Lecture 2: In-class problems

Three important equations in mechanics discussed during lecture.

1. Constitutive Equations
2. Equilibrium Equations
3. Kinematic Equations

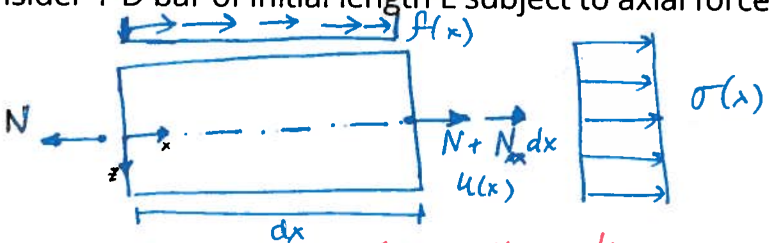
We also reviewed Euler Bernoulli beam theory, in which there are three kinematic assumptions to consider:

- normals remain straight
- normals remain unstretched
- normals remain normal

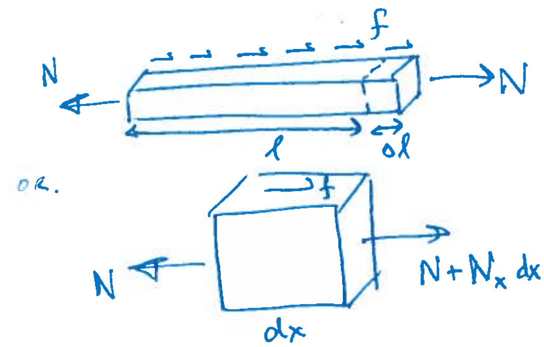
We can derive the properties for cytoskeleton biopolymers in extension and bending.

Extension

Consider 1-D bar of initial length L subject to axial force f.



OR.



Kinematic Equation $\varepsilon = \lim_{x\to 0} \frac{u}{x} = \frac{du}{dx}$

For homogeneous $\boxed{\varepsilon = \Delta l / l}$

Constitutive Equation $\sigma = \sigma(\varepsilon)$

if linear elastic $\boxed{\sigma = E\varepsilon}$

Stress Resultant $N = \iint \sigma \, dy\, dz$

if homogeneous $\boxed{\sigma = N/A}$

Equilibrium

$\Sigma f = 0$ in axial direction

$$\frac{dN}{dx} + f = \frac{A\,d\sigma}{dx} + f = \frac{EA\,d\varepsilon}{dx} = EA\frac{\partial^2 u}{\partial x^2} + f = 0$$

Differential Equation

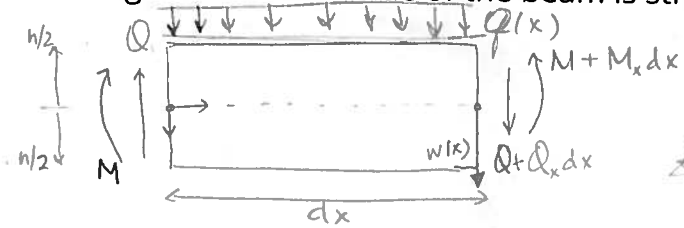
$$EA\frac{d^2u}{dx^2} + f = 0$$

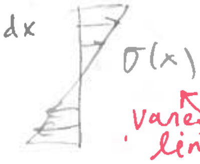
f homogeneous $u = \frac{NL}{EA}$

CSK protein	Radius [nm]	Area [nm²]	Young's Modulus (E) [N/m²]	Structural stiffness (EA) [N]
Microtubule	12.5	492 nm²	1.9×10^9	0.934×10^{-6} N
Intermediate Filament	5.0	79 nm²	2×10^9	0.16×10^{-6} N
Actin Filament	3.5	38 nm²	1.9×10^9	0.23×10^{-9} N

Bending

Consider 1-D bar. With beam bending: inner side of the beam compressed to shorter length while outer side of the beam is stretched.





inner side of beam: compressed
outer side of beam: stretched

Kinematic Equation

$$\varepsilon = \cancel{\frac{du}{dx}}^{\,0} - z\frac{\partial^2 w}{\partial x^2} \Longrightarrow \varepsilon = -\frac{\partial^2 w}{\partial x^2} z = \kappa z \quad \text{where } \kappa = 1/r$$

Constitutive Equation

$$\sigma = E\varepsilon = -E\frac{\partial^2 w}{\partial x^2} z = E\kappa z$$

Stress Resultant

$$M = \int_{-h/2}^{h/2} \sigma z\, dz = \int_{-h/2}^{h/2} E\kappa z^2 dz = \boxed{EI\kappa = M} \quad \text{where } I = \int_{-h/2}^{h/2} z^2 dz$$

Equilibrium

$$① \ \sum f_z = 0 \longrightarrow Q_{,x} + q = 0$$

$$② \ \sum M = 0 \longrightarrow M_{,x} - Q = 0$$

$$\frac{d^2M}{dx^2} + q = 0$$

Differential Equation

$$q = EI\frac{d^4 w}{dx^4}$$

$$\boxed{I = \frac{\pi r^4}{4}}$$

CSK protein	Radius [nm]	Inertia [nm⁴]	Young's Modulus (E) [N/m²]	Bending stiffness (EI) [Nm²]
Microtubule	12.5	19,175	1.9x10^9	364 x10^{-25}
Intermediate Filament	5.0	491	2x10^9	10 x10^{-25}
Actin Filament	3.5	118	1.9x10^9	2 x10^{-25}

From these numbers, what can you say about CSK proteins?

CSK proteins are relatively stiff in axial tension but flexible in bending.

What do you expect for these proteins in torsion? Would they be stiff or flexible?

Flexible.

Now let's have a bit of fun! Try to guess the persistence length of spaghetti at room temperature. State your assumptions clearly.

Assume $d = 2mm$

$E = 1\times10^{8} J/m^3 = 1\times10^{8} N/m^2$

$T = 300K$ $k_b = 1.38\times10^{-23} J/K$

$$A = \frac{EI}{k_B T} = \frac{\pi E r^4}{4 k_B T}$$

Plug in #s

$A = 1.8\times10^{16} m$!

Uncooked pasta changes its direction at length scales on that order $\longrightarrow$ consider this: $d_{EARTH \to MOON} = 3.8\times10^{8} m$!

Estimate the number of cells in the body. This is a classic Fermi problem. State your assumptions and write out your technique. Can you get close the actual number, 37.2 trillion cells?

Simplified version

Say human body ~ 70% H_2O, $m = 85 kg$

$\rho = 1000 \ kg/m^3$

$\rho = m/V \longrightarrow V_{HUMAN} = \frac{m}{\rho} = \frac{(0.7)(85 \ kg)}{1000}$

$V_{Human} \sim 0.0595 m^3$

Let's pretend a cell is a cube with sides of $20\mu m$

$V_{cell} = (20\mu m)^3 = 8\times10^{-15} m^3$

$N_{cell} = \frac{V_{Human}}{V_{cell}} \sim 10^{4}$ trillion.

$\boxed{\text{Fun facts}}$

~210 different cell

~10 μm typical cell size

smallest cell → d ≈ 2-4 μm

nerve cell can be 1 m long

typical m ~ 1 ng.