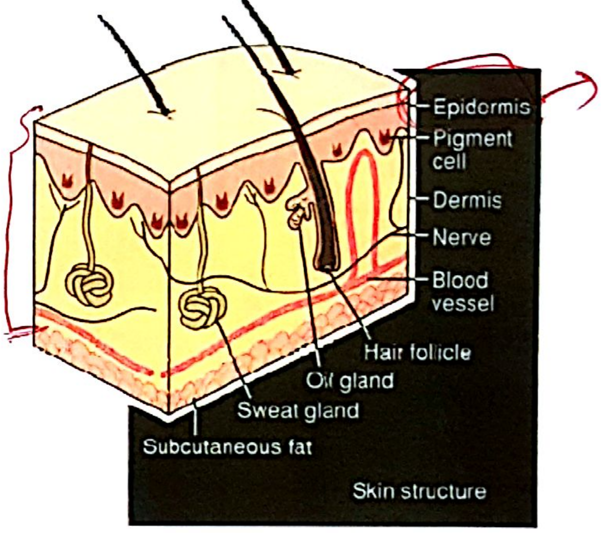
Session 2, Lecture 4, Skin, Strain Energy Function 5/3.

Skin

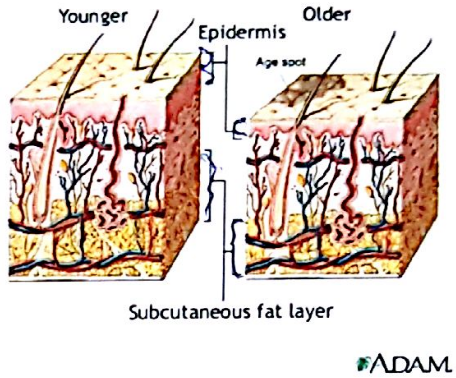
- Characteristics
  - 15% of body weight
  - 1.5 ~ 2.0 m^2 (~ 20 sqft),
  - In "one in^2": 650 sweat glands, 20 blood vessels, 60,000 melano-cytes. >1000 nerve endings.
  - 0.07 ~ 0.12 mm over the body, $\boxed{0.8-1.4\,mm}$ on palm and soles
- Functions
  - Insulation
  - Protection of underlying organs
  - Production of vitamins D and B
  - Sensation
  - Excretion
  - Heat regulation
  - Storage of lipids and H2O
  - Absorption of N2, O2, and CO2



- Epidermis: Outer layer; flatten and keratinized cells and fibers; cells die and flake off (30,000~40,000 cells per minutes or 10 pounds per year)
  - Stratum corneum: the outer layer of the epidermis composed mainly of keratinocytes
- Dermis: Inner layer; capillaries, sensory nerve endings, lymphatic vessels, sweat glands, and hair follicles; Fibroblasts are constantly dividing
- Subcutaneous Fat: hypodermis; 50% of body fat, padding and insulation

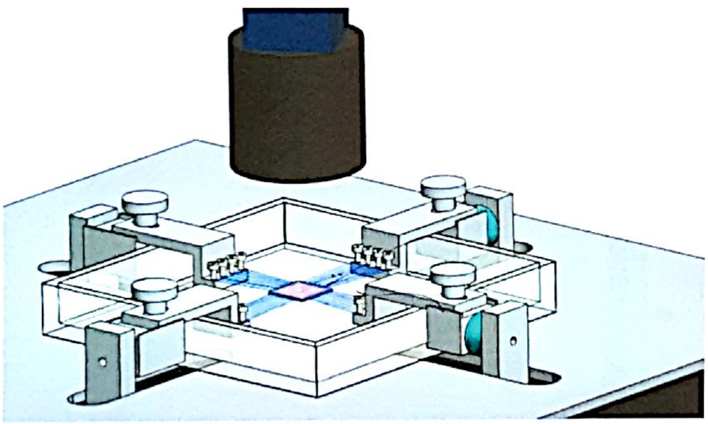
- Aging
  : thinner and more easily damaged

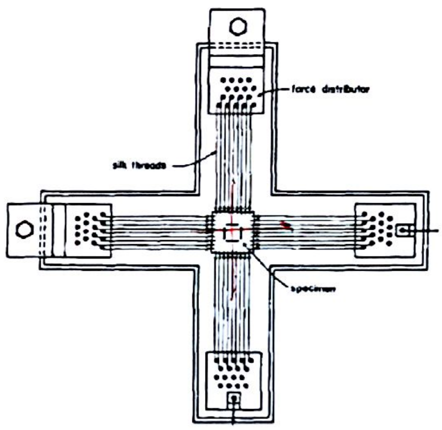
→ elasticity loss
less blood flow.
lower gland activity



Mechanical Properties of Skin

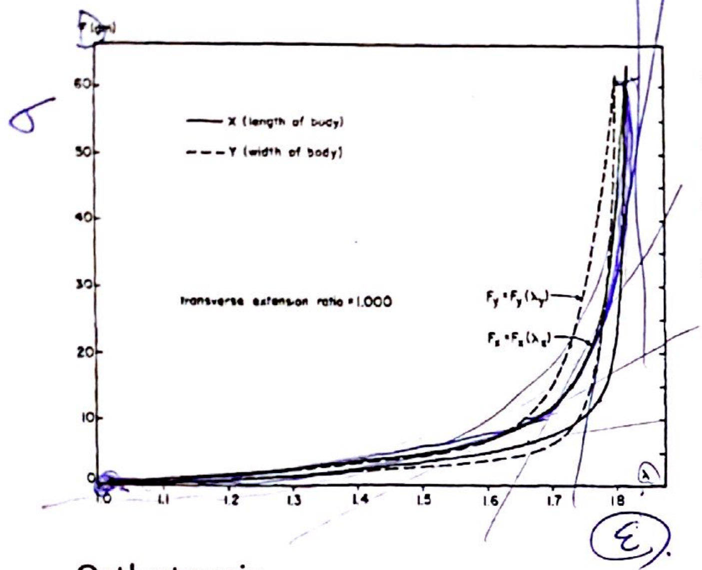
Biaxial stress and Strain



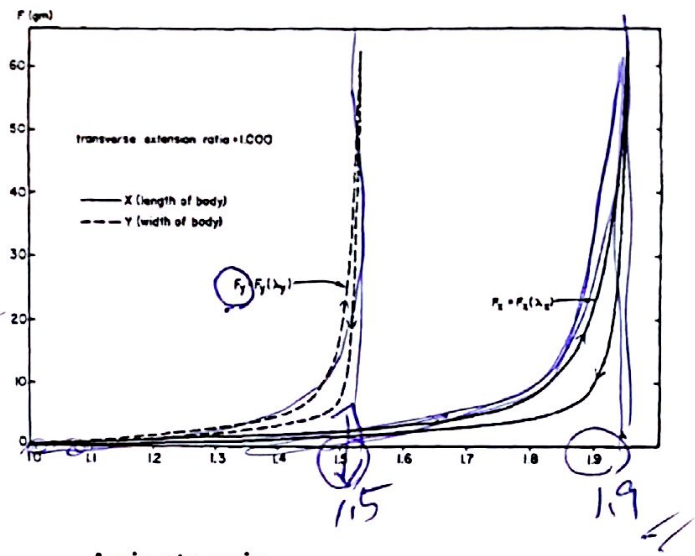


* Uniaxial loading experiments cannot provide the full relationship between all stress and strain component.
* Sufficient to yield a 2D constitutive equation for a membranous material under plane stress
* Still not sufficient to derive a 3D stress-strain relationship.

Lanir and Fung (1974)

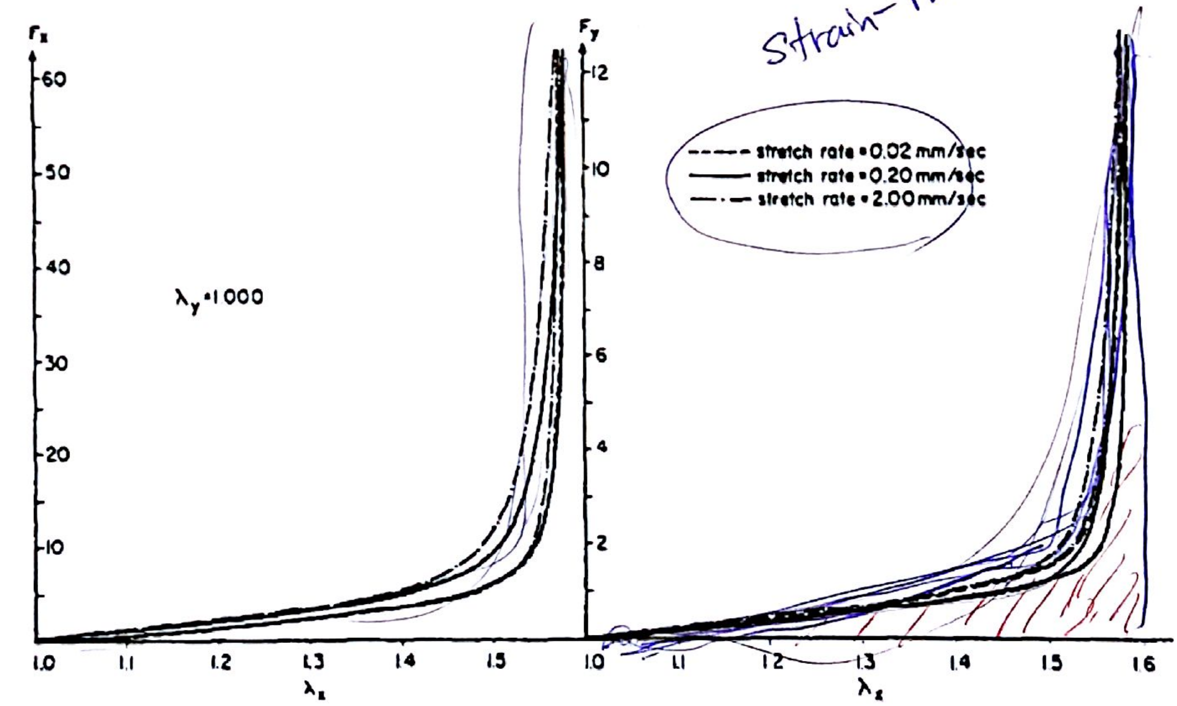


Orthotropic

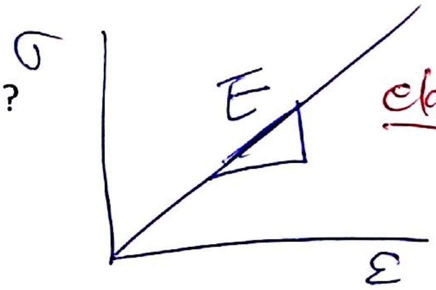


Anisotropic

strain-rate



How do you define the stress-strain relationship??



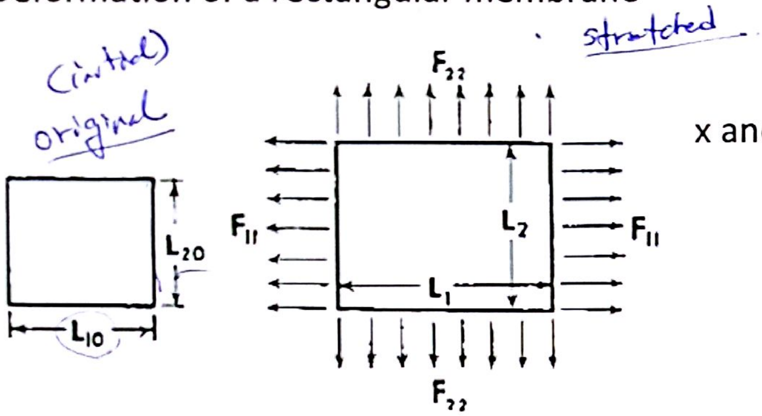
elastic

Assumptions for lumped parameter

* small strain. not for skin.
* Quasistatic, dynamic loading/unloading

Hyperelasticity

Deformation of a rectangular membrane



x and y are the principal directions

Finite strains

Principal Stretches $\lambda_1 = \frac{L_1}{L_{10}}$, $\lambda_2 = \frac{L_2}{L_{20}}$

Green (St. Venant) strain (Lagrangian reference system)

$$E_1 = \frac{L_1^2 - L_{10}^2}{2L_{10}^2} = \frac{1}{2}(\lambda_1^2 - 1) \quad E_2 = \frac{L_2^2 - L_{20}^2}{2L_{20}^2} = \frac{1}{2}(\lambda_2^2 - 1)$$

Almansi (Hamel) strain (Eulerian reference system)

$$e_1 = \frac{L_1^2 - L_{10}^2}{2L_1^2} = \frac{1}{2}\left(1 - \frac{1}{\lambda_1^2}\right) \quad e_2 = \frac{L_2^2 - L_{20}^2}{2L_2^2} = \frac{1}{2}\left(1 - \frac{1}{\lambda_2^2}\right)$$

$$\boxed{\begin{aligned} \varepsilon_1 &= \frac{L_1 - L_{10}}{L_{10}} = \lambda_1 - 1 \\ \varepsilon_2 &= \frac{L_2 - L_{20}}{L_{20}} = \lambda_2 - 1 \end{aligned} \Bigg\} \text{ infinitesimal strain}}$$

Conjugate stresses

h and h_0 : thicknesses of deformed and original tissue

ρ and ρ_0 : densities of the deformed and original (equal if incompressible)

Cauchy stress (Eulerian reference system)

$$S_{11} = \frac{F_{11}}{L_2 h} \quad , \quad S_{22} = \frac{F_{22}}{L_1 h}$$

SED.

1st Piola Kirchhoff stress $T_{11} = \frac{F_{11}}{L_{20} h_0} = \frac{\rho_0}{\rho} \frac{1}{\lambda_1} s_{11}$, $T_{22} = \frac{F_{22}}{L_{10} h_0} = \frac{\rho_0}{\rho} \frac{1}{\lambda_2} s_{22}$

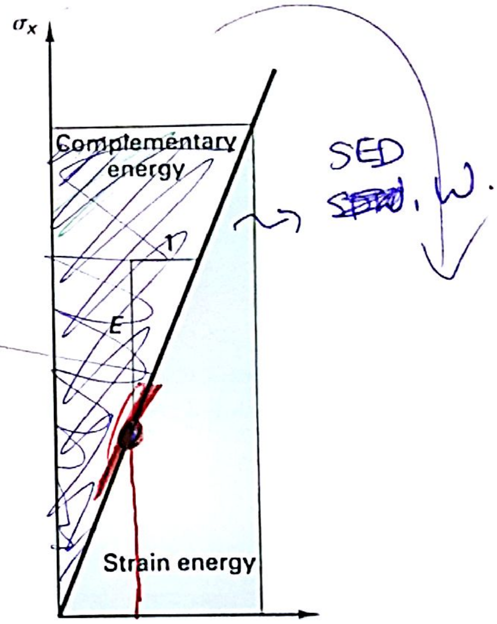
2nd Piola Kirchhoff stress

$S_{11} = \frac{1}{\lambda_1} T_{11} = \frac{\rho_0}{\rho} \frac{1}{\lambda_1^2} s_{11}$ $S_{22} = \frac{1}{\lambda_2} T_{22} = \frac{\rho_0}{\rho} \frac{1}{\lambda_2^2} s_{22}$

Small   Small

Strain Energy Density, W " SED.

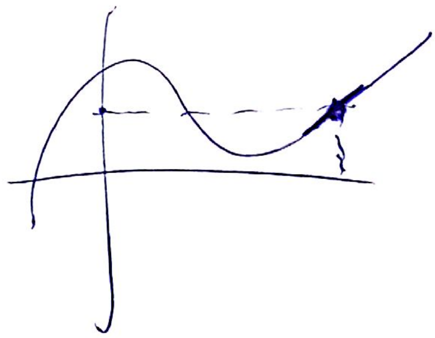
- Strain energy per unit of initial volume
- Area between the stress strain curve and the strain axis from energy conjugates

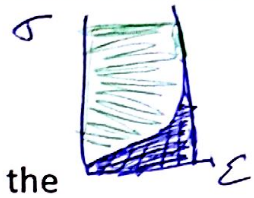


$$\sigma_{xx} = E \varepsilon_{xx}$$

$$W = \frac{1}{2} \sigma_{xx} \varepsilon_{xx} = \boxed{\frac{1}{2} E \varepsilon_{xx}^2}$$

$$\Rightarrow \frac{dW}{d\varepsilon_{xx}} = E \varepsilon_{xx} = \sigma_{xx}$$





Alternatively, area between the stress strain curve and the stress axis is the complementary strain energy density $\boxed{W^*}$

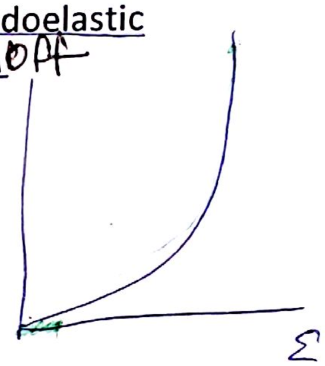
$$W^* = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{1}{2E}\sigma_{xx}^2 \rightarrow \frac{dW^*}{d\sigma_{xx}} = \frac{1}{E}\sigma_{xx} = \varepsilon_{xx}$$

Hyperelasticity:

$$S_{ij} = \frac{fW}{fE_{ij}}$$

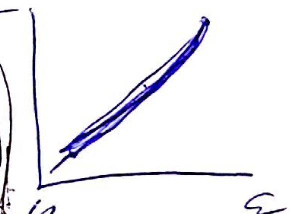
- Often with Green strains E_{ij} and 2nd Piola-Kirchhoff stresses S_{ij}
- Stress-strain relationship derives from a strain energy density function
- Rubber, non-linearly elastic, isotropic and independent of strain rate
- Biological tissues are modeled via hyperelasticity assuming pseudoelastic behavior

Saint-Venant Kirchhoff
Neo-Hookean "
Moony-Rivlin "
Ogden Model



Fung Model (for large stretches)

* Nonlinear, anisotropic material
* exponential strain constitutive equation for preconditioned soft tissue



Fung's postulate $\boxed{W = \frac{1}{2}[q + c(e^{Q} - 1)]}$

$$\boxed{W.}$$

$$S_{ij} = \frac{\partial W}{\partial E_{ij}}, \quad E_{ij} = \frac{\partial W^*}{\partial S_{ij}}$$

strain.

$$q = a_{ijkl} E_{ij} E_{kl}$$
$$Q = b_{ijkl} E_{ij} E_{kl}$$

) tensor expression.

$$\rightarrow \boxed{W^{(2)} = \cancel{\tfrac{1}{2}[Q]} + c(e^{Q}-1)]}$$

$$\cancel{Q = c_1 E_1^2 + c_2 E_2^2}$$

$$\boxed{W^{(2)} = c(e^{Q}-1)}$$ if we don't need for great accuracy at small stress levels

$$= c\exp\Big[c_1 E_{11}^2 + \cdots + c_9(E_{13}^2 + E_{31}^2) + \cdots$$

$$\cdots + \cancel{c E_{11}^4} + c\,\cancel{E_{31}^2 E_{13}^2}\cdots$$

Original pseudo strain energy function for skin by Fung

$$W^{(2)} = \frac{1}{2}(\alpha_1 E_1^2 + \alpha_2 E_2^2 + \cancel{\alpha_3 E_{12}^2} + \cancel{\alpha_3 E_{21}^2} + 2\alpha_4 E_1 E_2)$$ if no shear strain

$$+ \frac{1}{2} c \cdot \exp(a_1 E_1^2 + a_2 E_2^2 + \cancel{a_3 E_{12}^2} + \cancel{a_3 E_{21}^2} + 2a_4 E_1 E_2$$

$$+ \gamma_1 E_1^3 + \gamma_2 E_2^3 + \gamma_4 E_1^2 E_2 + \gamma_5 E_1 E_2^2)$$

Higherorder terms

Third-order terms are not important for practical purposes, then,

$$W = f(\alpha, \underset{\sim}{e}) + c\cdot\exp[F(a, \underset{\sim}{e})]$$

$\tilde{E}$, strain $\qquad \tilde{E}$, strain

Biphasic behavior.

at a lower stress level

at a higher stress level

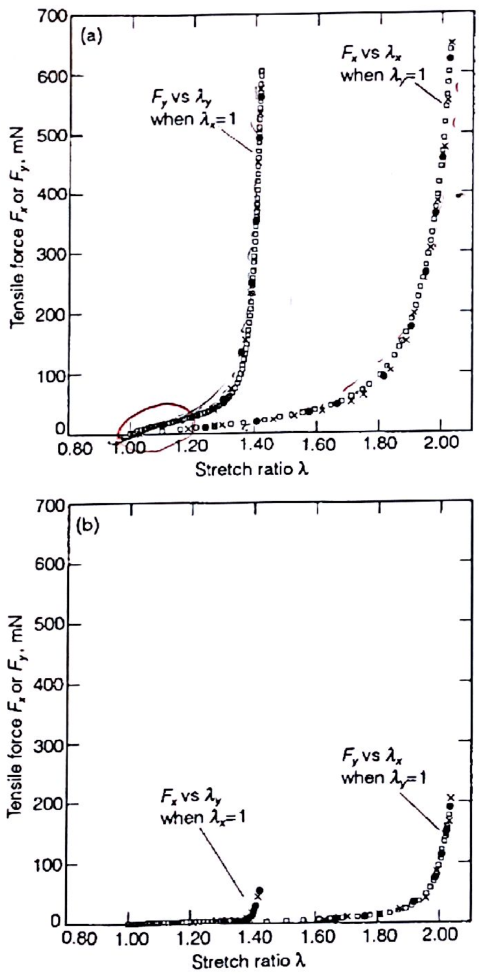
Rabbit abdominal skin experiment and "hyperelastic" curve fit

$$W = \rho_0 W^{(2)} = f(\alpha, E) + c \exp[F(a, E)], \tag{4}$$

where

$$f(\alpha, E) = \alpha_1 E_{11}^2 + \alpha_2 E_{22}^2 + \alpha_3 E_{12}^2 + \alpha_3 E_{21}^2 + 2\alpha_4 E_{11} E_{22}, \tag{5}$$

$$F(a, E) = a_1 E_{11}^2 + a_2 E_{22}^2 + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22}. \tag{6}$$



□ : exp. data

● : $\gamma = 0$ → higher ~~od~~ order term

× : $\gamma \neq 0$.

Figure 7.12:2 Comparison between experimental data and mathematical expression. The tensile forces F_x, F_y are given in milli Newton. Lagrange stress T_x is equal to F_x divided by A_x, the cross-sectional area perpendicular to the x axis. Squares: experimental data. Circles: from Eqs. (1) and (2) with $\alpha_1 = \alpha_2$, all γ's = 0. $a_1 = 3.79$, $a_2 = 12.7$, $a_4 = 0.587$, $c = 0.779$ N/m^2, $\alpha_1 = \alpha_2 = 1{,}020$ N/m^2, and $\alpha_4 = 254$ N/m^2. Crosses: From Eq. (2) with $\alpha_1 = \alpha_2$, $\gamma_1 = \gamma_2 = 0$, $\gamma_4 = \gamma_5 \neq 0$. $a_1 = 3.79$, $a_2 = 18.4$, $a_4 = 0.587$, $c = 0.779$ N/m^2, $\alpha_1 = \alpha_2 = 1{,}020$ N/m^2, and $a_4 = 254$ N/m^2. $\gamma_4 = \gamma_5 = 15.6$. From Tong and Fung (1976), by permission. Experimental data are from Lanir and Fung (1974b).

5/~~4~~3 Group work

List the names of group members:

Ignoring shear terms, Fung's strain energy density function for rabbit abdominal skin (assumed incompressible and biaxially loaded in principal directions) can be represented given by

$$W = \alpha_1 E_{11}^2 + \alpha_2 E_{22}^2 + 2\alpha_4 E_{11} E_{22} + c\exp\left[a_1 E_{11}^2 + a_2 E_{22}^2 + 2a_4 E_{11} E_{22}\right]$$

Q: Derive an expression for Cauchy stress s_{11} as a function of stretches λ_1 and λ_2.

$\lambda_1 = \frac{L_1}{L_{10}}$, $\lambda_2 = \frac{L_2}{L_{20}}$, $E_1 = \frac{1}{2}(\lambda_1^2 - 1)$, $E_2 = \frac{1}{2}(\lambda_2^2 - 1)$

Hint: First obtain an expression for 2nd Piola Kirchhoff stress as a function of Green strains. Then re-parameterize this Kirchhoff stress as a function of stretches λ1 and λ2 and finally convert Kirchhoff stress to Cauchy stress.

$$\boxed{S_{11} = \frac{\rho_0}{\rho} \frac{1}{\lambda_1^2} s_{11}}, \quad \boxed{S_{22} = \frac{\rho_0}{\rho} \frac{1}{\lambda_2^2} s_{22}}$$

$\frac{\rho_0}{\rho} = 1$ $\qquad$ 1

Question 1: Ignoring shear terms, Fung's strain energy density function for rabbit abdominal skin (assumed incompressible and biaxially loaded in principal directions) can be represented given by

$$W = \alpha_1 E_{11}^2 + \alpha_2 E_{22}^2 + 2\alpha_4 E_{11} E_{22} + c\exp\left[a_1 E_{11}^2 + a_2 E_{22}^2 + 2a_4 E_{11} E_{22}\right]$$

a) Derive an expression for <u>Cauchy</u> stress s_{11} as a function of stretches λ_1 and λ_2.

Hint: First obtain an expression for 2nd Piola Kirchhoff stress as a function of Green strains. Then re-parameterize the Kirchhoff stress and a function of stretches λ_1 and λ_2 and finally convert Kirchhoff stress to Cauchy stress.

$$S_{11} = \frac{\partial W}{\partial E_{11}} = 2\alpha_1 E_{11} + 2\alpha_4 E_{22} + c(2a_1 E_{11} + 2a_4 E_{22}) exp(a_1 E_{11}{}^2 + a_2 E_{22}{}^2 + 2a_4 E_{11} E_{22})$$

$$E = \frac{1}{2}(\lambda^2 - 1) \qquad \lambda \, , \lambda_2$$

$$S_{11} = \alpha_1(\lambda_1{}^2 - 1) + \alpha_4(\lambda_2{}^2 - 1) + c\left(a_1(\lambda_1{}^2 - 1) + a_4(\lambda_2{}^2 - 1)\right) exp\left(\frac{a_1}{4}(\lambda_1{}^2 - 1)^2 + \frac{a_2}{4}(\lambda_2{}^2 - 1)^2 + \frac{a_4}{2}(\lambda_1{}^2 - 1)(\lambda_2{}^2 - 1)\right)$$

$$\boxed{s = \lambda^2 S}$$

$$s_{11} = \lambda_1{}^2[\alpha_1(\lambda_1{}^2 - 1) + \alpha_4(\lambda_2{}^2 - 1)] + c\lambda_1{}^2\left(a_1(\lambda_1{}^2 - 1) + a_4(\lambda_2{}^2 - 1)\right) exp\left(\frac{a_1}{4}(\lambda_1{}^2 - 1)^2 + \frac{a_2}{4}(\lambda_2{}^2 - 1)^2 + \frac{a_4}{2}(\lambda_1{}^2 - 1)(\lambda_2{}^2 - 1)\right)$$

b) Using Fung's parameters (below) for rabbit abdominal skin, plot s_{11} when stretch λ_1 goes from 1 to 2 and $\lambda_2 = 1$.

$\alpha_1 = \alpha_2 = 1{,}020\ N/m^2$

$\alpha_4 = 254\ N/m^2$

$c = 0.779\ N/m^2$

$a_1 = 3.79$

$a_2 = 18.4$

$a_4 = 0.587$