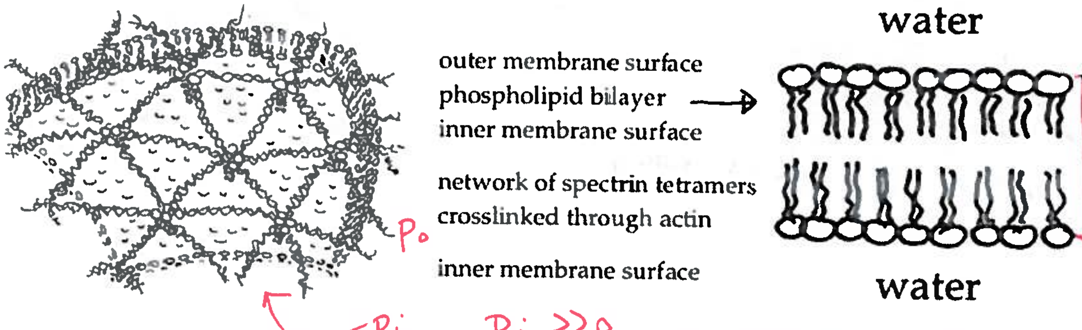
Lecture 4: Mechanics of Cell Membrane

Recall, the red blood cell.



head : hydrophilic

tail : hydrophobic

t ≈ 5nm vs. cell size [μms]
so membrane → shell.

Features
- flexible
- semipermeable
  - → Passive Transport
  - → Active Transport

$P_i \gg P_o$ (internal pressure) ⇒ Cell membrane strong enough to prevent explosion

Kinematic Assumptions

1. **Normal remain straight** → do not bend
2. **Normal remain unstretched** → keep same length
3. **Normal remain normal** → orthogonal to mid-surface

Kirchoff's hypothesis.

Let the total deformation in x and y be u^{tot} and v^{tot}, respectively.

$$u^{TOT}(x,y,z) = u(x,y) - z\, dw/d\cancel{z}$$

$$U^{TOT}(x,y,z) = v(x,y) - z\, dw/dy$$

$$w^{TOT}(x,y,z) = w(x,y)$$

Simple ~~Beam~~ Extension of Beam Kinematics.

From this we can calculate strain.

$$\varepsilon_{xx} = \frac{du^{TOT}}{dx} = \frac{du}{dx} - z\frac{d^2w}{dx^2}$$

$$\varepsilon_{yy} = \frac{dv^{TOT}}{dy} = \frac{dv}{dy} - z\frac{d^2w}{dy^2}$$

$$\varepsilon_{zz} = \frac{dw^{TOT}}{dz} = 0$$

NORMAL STRAIN

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{du^{TOT}}{dy} + \frac{dv^{TOT}}{dx}\right) = \frac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right) - z\frac{d^2w}{dxdy}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx} - z\frac{d^2w}{dxdz} + \frac{d^2w}{dx^2}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{dv}{dz} - \frac{dw}{dy} - z\frac{d^2w}{dydz} + \frac{d^2w}{dy^2}\right)$$

SHEAR STRAIN

We used 2/3rds of our assumptions so far: #1 and #3.
Let's apply assumption (2) → unstretched.
- **Thickness of shell does not change so any strain terms involving z→ 0** (b/c this is d/dz)

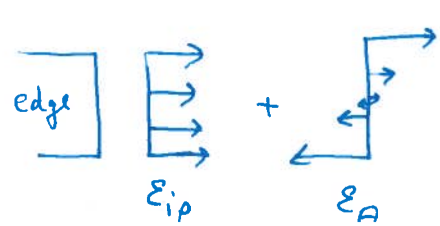
$$\varepsilon_{zz} = 0 \qquad \varepsilon_{zx} = \varepsilon_{zy} = 0 \quad \varepsilon_{xz} = \varepsilon_{yz} = 0.$$

$$\varepsilon_{xx} = \frac{du}{dx} - z\frac{d^2w}{dz^2}$$

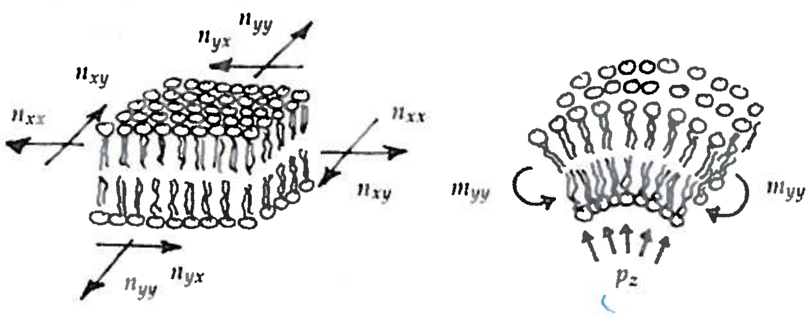
$$\varepsilon_{yy} = \frac{dv}{dy} - z\frac{d^2w}{dy^2}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right) - z\frac{d^2w}{dxdy}$$

Notice, the nonzero terms have two parts:
- **One constant through thickness (in plane)** ε_{ip}
- **One varies linearly with thickness (rotation)** ε_{θ}



The overall deformation of plates/shells can be understood as the superposition of 3 modes.



In plane tension
+
Shear
+
Out of plane bending

Let's try to understand these independently.
First, the constitutive model describes material behavior. Let's assume a generalized Hookean material response.

$$\sigma_{xx} = 2\mu\,\varepsilon_{xx} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\,\varepsilon_{yy})$$

$$\sigma_{yy} = 2\mu\,\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\,\varepsilon_{xx})$$

$$\tau_{xy} = 2\mu\,\varepsilon_{xy} = \frac{E}{1+\nu}\,\varepsilon_{xy}$$

Note.

$$\mu = \frac{E}{2(1+\nu)} \quad \text{Lamé constant}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Now consider equilibrium conditions. We will consider two loading cases as we have done in the past – shear and tension.

Note: I will not go through the math here but you can derive the following equations.

$$\sum f_x = 0 \rightarrow \frac{dn_{xx}}{dx} + \frac{dn_{xy}}{dy} = 0$$

$$\sum f_y = 0 \rightarrow \frac{dn_{xy}}{dx} + \frac{dn_{yy}}{dy} = 0$$

Stress resultant

$$n_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\,dz = \sigma h$$

$$n_{yy} = \sigma h$$

$$n_{xy} = \int_{-h/2}^{h/2} \tau_{xy}\,dz$$

$$\sum f_z = 0 \rightarrow n_{xx}\underbrace{\frac{d^2w}{dx^2}}_{\kappa_{xx}} + 2n_{xy}\underbrace{\frac{d^2w}{dx\,dy}}_{\kappa_{xy}} + n_{yy}\underbrace{\frac{d^2w}{dy^2}}_{\kappa_{yy}} + p_z = 0$$

curvature.

Note

$$\kappa_{xx} = 1/r_y$$

$$\kappa_{yy} = 1/r_x$$

Notation gets a bit weird because technically p_z is negative or some others like to say

$$-\frac{d^2w}{dx^2} = \kappa_{xx}, \text{ etc.}$$

Let's consider equibiaxial tension.

Assume: $\sigma_{xx} = \sigma_{yy} = \sigma$ uniform across membrane

$\tau_{xy} = 0 \rightarrow n_{xy} = 0$

$n_{xx} = n_{yy} = n$

$$n_{xx}\frac{d^2w}{dx^2} + \cancel{2n_{xy}\frac{d^2w}{dx\,dy}}^{\to 0} + n_{yy}\frac{d^2w}{dy^2} + p_z = 0$$

$$n\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right] + p_z = 0$$

$$\boxed{n\,\nabla^2 w + p_z = 0} \rightarrow \begin{aligned} p_z &= -n\,\nabla^2 w \\ p_z &= n\left[\frac{1}{r_x} + \frac{1}{r_y}\right]\end{aligned}$$

For equal radii $r_x = r_y = r$

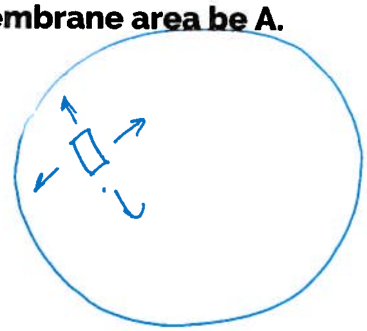
$$p_z = \frac{2n}{r}$$

n = surface tension

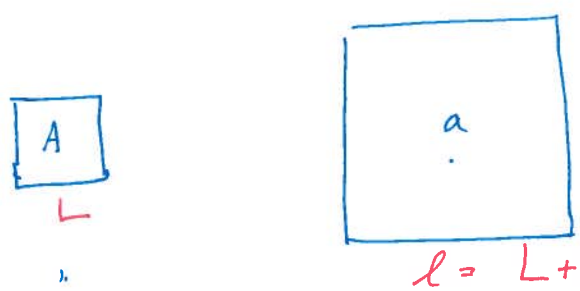
classic membrane definition of Sphere [soap bubbles]

Surface Strain

Let the membrane area be A.



Let's define a ~~membrane~~ measure for ΔA



$\varepsilon = \dfrac{\Delta L}{L}$

$l = L + (\Delta L)$

$= L + \varepsilon L$

$$\frac{\Delta A}{A} = \frac{a - A}{A} = \frac{(L + \varepsilon L)^2 - L^2}{L^2} = \frac{(1+\varepsilon)^2 L^2 - L^2}{L^2} \approx 2\varepsilon \qquad \varepsilon^2 \approx 0$$

$$\boxed{\varepsilon = \frac{1}{2}\frac{\Delta A}{A}}$$

Recall

$n = \sigma h$

$$n = \frac{Eh}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right]$$

$$n = \frac{Eh}{2(1-\nu^2)}\frac{\Delta A}{A} = K_A\frac{\Delta A}{A}$$

K_A = expansional modulus

Shear

$$\tau_{xy} = \frac{E}{1+\nu}\varepsilon_{xy} = 2\mu\varepsilon_{xy}$$

$$n_{xy} = \tau_{xy} h = 2\mu h\varepsilon_{xy}$$

$$\downarrow$$

$$K_s = \frac{Eh}{1+\nu}$$ membrane shear stiffness

$\sim G$

There is one more thing we need to consider – transverse deformation or bending.

$$p_z = K_B\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] \text{ where } K_B = \frac{Eh^3}{12[1-\nu^2]}$$

The equation above is the Kirchoff plate equation. You derive this equation in the homework.
For red blood cells,

$K_A \sim 0.45\ N/m$ $[0.1 \rightarrow 1]$] HUGE

$K_s \sim 6-9 \times 10^{-6}\ N/m$

$K_B \sim 10^{-19}\ N.m$] VERY SMALL

Based on these equations and some values I just gave you for the red blood cell, there are few things to understand.

1. **Because K_A is huge compared to other moduli which means the cell membrane is incompressible.**
2. **K_s is very small so the effect of shear usually neglected under static loading but it may play a significant role under dynamic loading.**
3. **K_b is very small so the effect of bending is very small in a biomembrane.**

Summary

$$n\left[\nabla^2 w\right] - K_B\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + p_z = 0$$

$$\frac{n}{K_B}$$ establish dominance

let w = transverse displacement
λ = characteristic length over which w varies

Membrane scales $\frac{nw}{\lambda^2}$ Bending scales $\frac{K_B w}{\lambda^4}$

$$\frac{K_B}{n\lambda^2}$$

$\ll 1$ tension

$\gg 1$ bending

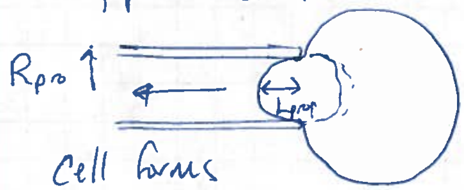
for cells.

$$\frac{10^{-19}}{(5 \times 10^{-5})(1 \times 10^{-6})^2} \quad 0.02$$

Micropipette Aspiration

$$\frac{L_{protrusion}}{R_{protrusion}}$$

pipette suction

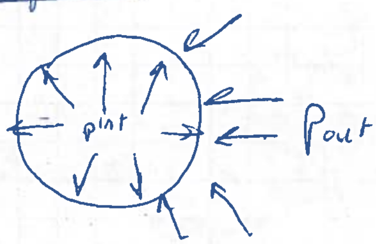


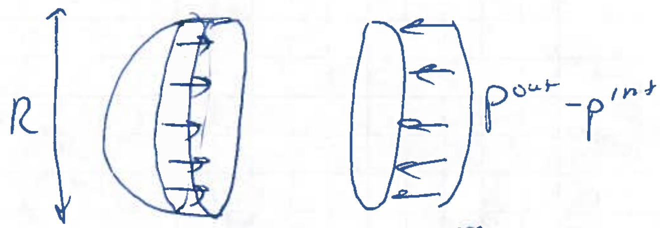
Cell forms protrusion

As pressure ↑.

$L_{pro} > R_{pro}$ or $\frac{L_{pro}}{R_{pro}} > 1$

Law of Laplace





$F^{mem} = n\,2\pi r$ membrane

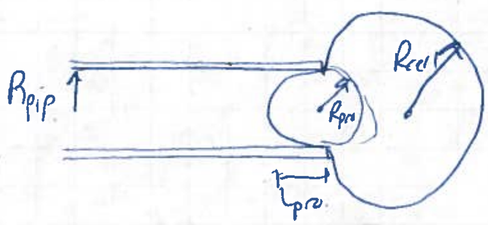
$F^{pre} = \Delta p\, \pi r^2$ pressure

$$F^{mem} = F^{pre}$$

$$n\,2\cancel{\pi R} = \Delta p \cancel{\pi} R^{\cancel{2}}$$

$$\boxed{\Delta p = \frac{2n}{R}}$$ law of Laplace

Look familiar?!



Limit case

$L^{pro} = R^{pro} = R^{pip}$

① $P^{pip} + P^{int} - P^{out} = 2n \frac{1}{R^{pip}}$

② $P^{int} - P^{out} = \frac{2n}{R_{cell}}$

$$\boxed{P^{pip} = 2n \left[\frac{1}{R_{pip}} - \frac{1}{R_{cell}}\right]}$$

Neutrophils behave like liquid drop.

Chondrocyte/Endothelial Cells behave as elastic solids.