Homework #2: Polymer Networks

Problem #1- Sixfold and Fourfold connectivity network model (25 pts)

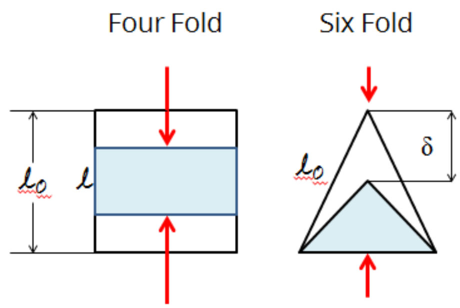
Use of micro W law (5 pts)

Identifying only two springs at work and representative area is 2x (5 pts)

Calculating length for the triangle with proper approximations (5 pts)

Recognizing strain is with respect to l vs. h in four vs six (5 pts)

Taking the derivative properly, calculating the correct modulus, and identifying the differences (or lack thereof) (5 pts)



<u>Four Fold</u>

$$\delta = l - l_0$$

$$W = \frac{\sum_{i=1}^{4} W_i^{sp}}{\sum_{i=1}^{4} A_i^{sp}} = \frac{W_{left} + W_{right}}{2A_{square}} = \frac{2[\frac{1}{2}k_{sp}(\delta^2)]}{2l_0^2} = \frac{k_{sp}}{2}\left(\frac{\delta}{l_0}\right)^2 = \frac{k_{sp}}{2}\varepsilon^2$$

$$K_a = \frac{\partial^2 W}{\partial \varepsilon^2} = k_{sp}$$

<u>Six Fold</u>

First, we need to work out the geometry here because the strain affects the height, and therefore the left and right spring but not the bottom.

$$h_0 = \frac{\sqrt{3}}{2} l_0$$

$$l = \sqrt{(h_0 - \delta)^2 + \left(\frac{l_0}{2}\right)^2} = \sqrt{h_0^2 - 2h_0\delta + \delta^2 + \frac{l_0^2}{4}} = \sqrt{\frac{3}{4}l_0^2 - 2h_0\delta + \delta^2 + \frac{l_0^2}{4}}$$

$$l = \sqrt{l_0^2 - \sqrt{3}l_0\delta + \delta^2} \text{ but in the small strain limit}, \delta^2 \approx 0 \text{ so } l = \sqrt{l_0^2 - \sqrt{3}l_0\delta}$$

$$\text{Now let's use the complete the square trick: } l = l_0\sqrt{1 - \sqrt{3}\frac{\delta}{l_0}} \approx l_0\sqrt{\left(1 - \frac{\sqrt{3}}{2}\frac{\delta}{l_0}\right)^2}$$

$$l = l_0\left(1 - \frac{\sqrt{3}}{2}\frac{\delta}{l_0}\right) = l_0 - \frac{\sqrt{3}}{2}\delta$$

It is important to note that this trick only works in the small strain limit.

$$W = \frac{\sum_{i=1}^{3} W_i^{sp}}{\sum_{i=1}^{3} A_i^{sp}} = \frac{W_{left} + W_{right} + W_{bottom}}{2A_{triangle}}$$

$$W_{left} = W_{right} \text{ and } W_{bottom} = 0$$

$$W = \frac{2 * \frac{1}{2}k_{sp}\left(l_0 - \frac{\sqrt{3}}{2}\delta - l_0\right)^2}{\frac{\sqrt{3}}{2}l_0^2} = \frac{\frac{3}{4}k_{sp}\delta^2}{\frac{\sqrt{3}}{2}l_0^2}$$

Now, what we need to do is rewrite this in terms of strain ε. What is strain in this case? If you drew the diagram above, it is easy to see

$$\varepsilon = \frac{\delta}{h_0} \quad OR \quad \delta = \varepsilon h_0$$

$$W = \frac{\frac{3}{4}k_{sp}(\varepsilon h_0)^2}{\frac{\sqrt{3}}{2}l_0^2} = \frac{\frac{3}{4}k_{sp}\left(\varepsilon\frac{\sqrt{3}}{2}l_0\right)^2}{\frac{\sqrt{3}}{2}l_0^2} = \frac{3\sqrt{3}k_{sp}}{8}\varepsilon^2$$

$$K_a = \frac{\partial^2 W}{\partial \varepsilon^2} = \frac{3\sqrt{3}k_{sp}}{4}$$

Finally, if you look at four fold and six fold, they are on the same order so in the case of uniaxial compression or normal stress, there is no big advantage. The six fold is slightly bigger.

Problem #2 – Micropipette Aspiration (30 points – 10 pts per question)

1. Explain the difference between the liquid drop vs. elastic solid model with respect to micropipette aspiration. Draw a graph to help your explanation.

In the liquid drop model, the cell is considered as a fluid filled balloon with an elastic cortical membrane whereas in the elastic solid model, the cell is considered as a homogeneous elastically deformable material. (i) Before the critical limit point L_{pro}/R_{pro} = 1, we can determine the surface tension n for all cell types. (ii) Beyond the critical limit point L_{pro}/R_{pro} = 1, we can tell whether the cell behaves more like a liquid drop, i.e., it rushes into the pipette without further resistance, or more like an elastic solid, i.e., the pressure can be further increased as the cell gets sucked in gradually. When cells that behave as solids are aspirated into a micropipette, obviously they do not flow into the pipette when the L_{pro}/R_{pro} > 1. Note, pro or pip are both acceptable. Graph should illustrate these phenomena by having L_{pro}/R_{pro} on the x axis with the critical point of 1 being a divergence between the two models.

2. Choose one of the papers this review highlights and describe:

Many cells were described in this paper, here is a summary chart.

Cell	Properties	Liquid/Solid
Neutrophil	T_c= 24-35 pN/um, K_A = 39 pN/um	Liquid
Chondrocyte	E= 650 pN/um^2	Solid
Red Blood Cell	μ = 6-9 pN/um, K_A = 500 nN/um, T_{mem} = 10 nN/um	Liquid
Endothelial	E= 400 pN/um^2	Solid

3. What are the limitations of micropipette aspiration?

The big limitation comes from the set up. The suction pressure is based on the height of a water reservoir. However, if there is evaporation of this water, then there is a shift in the zero value on the order of pN/um. A system like this also requires intense training and patience.

Problem #3 – Measuring 3D mechanical forces with droplet mechanics (30 points)
We derived many equations to understand membrane dynamics in class and problem 4. Now read *Quantifying cell-generated mechanical forces within living embryonic tissues* and see how it is used!

1. Explain in around 150 words, the gist of this paper. Pretend this is your project proposal and address all the points you would address in your presentation.
   Motivation: Understand the role of cell forces during tissue and organ formation in an embryo
   Approach: Using fluorescent oil microdropolets embedded into tissue and imaging with confocal microscopy to reconstruct the 3D stack. Then quantifying the mechanical stresses exerted locally based on membrane mechanics.
2. Why is this method more advantageous than other methodologies for measuring force? What are some of the disadvantages?
   The paper directly mentions literally all the different force application and sensing techniques to measure forces of cultured cells on a single or multicellular level in either 2D or 3D geometries. None of these give the capability of measuring mechanical forces of living tissue/organ in vivo. With any technique like this, imaging and reconstruction will always take time.
3. How were the measurements taken and calculated?
   They measure the curvature of the droplet in two principal directions in order to calculate a mean curvature. They defined anisotropic stress based on curvature and overall radius and derived that any changes in the stress would be linear to changes in the curvature. All of their main equations come from Laplace's Law in spherical coordinates.

Problem #4 – Derivation of Kirchhoff plate equation **(Possible total w/bonus: 30/20)**

$$p_z = K_B\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] \quad where\ K_B = \frac{Eh^3}{12[1-\nu^2]}$$

Recall the equation I stated in class as shown above. For a red blood cell this number was very small, on the order of 10^{-19} Nm. So where does this equation come from? Let's begin with the kinematic assumptions. What 3 assumptions have been making throughout this class? (2 pts)

1. Normal remain straight
2. Normal do not stretch
3. Normal remains normal

Based on these assumptions, we typically derive at three equations for deformation (in each axis). What are they (u, v, w)? (1 pts)

$$u^{tot}(x,y,z) = u(x,y) - z\frac{dw}{dx}$$
$$v^{tot}(x,y,z) = v(x,y) - z\frac{dw}{dy}$$
$$w^{tot}(x,y,z) = w(x,y)$$

Now, recall that there is a constant term and a term that varies linearly with thickness. Only the terms that vary linearly with thickness can produce a bending moment. Thus, you can drop the constant terms. This simplifies your deformation equations to (1 pts)

$$u^{tot}(x,y,z) = -z\frac{dw}{dx}$$
$$v^{tot}(x,y,z) = -z\frac{dw}{dy}$$

From this, calculate the strains from their continuum definition. (1 pts)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2 w}{dx^2}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{d^2 w}{dy^2}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -z\frac{d^2 w}{dxdy}$$

As before, we can use the definition of curvature to simplify these equations to

$\varepsilon_{xx} = \kappa_{xx} z \quad \varepsilon_{yy} = \kappa_{yy} z \quad \varepsilon_{xy} = \kappa_{xy} z$ (1 pts)

Alright then! Kinematic equations done, now let's move to the constitute model. We will assume generalized Hookean material behavior (linear and elastic). We will assume the material properties and thickness are constant (homogeneous). Because of our kinematic assumptions, what can we say about our z-direction strains and why? (1 pt)

z direction strains are all zero because normal do not stretch.

Write the simple form of the constitutive model that describes normal stress in x and y and shear stress. (In your lecture notes!). Now substitute the definitions of curvature to replace your expressions of strain. (Note I combined these two parts for solutions) (2 pts)

$$\sigma_{xx} = 2\mu\varepsilon_{xx} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) = 2\mu z\kappa_{xx} + \lambda z(\kappa_{xx} + \kappa_{yy})$$
$$\sigma_{yy} = 2\mu\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) = 2\mu z\kappa_{yy} + \lambda z(\kappa_{xx} + \kappa_{yy})$$
$$\tau_{xy} = 2\mu z\kappa_{xy}$$

Alright, kinematic assumptions done. Constitutive model in place. What's next? Equilibrium conditions! We need to consider the resultant forces first. In this particular situation, we are considering bending only, so we need to derive the stress moment resultants. For a plate, we need three moments {xx, yy, xy}. Write what the moment equations would be (keep it in integral form). (2 pts)

$$m_{xx}(x,y) = \int_{-h/2}^{h/2} \sigma_{xx}(x,y,z)zdz$$

$$m_{yy}(x,y) = \int_{-h/2}^{h/2} \sigma_{yy}(x,y,z)zdz$$

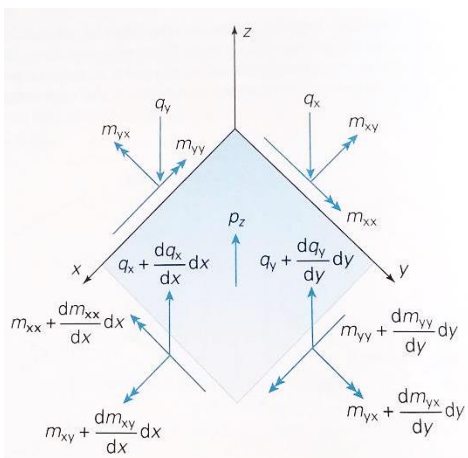
$$m_{xy}(x,y) = \int_{-h/2}^{h/2} \sigma_{xy}(x,y,z)zdz$$

With moment resultants defined, consider the drawing below. What's missing? We still need to define shear resultants in order to satisfy equilibrium conditions. Go ahead and write the integral form for q_x and q_y. (2 pts)

$$q_x(x,y) = \int_{-h/2}^{h/2} \sigma_{xz}(x,y,z)dz$$

$$q_y(x,y) = \int_{-h/2}^{h/2} \sigma_{yz}(x,y,z)dz$$

Now we can apply equilibrium equations. In class, we used force equilibrium to derive the governing equations and moment equilibrium about the z-axis to show a symmetry condition. For the plate bending, we need to add the moment about x and y axes. Note: force equilibrium in x and y does not apply since there are no forces in those directions. Let's start with a free body diagram. I labeled a few things, you label the rest! (2 pts – only look at z since there were some inconsistencies)



Using the FBD, let's write the force equilibrium condition in the z direction. (2 pts)

$$p_z dxdy + \left[q_y + \frac{dq_y}{dy}dy\right]dx - q_y dx + \left[q_x + \frac{dq_x}{dx}dx\right]dy - q_x dy$$

Cancel terms and divide by dxdy to get:

$$\frac{dq_x}{dx} + \frac{dq_y}{dy} + p_z = 0$$

Must show work and cannot simply state that equation for full credit. This is not the only way to do force equilibrium, you can start with general equation as well.

$$\frac{d\tau_{xz}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_{zz}}{dz} = 0$$

Now write the moment equilibrium equations in the x and y direction. The z axis moment condition illustrates that two quantities are equal to each other. Write this down as well. **This is bonus if someone figured it out. You do not need to use moment equilibrium, just force. (Extra Credit: +5 pts split across this)**

Start with general equilibrium equations for force. For example, x direction is

$$\frac{d\sigma_{xx}}{dx} + \frac{d\tau_{yx}}{dy} + \frac{d\tau_{zx}}{dz} = 0$$

You can apply an integral for zdz across all terms. Since the derivative for the first two are not z, those come out but the third stays in.

$$\frac{d}{dx}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z dz + \frac{d}{dy}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yx} z dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{zx}}{dz} z dz = 0$$

Recognize the first two integral terms as what you defined for moment equations. The third integral needs integration by parts.

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{zx}}{dz} z dz = z\tau_{zx} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx} dz = 0 - q_x$$

Final solution for x direction force, which would be y direction moment:

$$\frac{dm_{xx}}{dx} + \frac{dm_{yx}}{dy} - q_x = 0$$

Similarly, y direction force, which would be x direction moment, force equilibrium and the resulting final solution.

$$\frac{d\tau_{xy}}{dx} + \frac{d\sigma_{yy}}{dy} + \frac{d\tau_{zy}}{dz} = 0$$

$$\frac{dm_{yy}}{dy} + \frac{dm_{xy}}{dx} - q_y = 0$$

And finally, this last equilibrium should be obvious:

$$m_{yx} = m_{xy}$$

<u>Bonus allocation:</u> +1 for stating traditional force equilibrium. +1 for using this integral technique. +2 for showing the work to solve it completely. Simply stating does not count. +1 for the last equilibrium.

Wonderful! Now use these equations to form the classical equilibrium equation for the plate. **<u>(Bonus allocation: +2 pts must show all work below since the final solution was given)</u>**

In order to do this, remember that when you have coupled set of first order differential equations, it is possible to write them into a single higher order differential equation.

First take the moment equations and solve for shear components. For example:

$$\frac{dm_{xx}}{dx} + \frac{dm_{yx}}{dy} = q_x$$

Then take the derivative

$$\frac{d^2 m_{xx}}{dx^2} + \frac{d^2 m_{yx}}{dxdy} = \frac{dq_x}{dx}$$

Remember

$$m_{yx} = m_{xy}$$

Plug everything into your z-direction force equilibrium:

$$\frac{dq_x}{dx} + \frac{dq_y}{dy} + p_z = 0$$

Finally, you get

$$\frac{d^2 m_{xx}}{dx^2} + 2\frac{d^2 m_{xy}}{dxdy} + \frac{d^2 m_{yy}}{dy^2} + p_z = 0$$

Now let's combine the original definition of moments with the constitutive law.

If you substitute in the definitions and use the constants I gave you in class you should get the following. (1 pts if m_{xy} is correct, bonus of 3 pts if work shown to get all 3 with work).

The way I went about this was recognizing the original form of a planar stress

$$\begin{bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\left(\int_{h/2}^{h/2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix} z^2 dz\right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

$$\begin{bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

The final solution is this:

$$m_{xx}(x,y) = \int_{-h/2}^{h/2} \sigma_{xx} z dz = K_B(\kappa_{xx} + \nu\kappa_{yy})$$

$$m_{yy}(x,y) = \int_{-h/2}^{h/2} \sigma_{yy} z dz = K_B(\kappa_{yy} + \nu\kappa_{xx})$$

$$m_{xy}(x,y) = \int_{-h/2}^{h/2} \sigma_{xy} z dz = K_B\left(\frac{1-\nu^2}{1+\nu}\right)\kappa_{xy} = K_B(1-\nu)\kappa_{xy}$$

$$K_B = \frac{Eh^3}{12(1-\nu^2)}$$

Now use these expressions for the moments in terms of curvature into the equilibrium equation you derived and voila! You have now derived the fourth order differential equation for plate bending. You can write it in terms of curvature or in terms of displacement, like below. (2 pts for all work shown)

$$p_z = K_B\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] \; where \; K_B = \frac{Eh^3}{12[1-\nu^2]}$$

$$p_z = -\left(\frac{d^2 m_{xx}}{dx^2} + 2\frac{d^2 m_{xy}}{dxdy} + \frac{d^2 m_{yy}}{dy^2}\right)$$

$$p_z = -\left(\frac{d^2}{dx^2}\left(K_B(\kappa_{xx} + \nu\kappa_{yy})\right) + 2\frac{d^2}{dxdy}\left(K_B\left(\frac{1-\nu^2}{1+\nu}\right)\kappa_{xy}\right) + \frac{d^2}{dy^2}\left(K_B(\kappa_{yy} + \nu\kappa_{xx})\right)\right)$$

$$p_z = -K_B\left(\frac{d^2}{dx^2}\left((\kappa_{xx} + \nu\kappa_{yy})\right) + 2\frac{d^2}{dxdy}(1-\nu)\kappa_{xy} + \frac{d^2}{dy^2}\left((\kappa_{yy} + \nu\kappa_{xx})\right)\right)$$

$$\kappa_{xx} = -\frac{d^2 w}{dx^2} \; and \; \kappa_{yy} = -\frac{d^2 w}{dy^2} \; and \; \kappa_{xy} = -\frac{d^2 w}{dxdy}$$

$$p_z = K_B\left(\frac{d^2}{dx^2}\left(\left(\frac{d^2 w}{dx^2} + \nu\frac{d^2 w}{dy^2}\right)\right) + 2(1-\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{d^2}{dy^2}\left(\left(\frac{d^2 w}{dy^2} + \nu\frac{d^2 w}{dx^2}\right)\right)\right)$$

If you work on the algebra, you will see that the following term disappears and your bending remains:

$$2\nu\frac{\partial^4 w}{\partial x^2 \partial y^2}$$

Leaving the equation with curvature is also acceptable but make sure it is a second derivate instead of fourth. It takes the same form essentially.