We aim to provide you some familiarity with these tools, but the presentation is far from complete.

As we described before, there are three parts to a problem in continuum mechanics: kinematics, equilibrium, and constitutive behavior. So, let us take each one in turn.

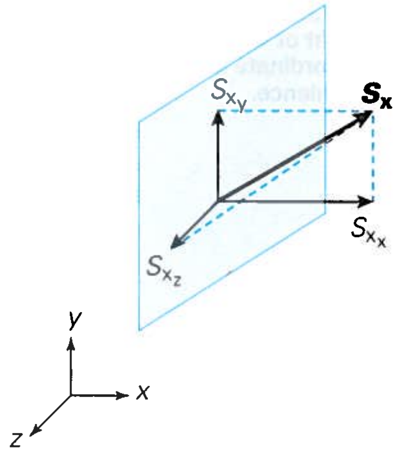

Figure 3.23 Forces on an imaginary cut-plane in the *y-z*-plane.

Equilibrium implies conditions on stress

In our discussion of stress and equilibrium, let us first introduce a notation system for stress. In our examples, we defined stress conceptually to be the distributed force per unit area through a given test plane. In the example, the location of the test plane was obvious. Now we want to create a general definition and notation system associated with our x-, y-, z-coordinate system. To specify the orientation of the test plane, we will make use of the normal vector to the plane.

Next consider the resultant force denoted by $\boldsymbol{S}_x$ (**Figure 3.23**). The $\boldsymbol{x}$ refers to the cut-plane we selected, which has a surface normal aligned in the x-direction. Like any vector, $\boldsymbol{S}_x$ can be broken into its three components, $S_{x_x}, S_{x_y}, S_{x_z}$. So, for this face, we have three potential stresses, one associated with each of the components of $\boldsymbol{S}_x$. To express the components of stress, we need a double subscript notation system with the first referring to the direction of the cut-plane normal, and the second referring to the direction of the internal force acting through that plane. Now we have everything we need to define the stress components in this coordinate system. For a cut-plane oriented perpendicular to the x-direction,

$$\sigma_{xx} = \lim_{A\to 0} \frac{S_{x_x}}{A}. \tag{3.39}$$

Similarly, for the other components for the x-cut-plane

$$\tau_{xy} = \lim_{A\to 0} \frac{S_{x_y}}{A} \text{ and } \tau_{xz} = \lim_{A\to 0} \frac{S_{x_z}}{A}. \tag{3.40}$$

And for the y-cut-plane

$$\tau_{yx} = \lim_{A\to 0} \frac{S_{y_x}}{A}, \quad \sigma_{yy} = \lim_{A\to 0} \frac{S_{y_y}}{A} \text{ and } \tau_{yz} = \lim_{A\to 0} \frac{S_{y_z}}{A} \tag{3.41}$$

and for the z-cut-plane

$$\tau_{zx} = \lim_{A\to 0} \frac{S_{z_x}}{A}, \quad \tau_{zy} = \lim_{A\to 0} \frac{S_{z_y}}{A} \text{ and } \sigma_{zz} = \lim_{A\to 0} \frac{S_{z_z}}{A}. \tag{3.42}$$

Now that we have a consistent notation system, let us see what equilibrium can tell us about stress. Remember that the equilibrium condition is a condition on forces. Specifically, for a nonaccelerating body, the forces must sum to zero. Imagine a small piece of material within a general solid body, and consider the forces on the infinitesimal element as shown (**Figure 3.24**). We assume that the dimensions of the element are dx, dy, and dz respectively. Each face has one normal force and two shear forces. However, because the element is vanishingly small, we can approximate the force on the faces far from the origin as a function of the force on the closer face and its derivative. Taking the first term in a Taylor series,

$$S_{x_x}(x + \mathrm{d}x) = S_{x_x}(x) + \frac{\mathrm{d}S_{x_x}(x)}{\mathrm{d}x}\mathrm{d}x. \tag{3.43}$$

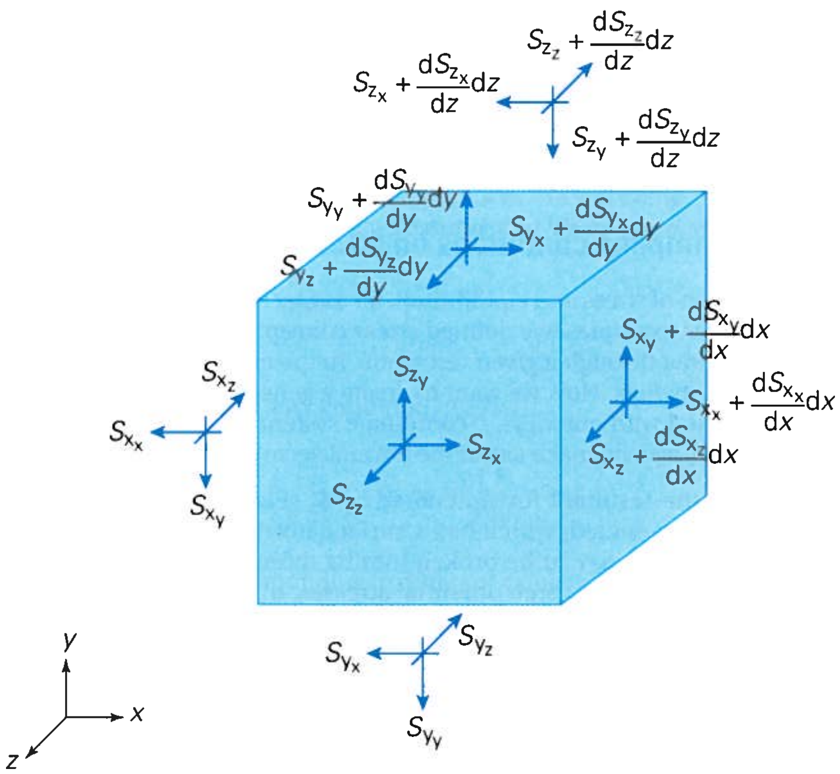

Figure 3.24 All the forces on a small element of volume oriented with the coordinate axes for convenience.

Now we simply sum the forces in the x-, y-, and z-directions in turn:

$$\sum f_x = 0 \Rightarrow \left(-S_{x_x} + S_{x_x} + \frac{dS_{x_x}}{dx}dx\right) + \left(-S_{y_x} + S_{y_x} + \frac{dS_{y_x}}{dy}dy\right) + \left(-S_{z_x} + S_{z_x} + \frac{dS_{z_x}}{dz}dz\right)$$

$$\sum f_y = 0 \Rightarrow \left(-S_{x_y} + S_{x_y} + \frac{dS_{x_y}}{dx}dx\right) + \left(-S_{y_y} + S_{y_y} + \frac{dS_{y_y}}{dy}dy\right) + \left(-S_{z_y} + S_{z_y} + \frac{dS_{z_y}}{dz}dz\right)$$

$$\sum f_z = 0 \Rightarrow \left(-S_{x_z} + S_{x_z} + \frac{dS_{x_z}}{dx}dx\right) + \left(-S_{y_z} + S_{y_z} + \frac{dS_{y_z}}{dy}dy\right) + \left(-S_{z_z} + S_{z_z} + \frac{dS_{z_z}}{dz}dz\right). \tag{3.44}$$

Notice that the first two terms in each quantity in parentheses cancel. Next, we can divide each row by the infinitesimal volume ($dx\,dy\,dz$) and simplify

$$\frac{\left(\frac{dS_{x_x}}{dx}\right)}{dydz} + \frac{\left(\frac{dS_{y_x}}{dy}\right)}{dxdz} + \frac{\left(\frac{dS_{z_x}}{dz}\right)}{dxdy} = 0$$

$$\frac{\left(\frac{dS_{x_y}}{dx}\right)}{dydz} + \frac{\left(\frac{dS_{y_y}}{dy}\right)}{dxdz} + \frac{\left(\frac{dS_{z_y}}{dz}\right)}{dxdy} = 0 \tag{3.45}$$

$$\frac{\left(\frac{dS_{x_z}}{dx}\right)}{dydz} + \frac{\left(\frac{dS_{y_z}}{dy}\right)}{dxdz} + \frac{\left(\frac{dS_{z_z}}{dz}\right)}{dxdy} = 0.$$

Now, notice that in each term we have a differential area. In each case, this differential area does not depend on the derivative in the numerator. Therefore, we can write

$$\frac{d}{dx}\left(\frac{S_{x_x}}{dydz}\right)+\frac{d}{dy}\left(\frac{S_{y_x}}{dxdz}\right)+\frac{d}{dz}\left(\frac{S_{z_x}}{dydz}\right)=0$$

$$\frac{d}{dx}\left(\frac{S_{x_y}}{dydz}\right)+\frac{d}{dy}\left(\frac{S_{y_y}}{dxdz}\right)+\frac{d}{dz}\left(\frac{S_{z_y}}{dydz}\right)=0 \qquad (3.46)$$

$$\frac{d}{dx}\left(\frac{S_{x_z}}{dydz}\right)+\frac{d}{dy}\left(\frac{S_{y_z}}{dxdz}\right)+\frac{d}{dz}\left(\frac{S_{z_z}}{dydz}\right)=0.$$

The differential area for each term in parentheses is the area normal to the respective force vector component. Therefore, each of these terms is simply our definition of stress. The equilibrium equations then take the following remarkably simple form:

$$\frac{d\sigma_{xx}}{dx}+\frac{d\sigma_{yx}}{dy}+\frac{d\sigma_{zx}}{dz}=0$$

$$\frac{d\sigma_{xy}}{dx}+\frac{d\sigma_{yy}}{dy}+\frac{d\sigma_{zy}}{dz}=0 \qquad (3.47)$$

$$\frac{d\sigma_{xz}}{dx}+\frac{d\sigma_{yz}}{dy}+\frac{d\sigma_{zz}}{dz}=0.$$

Example 3.4: Symmetry of stress

Our free-body analysis of the resultant forces on an infinitesimal element can tell us one other important fact about stress. Notice that the equilibrium equation is the result of requiring the forces to sum to zero. What about the moments? Remember that in a body that is not accelerating, the moments about any arbitrary axis must also sum to zero. In our example, calculate the moments about an axis passing through the center of the element in the x-direction (**Figure 3.25**).

Summing moments implies

$$\sum M_x = 0 \Rightarrow 2S_{y_z} + 2S_{z_y} = 0$$

or $S_{y_z} = S_{z_y}$. In terms of stress

$$\iint \sigma_{zy}dxdy = \iint \sigma_{yz}dxdy \text{ or } \sigma_{zy} = \sigma_{yz}.$$

Likewise for the y- and z-axes, we obtain

$$\sigma_{xy} = \sigma_{yx} \text{ and } \sigma_{zx} = \sigma_{xz}.$$

This important property of stress is that it must be symmetric for any nonaccelerating body. Because there is no additional information in the σ_{yx}, σ_{zx}, and σ_{zy} terms they are typically replaced by σ_{xy}, σ_{xz}, and σ_{yz}, respectively.

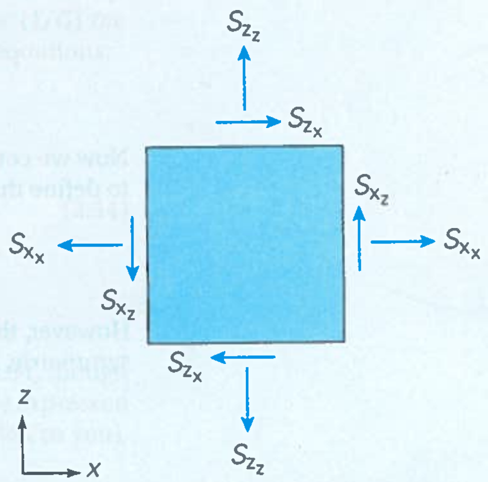

Figure 3.25 The forces on a small two-dimensional element.

Kinematics relate strain to displacement

What is strain? The equations that relate strain and displacement are the kinematic equations and serve as the formal definition of strain. Unlike in our simple examples above, these are general equations that characterize the deformation of a physical body. We begin our discussion by assuming we have specified a deformable body in a coordinate system specified x, y, and z. There is a deformation at each point in the body given as u, v, w, the displacements in the x-, y-, and z-directions respectively.

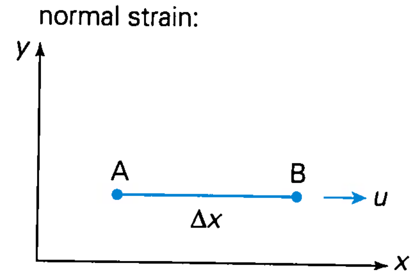

Figure 3.26 An imaginary line in the x-direction embedded in the material being analyzed.

Let us start by defining normal strains. Imagine a general body undergoing a small deformation. We can define a test line within the body in the undeformed condition and ask what happens to it during the deformation (Figure 3.26). We are going to consider every possible deformation and orientation of the test line in turn, but, for now, we assume that the test line is oriented along the x-direction. The ends of the test line are denoted by A and B, and are able to displace independently.

Now we need to determine how much the test line is elongated. In general, this will be a function of u, v, and w. However, if the deformations are "small" the extension of the test line is dominated by u. The strain quantity we are defining here is the so-called *infinitesimal* or *small deformation* strain. Technically, it is known as the *Cauchy* strain. The extension of the line is given by $u_A - u_B$ and the average strain in the test line is simply the change in length over the original length $(u_A - u_B)/(x_A - x_B) = \Delta u/\Delta x$. The strain at any given point can now be defined as the average strain in the test line as the length of the line shrinks to zero:

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}, \tag{3.48}$$

which is the definition of the derivative. To specify that we are referring to the displacement in the x-direction of a test line originally oriented along the x-axis, strain components are typically denoted with two subscripts,

$$\varepsilon_{xx} = \frac{du}{dx}. \tag{3.49}$$

Similarly in the y- and z-directions,

$$\begin{aligned} \varepsilon_{yy} &= \frac{dv}{dy}, \\ \varepsilon_{zz} &= \frac{dw}{dz}. \end{aligned} \tag{3.50}$$

Now we consider the shear strain. Perhaps the most logical thing to do would be to define them as similar to normal strains. We could define

$$\varepsilon_{xy} = \frac{dv}{dx}. \tag{3.51}$$

However, there is a problem with this approach. Strain defined in this way is not symmetric, because in general,

$$\frac{dv}{dx} \neq \frac{du}{dy}. \tag{3.52}$$

It will simplify things greatly to define strain as symmetric, and this can be easily achieved by taking our definition of shear strain to be

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right), \tag{3.53}$$

which preserves the symmetry condition, because $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{xz} = \varepsilon_{zx}$, and $\varepsilon_{yz} = \varepsilon_{zy}$.

To clearly distinguish normal and shear strains, the symbol γ is sometimes used for the components of the shear strain. The notation γ refers to the engineering strain defined in our pure shear and torsion examples. It differs from the continuum strain ε by a factor of two, that is, $\gamma_{xy} = 2\varepsilon_{xy}$, $\gamma_{xz} = 2\varepsilon_{xz}$, and $\gamma_{yz} = 2\varepsilon_{yz}$.

The constitutive equation or stress–strain relation characterizes the material behavior

How are strain and stress related? As we have noted, the equations that relate stress and strain are the constitutive equations. These are the equations that capture the behavior of the material. They will change depending on the material being considered. Earlier in this chapter, we introduced Hooke's law in the one-dimensional case, $\sigma = E\varepsilon$. Let us see if we can generalize Hooke's law to describe the material behavior of a three-dimensional, isotropic, linearly *xy* elastic solid. In fact, we have already described three parts of Hooke's law in simple example cases. We described the uniaxial behavior $\sigma = E\varepsilon$, the transverse contraction due to Poisson's ratio $\varepsilon_p = -\nu\varepsilon_a$, and the shear behavior $\tau = G\gamma$. Now, let us see if we can figure out the general case. If we are given a set of six stresses (three normal stresses and three shear stresses) can we figure out the set of six strains? In other words, what are the 36 (6 × 6) coefficients that must multiply the stresses to get the strains. Note, it is easier to determine the coefficients that multiply the stresses to get the strains, because the situations in which the material's properties were defined were given with several components of stress being zero. Because we are describing a linear material, we can simply apply each of these behaviors in our new notation system and add them up. We know from our example and definition of Young's modulus, when we hold all other stresses to be zero and apply a stress in a normal direction, the coefficient multiplying the strain in the same direction is $1/E$. This gives us three of our coefficients.

Now, what can Poisson's ratio tell us? If the stress is applied in the x-direction, it tells us that the normal strain in the y- and z-directions are $-\nu$ times the strain in the x-direction. However, we already know that strain is $1/E$ times the applied stress. This gives us six more coefficients. For shear strain, we know that for an applied pure shear stress, the normal strains and the shear strains in the other directions are all zero. This gives us 24 more coefficients that must be zero. Finally, we know from the relationship between shear strain and shear stress $(1/G)$ the final three coefficients we need. We can write the general form of the equations:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - v\sigma_{yy} - v\sigma_{zz}\right], & \gamma_{xy} &= (1/G)\tau_{xy} \\
\varepsilon_{yy} &= \frac{1}{E}\left[-v\sigma_{xx} + \sigma_{yy} - v\sigma_{zz}\right], & \gamma_{xz} &= (1/G)\tau_{xz} \\
\varepsilon_{zz} &= \frac{1}{E}\left[-v\sigma_{xx} - v\sigma_{yy} + \sigma_{zz}\right], & \gamma_{yz} &= (1/G)\tau_{yz}.
\end{aligned}
\tag{3.54}
$$

Notice that there are three material constants (E, ν, G) in Equation 3.54, though only two of these are independent constants. The shear modulus can be expressed in terms of Young's modulus and Poisson's ratio (the proof of this is left to you), $G = E/2(1+\nu)$.

Therefore,

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - v\sigma_{yy} - v\sigma_{zz}\right], & \tau_{xy} &= \frac{2(1+v)}{E}\gamma_{xy} \\
\varepsilon_{yy} &= \frac{1}{E}\left[-v\sigma_{xx} + \sigma_{yy} - v\sigma_{zz}\right], & \tau_{xz} &= \frac{2(1+v)}{E}\gamma_{xz} \\
\varepsilon_{zz} &= \frac{1}{E}\left[-v\sigma_{xx} - v\sigma_{yy} + \sigma_{zz}\right], & \tau_{yz} &= \frac{2(1+v)}{E}\gamma_{yz}.
\end{aligned}
\tag{3.55}
$$

These equations can be inverted to yield the more traditional form of stress as a function of strain:

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}\right] & \tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}\right] & \tau_{xz} &= \frac{E}{2(1+\nu)}\gamma_{xz} \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}\right] & \tau_{yz} &= \frac{E}{2(1+\nu)}\gamma_{yz}.
\end{aligned}
\qquad (3.56)
$$

Vector notation is a compact way to express equations in continuum mechanics

Writing out in detail and manipulating the continuum equations of solid mechanics can become quite cumbersome, therefore many compact forms of notation have been introduced. One very powerful and extensively used system is known as *Voigt notation* or *vector notation.* In this approach, the components of stress and strain are organized into vectors,

$$
\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}. \qquad (3.57)
$$

Using this notation the stress-strain relationship Equation (3.55) is simply

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
(1-\nu) & \nu & \nu & 0 & 0 & 0 \\
\nu & (1-\nu) & \nu & 0 & 0 & 0 \\
\nu & \nu & (1-\nu) & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad \text{or} \quad \boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}
\qquad (3.58)
$$

and

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \quad \text{or} \quad \boldsymbol{\varepsilon} = \boldsymbol{D\sigma} \tag{3.59}$$

Nota Bene: Lamé constants allow a compact form of Hooke's law

Equation 3.58 can be expressed in a more compact form.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix},$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

and

$$\mu = \frac{E}{2(1+\nu)},$$

note that μ and λ are known as the *Lamé constants*.

Advanced Material: Coordinate rotations

It is important to note that the vector notation we use is a notational system only. We arrange the components of stress and strain in a vector to make them easy to manipulate. However, stress and strain are not mathematically vectors. As you remember, a vector is a quantity that has both magnitude and direction, like force, deformation, or velocity. An implication of this is that a vector maintains its magnitude and direction from one coordinate system to another. This implies that the components of a vector must behave in a very specific way in terms of how they relate to one another when expressed in different coordinate systems. Assume that we have two (orthogonal) coordinate systems, one specified by the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ and another specified by another set of vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, and $\boldsymbol{z}'$ ($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and $\boldsymbol{x}'$, $\boldsymbol{y}'$, and $\boldsymbol{z}'$ are known as basis vectors). Also, define the angle between any of these vectors to be $\theta_{xx'}$, $\theta_{xy'}$, etc. A rotation matrix $\boldsymbol{Q}$ can be defined such that

$$\boldsymbol{Q} = \begin{bmatrix} \cos(\theta_{xx'}) & \cos(\theta_{xy'}) & \cos(\theta_{xz'}) \\ \cos(\theta_{yx'}) & \cos(\theta_{yy'}) & \cos(\theta_{yz'}) \\ \cos(\theta_{zx'}) & \cos(\theta_{zy'}) & \cos(\theta_{zz'}) \end{bmatrix}.$$

Then any vector can be expressed in the new coordinate system by multiplying the vector expressed in the old