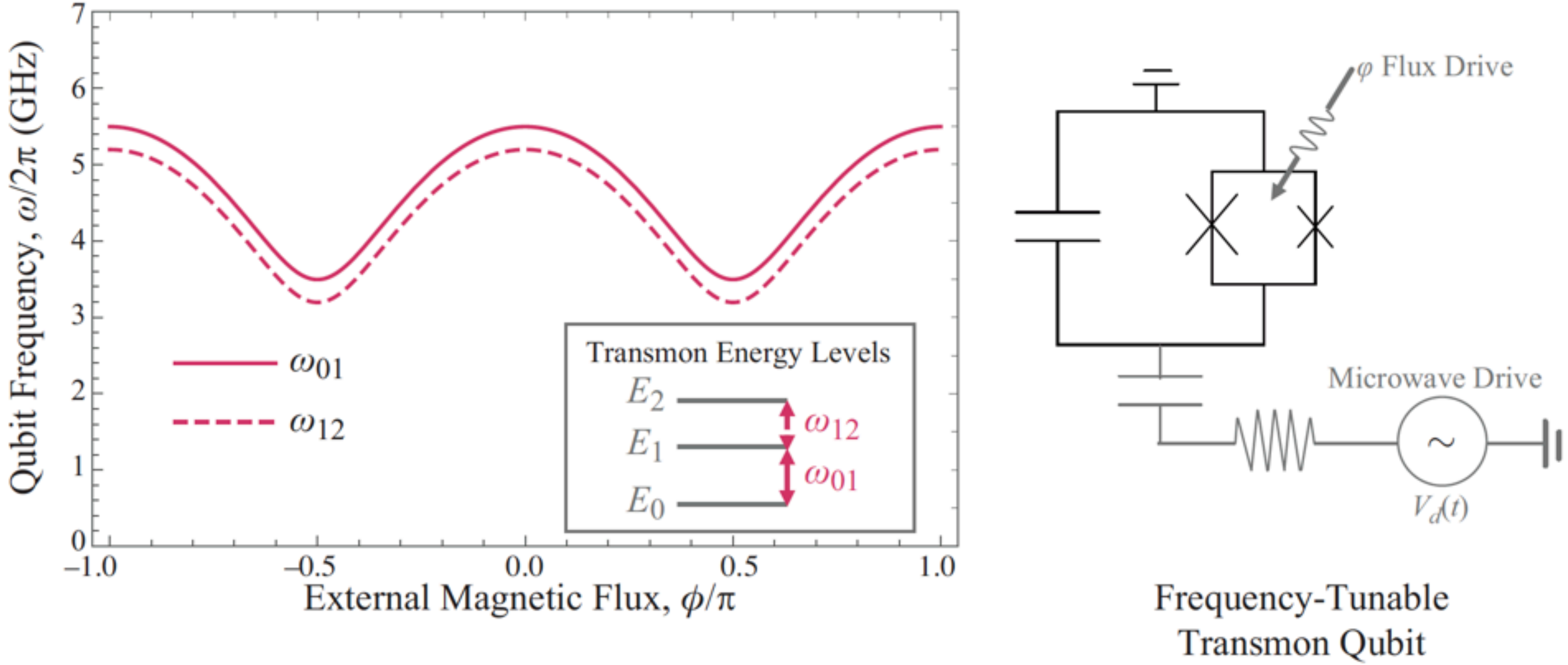


Figure 2.11: Types of superconducting qubits. **Left:** Circuit diagram for charge qubits (when $E_J \leq E_C$) and transmon qubit (when $E_J \gg E_C$), consisting of capacitor C and Josephson junction J. **Center:** Circuit diagram for a c-shunted flux qubit, where a junction is shunted with a number of junctions. **Right:** Circuit diagram for a phase qubit with current bias I_0.



Frequency-Tunable Transmon Qubit

Figure 2.12: **Left:** Qubit frequencies as a function of external magnetic flux. The first three levels of the transmon, ω_{01} and ω_{12}, are plotted. **Right:** Circuit diagram for a frequency-tunable (asymmetric) transmon qubit (highlighted in black), consisting of a capacitor and two asymmetric Josephson junctions. Highlighted in gray are two control lines: the external magnetic flux control φ and microwave voltage drive line $V_d(t)$ for each transmon qubit.

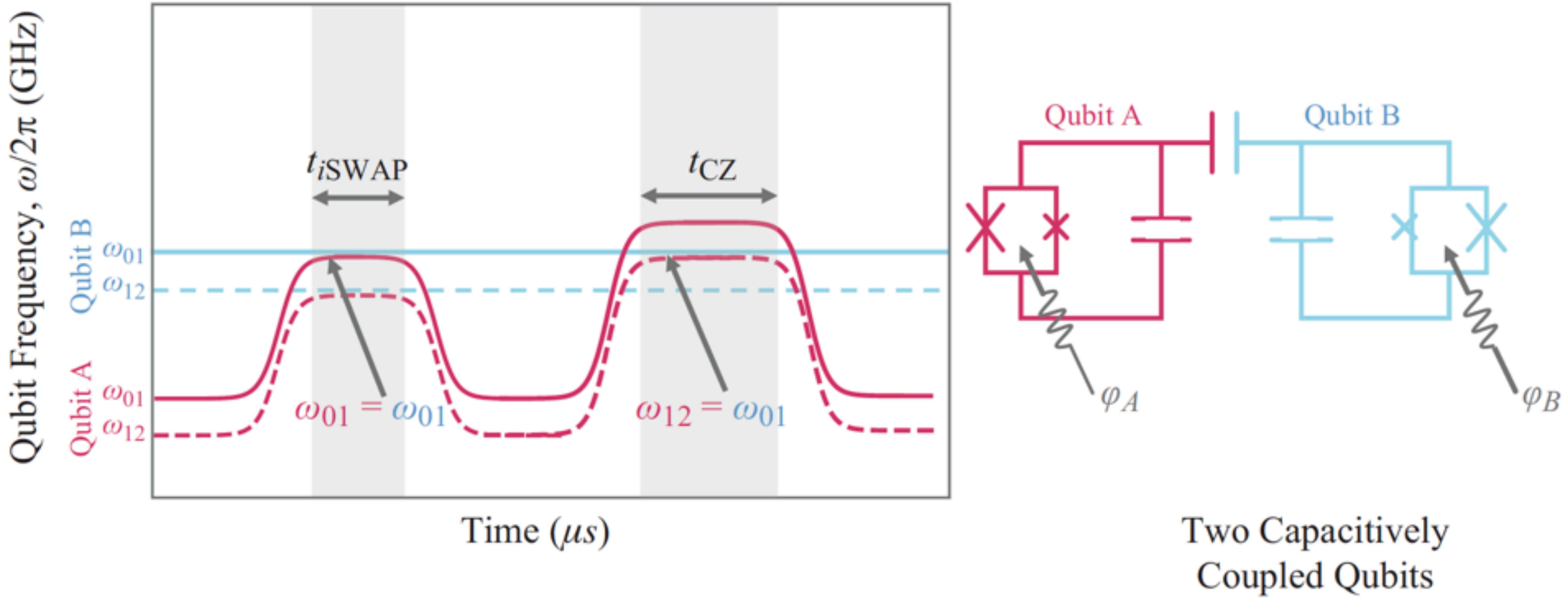


Figure 2.13: Two-qubit interactions for two capacitively coupled transmons. **Left:** Two-qubit gates are implemented with resonance of qubit frequencies. Shown here are how qubit frequencies are tuned for *i*`SWAP` gate and `CZ` gate. **Right:** Circuit diagram of two capacitively coupled transmon qubits.