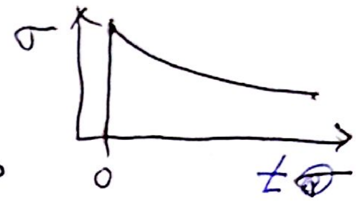
Session 2, Lecture 3, Mechanical behavior of soft tissue - cont

- Viscoelastic responses of those models

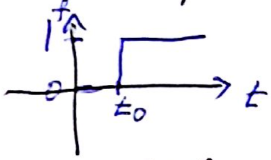


(1) Stress relaxation

$u(t-t_0) \rightarrow$ Heaviside step function

$\varepsilon(t) = \varepsilon_0 \, u(t)$

$H(t)$ or $\phi(t)$.



$$\begin{cases} \sigma(0) = 0 \\ \varepsilon(0) = 0 \end{cases} \text{ I.C.}$$

$$X(s) = \mathcal{L}\{\varepsilon(t)\}, \quad F(s) = \mathcal{L}\{\sigma(t)\}$$

$$\mathcal{L}\left\{\frac{d\varepsilon}{dt}\right\} = sX(s) - \cancel{\varepsilon(0)} \rightarrow 0.$$

$$\mathcal{L}\left\{\frac{d\sigma}{dt}\right\} = sF(s) - \cancel{\sigma(0)} \rightarrow 0$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

A. Maxwell. $\dot{\varepsilon} = \frac{\dot{\sigma}}{k} + \frac{\sigma}{\eta}$ $\qquad \varepsilon(t) = \varepsilon_0 u(t)$

$$\mathcal{L}: \quad \varepsilon_0 = \frac{1}{k} sF(s) + \frac{F(s)}{\eta}, \quad F(s) = \boxed{\frac{k\varepsilon_0}{\left(s + \frac{k}{\eta}\right)}}$$

$$\mathcal{L}^{-1}: \boxed{\sigma(t) = k\varepsilon_0 e^{-k/\eta \, t}}$$

SV : $\cancel{\dot{\varepsilon}} = \frac{\dot{\sigma}}{k} + \frac{\sigma}{\eta}$ $\qquad \dot{\varepsilon} = 0 \Big|_{t>0.}$

0

$$\dot{\sigma} = -\frac{k}{\eta}\sigma = \frac{d\sigma}{dt}, \quad \int \frac{1}{\sigma} d\sigma = \int -\frac{k}{\eta} dt$$

$$\ln \sigma = -\frac{k}{\eta} t + C_0$$

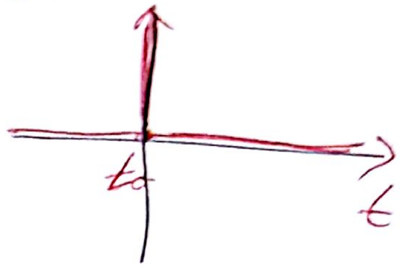
$$\Rightarrow \boxed{\sigma = C_0 e^{-\frac{k t}{\eta}}}.$$

B. Voigt Model. $\sigma = \underline{k\varepsilon} + \eta\dot{\varepsilon}$, $\varepsilon(t) = \varepsilon_0 \cdot u(t)$

$\mathcal{L}: F(s) = \frac{k\varepsilon_0}{s} + \eta\varepsilon_0$

$\mathcal{L}^{-1}: \sigma(t) = \varepsilon_0 [k \boxed{u(t)} + \eta \boxed{\delta(t)}]$

Dirac delta function



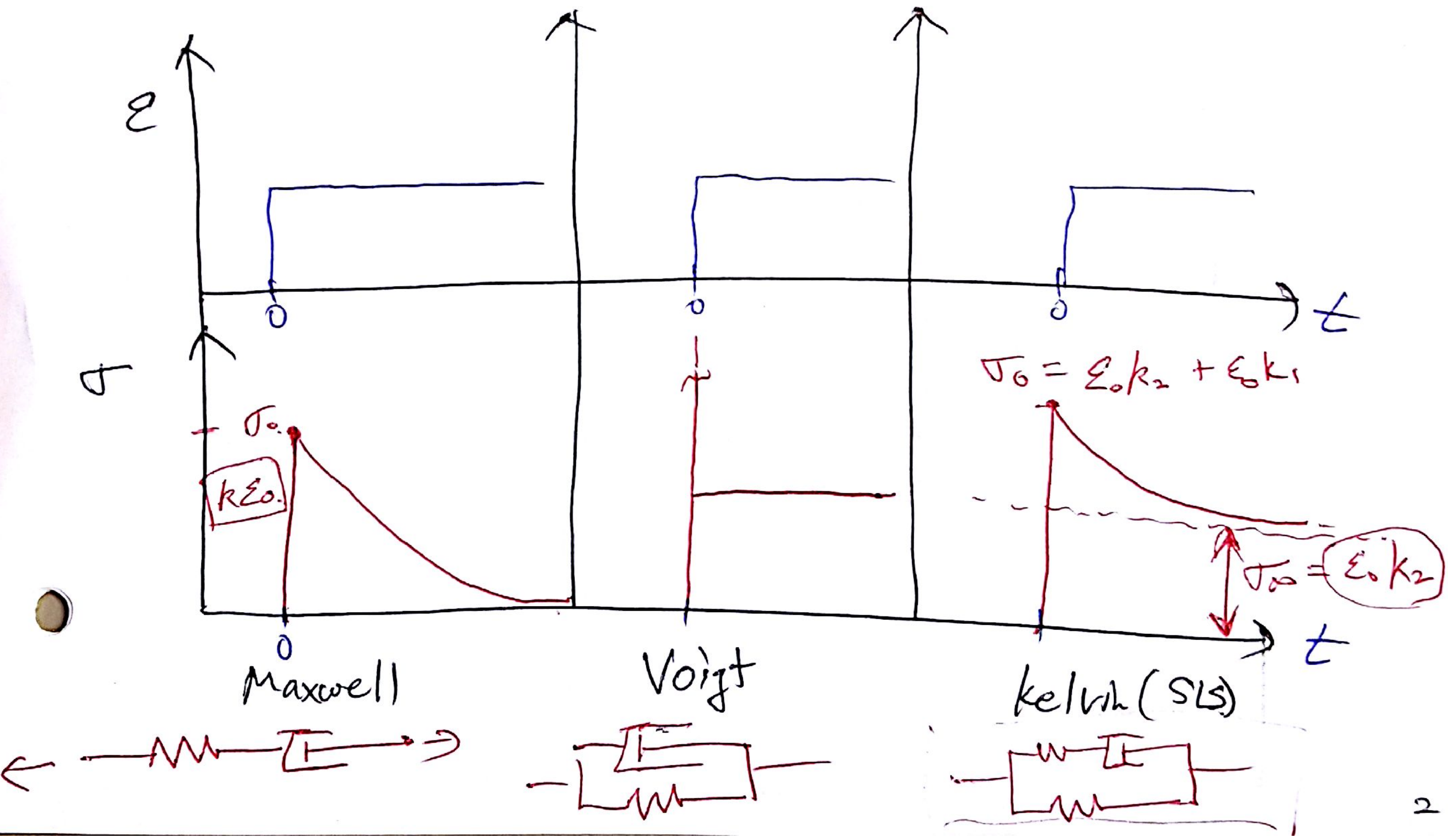
C. Kelvin (SLS)

$$\boxed{\sigma = \varepsilon_0 k_2 \left[1 + \frac{k_1}{k_2} e^{-\frac{k_1}{\eta}t}\right]}$$

D. Voigt (SLS)

$$\sigma = \frac{k_1 k_2}{k_1 + k_2} \varepsilon_0 \left[1 - e^{-\left(\frac{k_1+k_2}{\eta}\right)t}\right] + \sigma_0 e^{-\left(\frac{k_1+k_2}{\eta}\right)t}$$

A, B, C



(2) Creep → constant stress applied. ∴ $\sigma_0 = \text{const}$

$$\sigma(t) = \sigma_0 \, u(t), \quad \begin{cases} \sigma(0) = 0 \\ \varepsilon(0) = 0 \end{cases}$$

A. Maxwell. $\dot{\varepsilon} = \dfrac{\dot{\sigma}}{k} + \dfrac{\sigma}{\eta}$

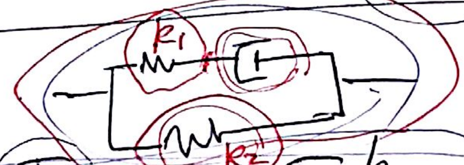
$\mathcal{L}$: $sX(s) = \dfrac{\sigma_0}{k} + \dfrac{\sigma_0}{\eta s}$, $\quad X(s) = \dfrac{\sigma_0}{ks} + \dfrac{\sigma_0}{\eta s^2}$

$\mathcal{L}^{-1}$: $\varepsilon(t) = \sigma_0 \left[\dfrac{u(t)}{k} + \dfrac{1}{\eta} t \right]$

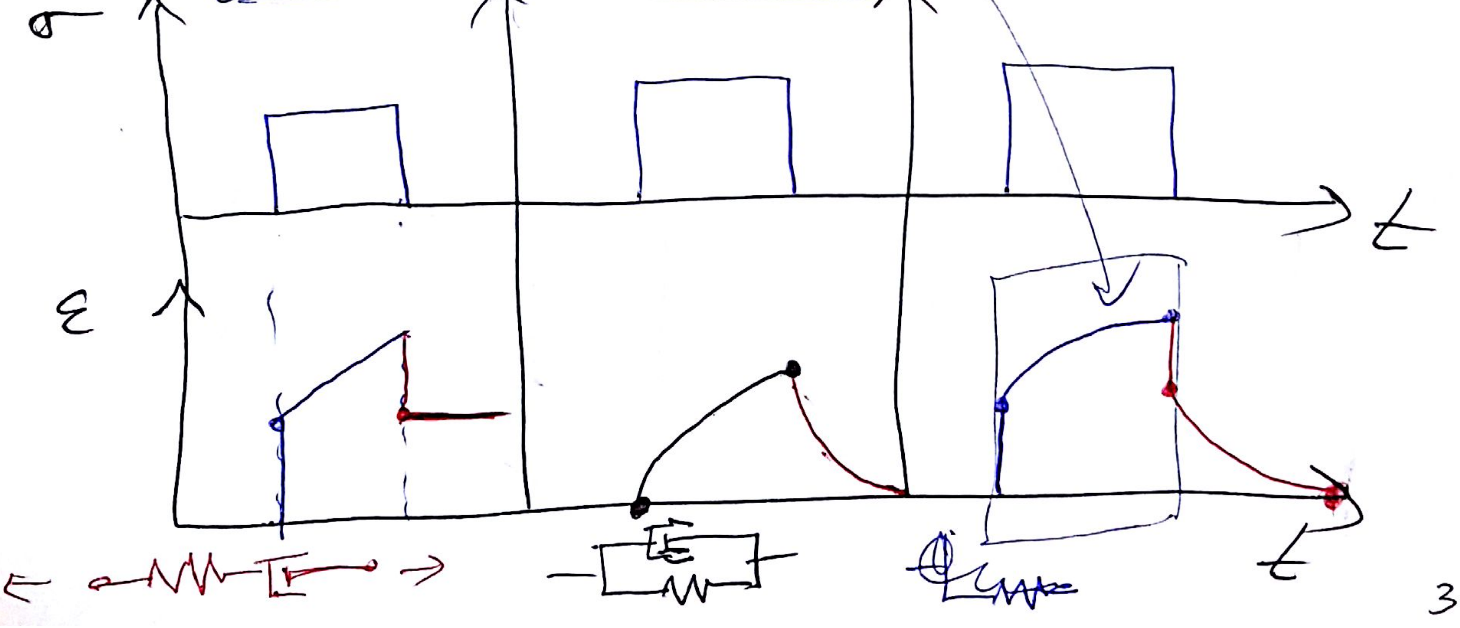
B. Voigt $\sigma = k\varepsilon + \eta\dot{\varepsilon}$

$$\varepsilon(t) = \frac{\sigma_0}{\eta}\left[1 - e^{-\frac{k}{\eta}t}\right]$$

C. Kelvin.



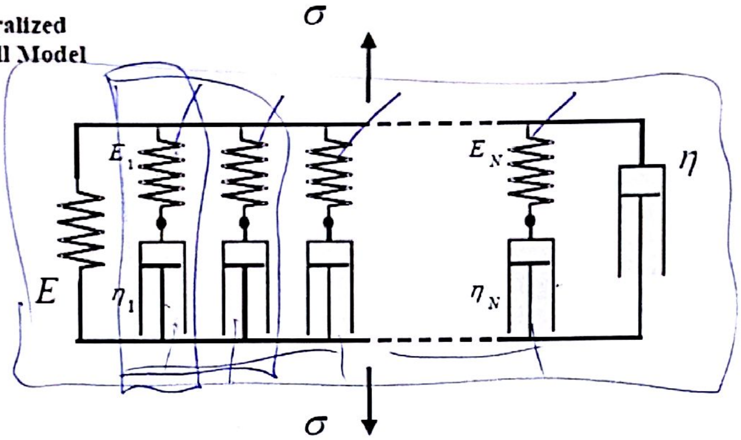
$$\varepsilon(t) = \frac{\sigma_0}{k_2} - \frac{\sigma_0 k_1}{k_2(k_1+k_2)} e^{-\frac{k_1 k_2}{\eta(k_1+k_2)} t}$$



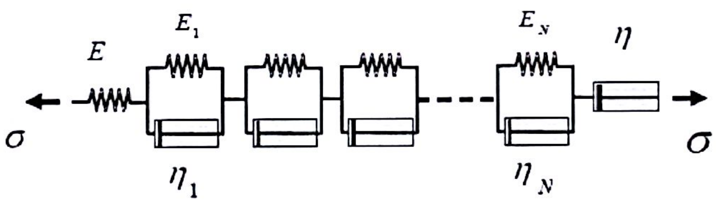
- Generalized Models

  : More complex models can be constructed by using more elements. The more elements one has, the more accurate a model will be in describing the response of real materials. There are more parameters that need to be evaluated by experiment.

Generalized Maxwell Model



Generalized Kelvin Chain



The general form of a linear viscoelastic constitutive equation

$$\boxed{p_o\sigma + p_1\dot{\sigma}} + p_2\ddot{\sigma} + p_3\dddot{\sigma} + p_4\sigma^{(IV)} + \cdots = \boxed{q_o\varepsilon + q_1\dot{\varepsilon}} + q_2\ddot{\varepsilon} + q_3\dddot{\varepsilon} + q_4\varepsilon^{(IV)} + \cdots$$

- Non-linear Models

  : Better matched to real data, but difficult to solve

$$\sigma = E\varepsilon^n$$

$$\boxed{\sigma = A\varepsilon'\dot{\varepsilon}}$$

$$\boxed{\sigma = Ae^{\left(-\varepsilon/\varepsilon_0\right)}\dot{\varepsilon}}$$

→ time deriv

→ strain ~~tit~~ history

Dynamic mechanical analysis

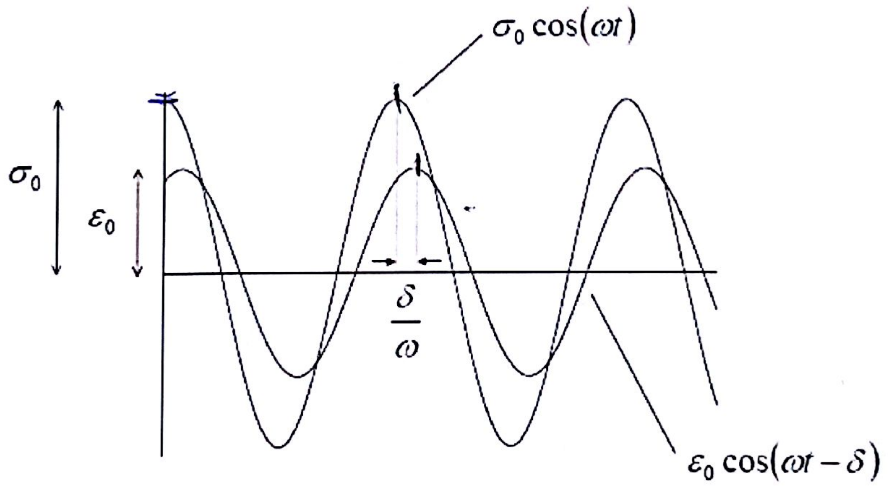
- Creep and stress relaxation tests are convenient at long times (minutes to days), but less accurate at shorter times (seconds and less)
- Ex) duration of the impact of a steel ball on a viscoelastic block ~ 10^-5 sec
- Dynamic experiments usually can provide from 10^-8 ~ 10^3 sec
- Applying sinusoidal stress, measuring the phase lag, δ, between the stress and strain after those reach to a steady state at the same angular frequency

Dynamic load $\sigma(t) = \sigma_0 \cos(\omega t)$.

stress amplitude : σ_0 angular frequency : ω.

Resulting strain $\varepsilon(t) = \varepsilon_0 \cos(\omega t - \delta)$

Phase lag, loss angle δ



- Complex Modulus

: As it is steady-state, we can regard of the strain as the input and the stress as the output.

$$= \mathrm{Re}\left[\varepsilon_0 e^{i\omega t}\right]$$

input $\quad \varepsilon(t) = \boxed{\varepsilon_0 \cos(\omega t)}$

$$\sigma(t) = \sigma_0 \cos(\omega t + \delta)$$

$$= \sigma_0\left[\cos\delta \boxed{\cos\omega t} - \sin\delta \boxed{\sin\omega t}\right]$$

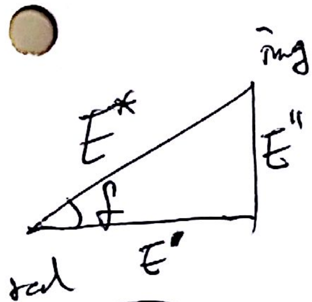
And similar to the analysis of reactive electrical circuits and other harmonic systems, we use a complex formulation (Euler's formula) and now define two different dynamic moduli, that both being ratios of stress to strain.

$$\boxed{e^{i\omega t} = \cos\omega t + i\sin\omega t}$$

$$\varepsilon(t) = \varepsilon_0 e^{i\omega t}$$

$$\sigma(t) = \sigma_0 e^{i(\omega t+\delta)}$$

$$= \sigma_0 e^{i\omega t} \cdot e^{i\delta}$$



$$E = \frac{\sigma}{\varepsilon}$$

$$E^* = E' + iE'' \quad (\text{or} \quad E_1 + iE_2)$$

$$E^* = \frac{\sigma(t)}{\varepsilon(t)} = \frac{\sigma_0}{\varepsilon_0} e^{i\delta} = \frac{\sigma_0}{\varepsilon_0}(\cos\delta + i\sin\delta)$$

$$E' = \frac{\sigma_0}{\varepsilon_0}\cos\delta \quad : \text{Storage M}$$

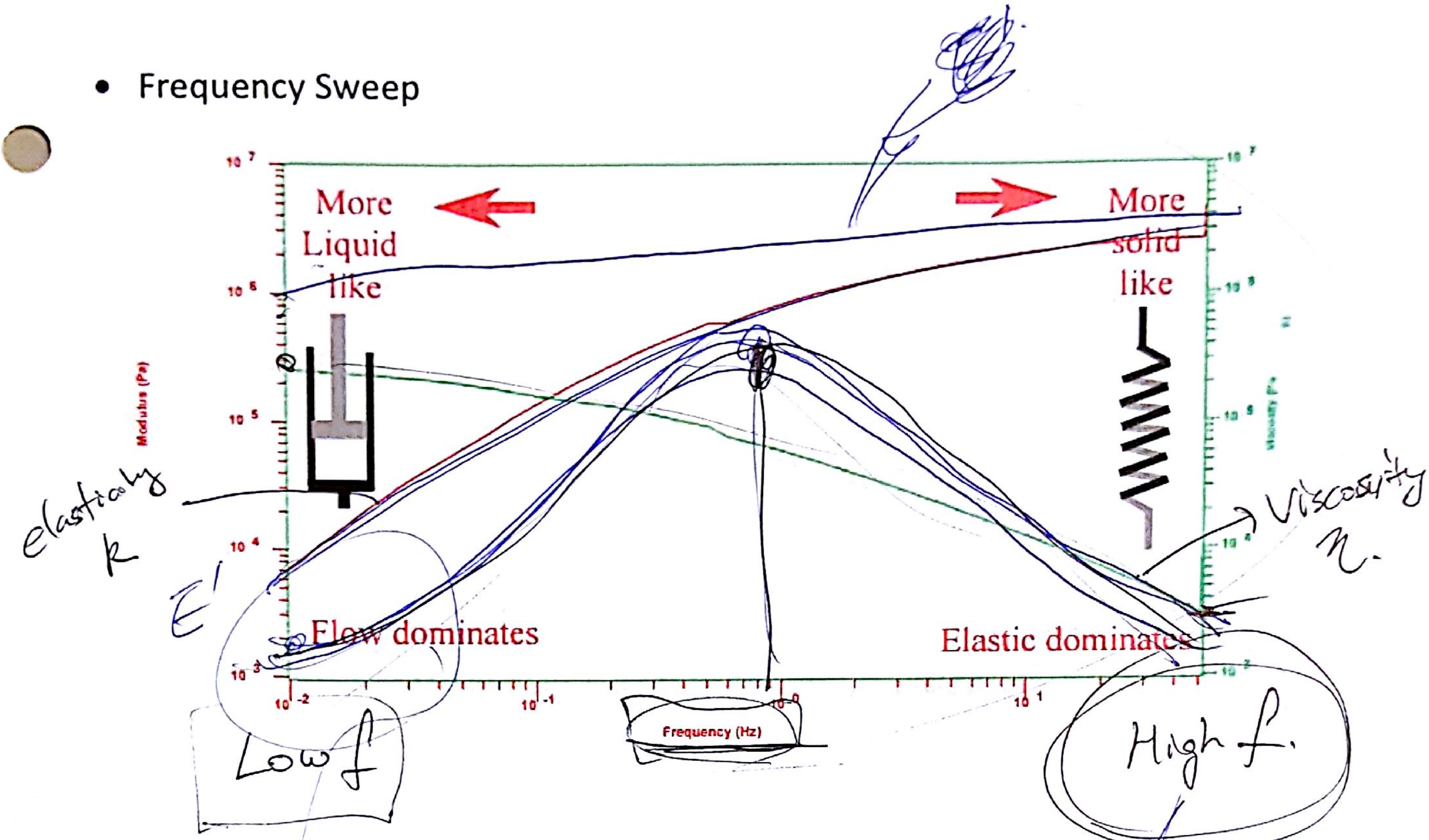
$$E'' = \frac{\sigma_0}{\varepsilon_0}\sin\delta \quad : \text{loss M.}$$

$$\boxed{G^* = \frac{E^*}{2(1+\nu)}}$$

elastic
$$\left[\begin{array}{l} E' = \frac{\sigma_0}{\varepsilon_0} = E. \\ E'' = 0 \end{array}\right.$$

viscous fluid
$$\left[\begin{array}{l} E' = 0 \\ E'' = \frac{\sigma_0}{\varepsilon_0} \end{array}\right.$$

- Frequency Sweep



E' and E''