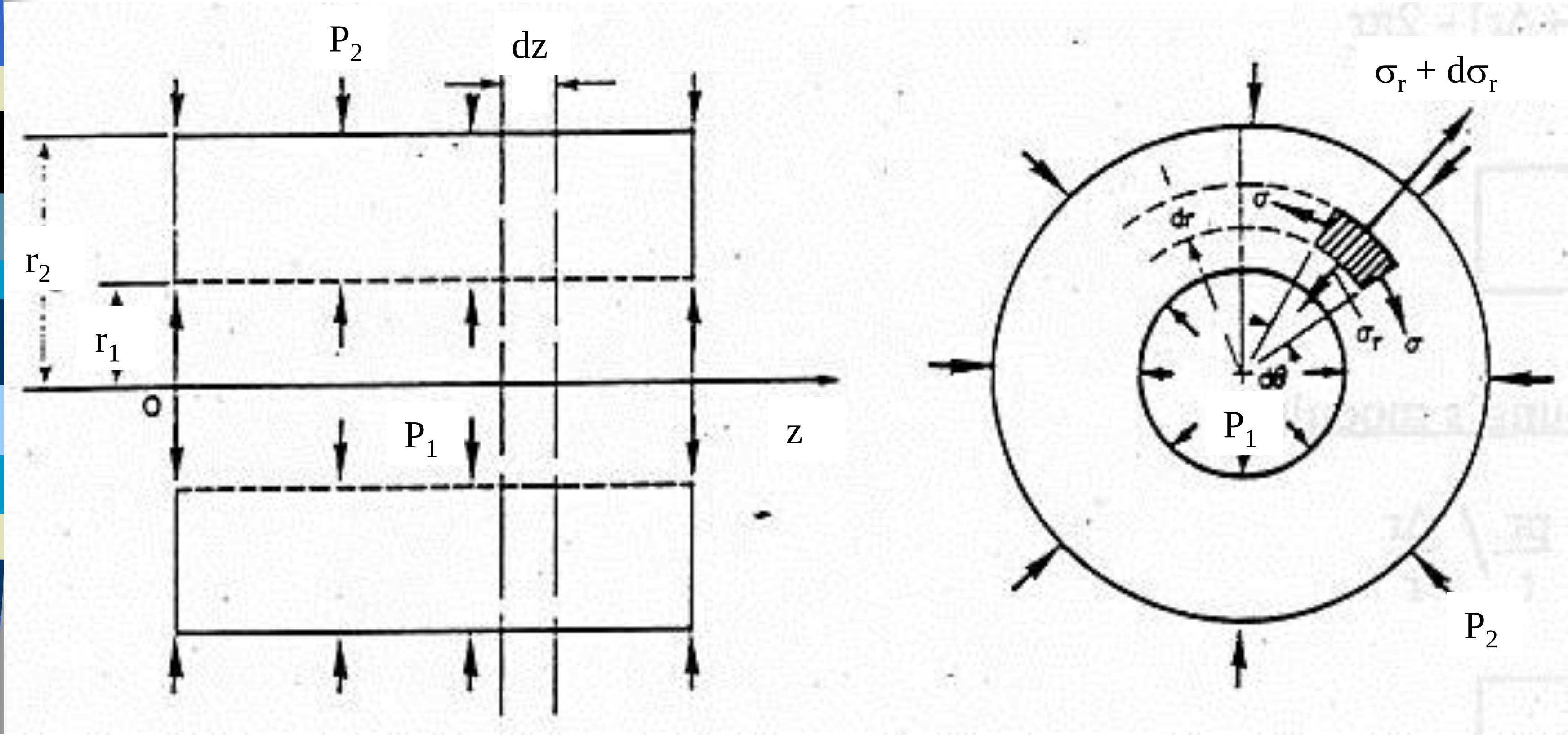
# Stress-strain relationship for cylindrical vessels

- **Thick-walled vessels**
  - $r_1$ = internal radius; $r_2$ = external radius
  - $p_1$ = internal pressure; $p_2$ = external pressure



# Stress

2. Stress

Equilibrium equation – radial direction

$(\sigma_r + d\sigma_r)\, dA - \sigma_r\, dA - 2\sigma_\theta\, dA \sin(d\theta/2) = 0$

$(\sigma_r + d\sigma_r)(r + dr)\, d\theta\, dz - \sigma_r\, d\theta\, dz - 2\,\sigma_\theta\, dr\, dz \sin(d\theta/2) = 0$

For small angles: $\sin(d\theta/2) = d\theta/2$

Gather terms, discard higher order, and divide by $(r\, dr\, d\theta\, dz)$

$$d\sigma_r/dr + (\sigma_r - \sigma_\theta)/r = 0$$

(equilibrium equation for axisymmetric cylinders)

# Strain

3. Strain

Compatibility condition

Displacements, function of r only, all points deform radially

Points at different radii will deform by different amounts

u = displacement

$\varepsilon_r = du/dr$

$\varepsilon_\theta = (2\pi\,[r + u] - 2\pi r)/2\pi r$

$\varepsilon_\theta = u/r$

# Young's Modulus

4. Young's Modulus

Substitute strain equation into stress-strain relations

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right) \qquad \sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \qquad \sigma_r = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

Substitute above equation into:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

Solvable differential equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = 0$$

$$u = c_1 r + \frac{c_2}{r}$$

$c_1$ and $c_2$ are constants of integration

# Determine constants

Constants evaluated at boundary conditions

At inner surface: $\sigma_r = -p_1, \qquad r = r_1$

At outer surface: $\sigma_r = -p_2, \qquad r = r_2$

$$c_1 = \frac{1-\nu}{E}\frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

$$c_2 = \frac{1+\nu}{E}\frac{r_1^2 r_2^2 \left[p_1 - p_2\right]}{r_2^2 - r_1^2}$$

Sub into u equation – get relationship for E in terms of radial displacement and pressure

# Stress (Lame Equations)

Stress (Lame equations)

$$\sigma_r = p_1\left(\frac{r_1^2}{r_2^2 - r_1^2}\right)\left(1 - \frac{r_2^2}{r^2}\right) - p_2\left(\frac{r_2^2}{r_2^2 - r_1^2}\right)\left(1 - \frac{r_1^2}{r^2}\right)$$

$$\sigma_\theta = p_1\left(\frac{r_1^2}{r_2^2 - r_1^2}\right)\left(1 + \frac{r_2^2}{r^2}\right) - p_2\left(\frac{r_2^2}{r_2^2 - r_1^2}\right)\left(1 + \frac{r_1^2}{r^2}\right)$$

For the case of no external pressure ($p_2=0$)

$$\sigma_r = p_1\left(\frac{r_1^2}{r_2^2 - r_1^2}\right)\left(1 - \frac{r_2^2}{r^2}\right)$$

$$\sigma_\theta = p_1\left(\frac{r_1^2}{r_2^2 - r_1^2}\right)\left(1 + \frac{r_2^2}{r^2}\right)$$

# For plane stress state ($\sigma_z$=0)

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$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_r + \sigma_\theta\right)$$

$$\varepsilon_z = -\frac{2\nu}{E}\frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

$$\boxed{\varepsilon_z = -\frac{2\nu}{E}\frac{p_1}{\left({r_2}/{r_1}\right)^2}}$$

$\varepsilon_z$ is constant and does not depend on r

# Special Case

- **For a thick wall cylinder with both ends fixed, and $p_2$=0, no longitudinal extension ($\varepsilon_z$=0) – plane strain condition**

$$c_1 = \frac{p_1 r_1^2}{2(\lambda + \beta)(r_2^2 - r_1^2)} \qquad c_2 = \frac{p_1 r_2^2 r_1^2}{2\beta(r_2^2 - r_1^2)}$$

where $$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \beta = \frac{E}{2(1+\nu)}$$

$$E = \left( \frac{p_1 r_1^2 (1+\nu)(1-2\nu)}{r_2^2 - r_1^2} \right) \frac{r}{u} + \left( \frac{p_1 r_2^2 + r_1^2 (1+\nu)}{(r_2^2 - r_1^2) r^2} \right) \frac{r}{u}$$

# Incremental modulus

To calculate incremental modulus measure:
$\Delta p$ internal, $\Delta r_2$ (at outside)

$p_1 = \Delta p$ $r = r_2$ $u = \Delta r_2$

$$E_{inc} = \frac{2\left(1-\nu^2\right) r_1^2 r_2}{\left(r_2^2 - r_1^2\right)\Delta r_2} \Delta p$$

To calculate, need internal pressure, external radius and wall thickness of the blood vessel *in situ*. $(r_2^2 - r_1^2)$ is constant if the wall is incompressible and the vessel wall is tetherered

-Magnitude of $E_{inc}$ will depend on $p$ at computation

# Stress (Lame Equations)

Stress (Lame equations)

$$\sigma_r = p_1\left(\frac{r_1^2}{r_2^2 - r_1^2}\right)\left(1 - \frac{r_2^2}{r^2}\right) - p_2\left(\frac{r_2^2}{r_2^2 - r_1^2}\right)\left(1 - \frac{r_1^2}{r^2}\right)$$

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For the case of no external pressure ($p_2=0$)

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$$\boxed{\varepsilon_z = -\frac{2\nu}{E}\frac{p_1}{\left(r_2/r_1\right)^2}}$$

$\varepsilon_z$ is constant and does not depend on r

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where $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ $\qquad \beta = \dfrac{E}{2(1+\nu)}$

$$E = \left(\frac{p_1 r_1^2 (1+\nu)(1-2\nu)}{r_2^2 - r_1^2}\right)\frac{r}{u} + \left(\frac{p_1 r_2^2 + r_1^2 (1+\nu)}{(r_2^2 - r_1^2) r^2}\right)\frac{r}{u}$$

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