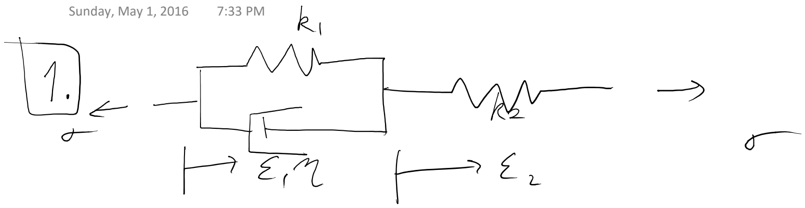
HW5 solution

Sunday, May 1, 2016 7:33 PM

1.



Voigt : $\sigma_1 = k_1 \varepsilon_1 + \eta \dot{\varepsilon}_1$

$\varepsilon_t = \varepsilon_1 + \varepsilon_2 , \quad \sigma_t = \sigma_1 = \sigma_2$

$\sigma_2 = \sigma_t = k_2 \varepsilon_2$

$$\begin{cases} \varepsilon_1 = \varepsilon_t - \varepsilon_2 = \varepsilon_t - \dfrac{\sigma_t}{k_2} \\ \dot{\varepsilon}_1 = \dot{\varepsilon}_t - \dfrac{\dot{\sigma}}{k_2} \end{cases}$$

plug into the Voigt Model

$$\sigma = k_1 \left[\varepsilon - \frac{\sigma}{k_2} \right] + \eta \left[\dot{\varepsilon} - \frac{\dot{\sigma}}{k_2} \right]$$

$$\sigma + \frac{k_1}{k_2}\sigma + \frac{\eta}{k_2}\dot{\sigma} = k_1 \varepsilon + \eta \dot{\varepsilon}$$

$$\boxed{\left(1 + \frac{k_1}{k_2}\right)\sigma + \frac{\eta}{k_2}\dot{\sigma} = k_1 \varepsilon + \eta \dot{\varepsilon}}$$

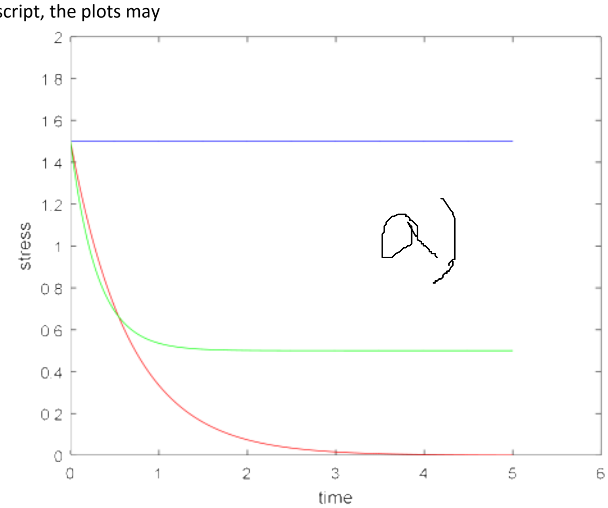
or

$$\boxed{\sigma + \frac{\eta}{(k_1 + k_2)}\dot{\sigma} = \frac{k_1 k_2}{(k_1 + k_2)}\varepsilon + \frac{\eta k_2}{(k_1 + k_2)}\dot{\varepsilon}}$$

2.

I didn't ask the code for the plots. But if there is no coding script, the plots may need to be same as the solution.

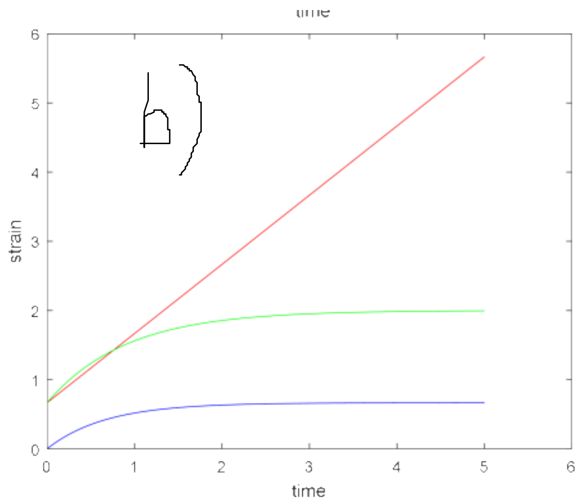
```
%Maxwell and Voigt
k=1.5;
eta=1;
%Kelvin
k1=1;
k2=0.5;
eta1=0.3;
e0=1;
s0=1;

t=[0:0.01:5];

%stress relaxation
f1=k*exp(-k/eta*(t));
f2=k+eta*dirac(t);
f3=k2*(1+k1/k2*exp(-k1/eta1*(t)));
%creep
g1=1/k+1/eta*t;
g2=1/k-1/k*exp(-k/eta*t);
g3=1/k2-k1/k2/(k1+k2)*exp(-k1*k2/eta1/(k1+k2)*t);

figure(1);plot(t,f1,'r',t,f2,'b',t,f3,'g');
```



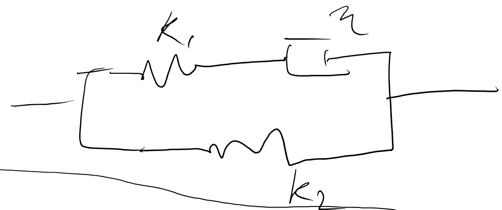
```
g1=1/k+1/eta*t;
g2=1/k-1/k*exp(-k/eta*t);
g3=1/k2-k1/k2/(k1+k2)*exp(-k1*k2/eta1/(k1+k2)*t);

figure(1);plot(t,f1,'r',t,f2,'b',t,f3,'g');
axis([0 6 0 2]);
xlabel('time');ylabel('stress');
figure(2);plot(t,g1,'r',t,g2,'b',t,g3,'g');
axis([0 6 0 6]);
xlabel('time');ylabel('strain');
```

a)

b)



$\boxed{3}$ corrected version



$$\boxed{\sigma + \frac{\eta}{k_1}\dot{\sigma} = k_2 \varepsilon + \eta\left[1 + \frac{k_2}{k_1}\right]\dot{\varepsilon}}$$

lecture note.

a) $\alpha = \dfrac{\eta}{k_1}$, $\beta = \eta\left[1 + \dfrac{k_2}{k_1}\right]$

b)

$$E^* = E' + iE''$$

$$= \frac{(\overset{k_2}{\cancel{k_1}} + i\omega\beta)}{(1 + i\omega\alpha)}$$

$$= \underbrace{\frac{\overset{k_2}{\cancel{k_1}} + \omega^2\alpha\beta}{(1+\omega^2\alpha^2)}}_{E'} + i\underbrace{\left[\frac{\omega\beta - \overset{k_2}{\cancel{k_1}}\,\omega\alpha}{(1+\omega^2\alpha^2)}\right]}_{E''}$$

c) low freq: $\omega \to 0$

$\cancel{\omega^2\alpha\beta}^{\,0}$

c) low freq: ω

$$E' = \frac{k_2}{1+\cancel{\omega^2\alpha^2}} + \cancel{\frac{\omega^2\alpha\beta}{1+\omega^2\alpha^2}} = \boxed{k_2}$$

$$E'' = \frac{\cancel{\omega\beta - k_2\omega\alpha}}{\cancel{1+\omega^2\alpha^2}} = \boxed{0}$$

high freq: $\omega \to \infty$

$$E' = \cancel{\frac{k_2}{1+\omega^2\alpha^2}} + \frac{\cancel{\omega^2}\alpha\beta}{\cancel{1+\omega^2}\alpha^2} = \frac{\alpha\beta}{\alpha^2} = \frac{\beta}{\alpha}$$

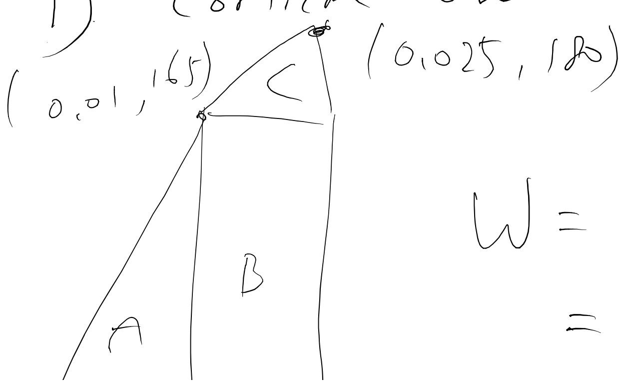
$$= \frac{\eta\left[1+\frac{k_2}{k_1}\right]}{\frac{\eta}{k_1}} = \boxed{k_1+k_2}$$

$$E'' = \frac{\cancel{\omega\beta - k_2\omega\alpha}}{\cancel{1+\omega^2\alpha^2}} = \boxed{0}$$

$$\boxed{A = k_2\ ,\ B = k_1+k_2}$$

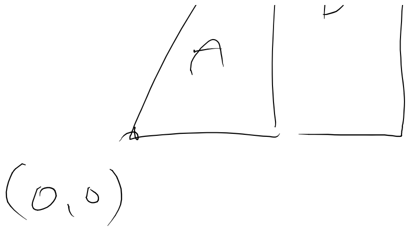
4

1) *Cortical Bone



$$W = A + B + C$$

$$= 0.825 + 2.475 + 0.1125$$



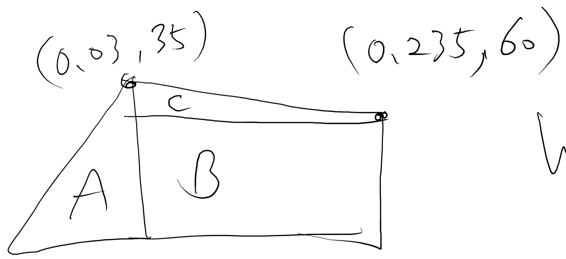
$$= 0.825 + 2.475 + 0.1125$$

$$= 3.41 \text{ MPa}$$

$$\text{Note: MPa} = \frac{MN}{m^2} = \frac{MN\cdot m}{m^3} = \frac{\cancel{M}J}{\cancel{10^6}\cdot cm^3} = \boxed{J/cm^3}$$

$$\boxed{W_{cortical} = 3.41 \text{ J/cm}^3}$$

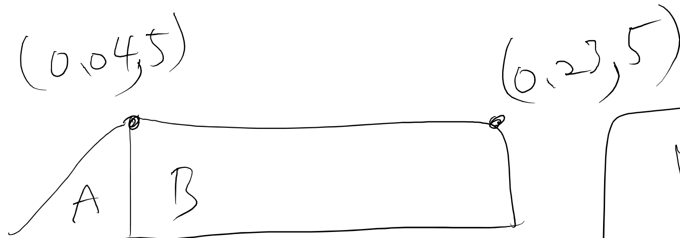
* Health TB



$$W_{H,TB} = A + B + C$$

$$= \boxed{10.26 \text{ J/cm}^3}$$

* Osteopenic TB



$$\boxed{\begin{aligned} W_{OS,TB} &= A + B \\ &= 1.05 \text{ J/cm}^3 \end{aligned}}$$

2) The strain energy density of healthy trabecular bone is four times higher than cortical bone. So, trabecular bone is more capable to absorb the energy under compression, but it does not necessarily mean that it can resist higher stress.

3) Cortical bone is tightly packed, so it is strong and brittle.
Trabecular bone has a porous and spongy structure,
therefore, it is more deformable under compression

4) Obviously, the lower density trabecular bone can absorb about 1/10 energy under compression with a large strain, which also means that the strength of bone is very lower than the healthy trabecular bone.
The loss of trabeculae in osteoporosis is more damaging for the overall structural integrity and strength of a trabecular bone structure than thinning of the trabeculae because lamellar new bone can only form on existing surfaces