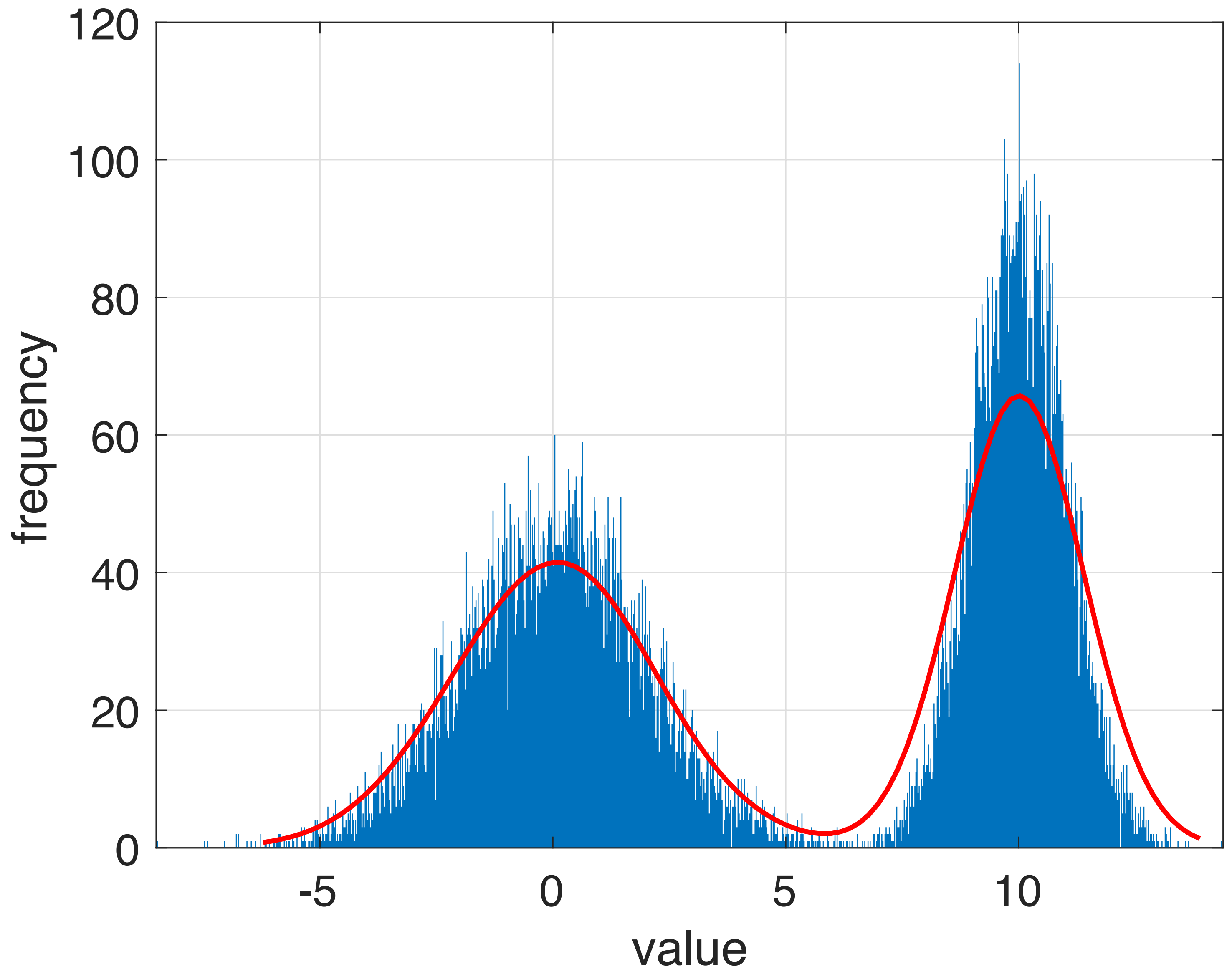
ME170b Lecture 5

2/22/23

Experimental Techniques

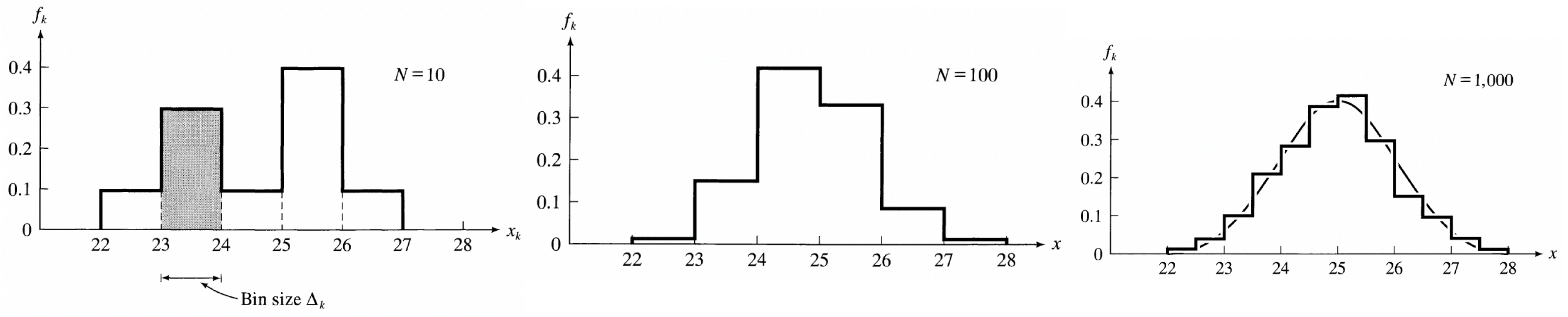
Last time:



Today:

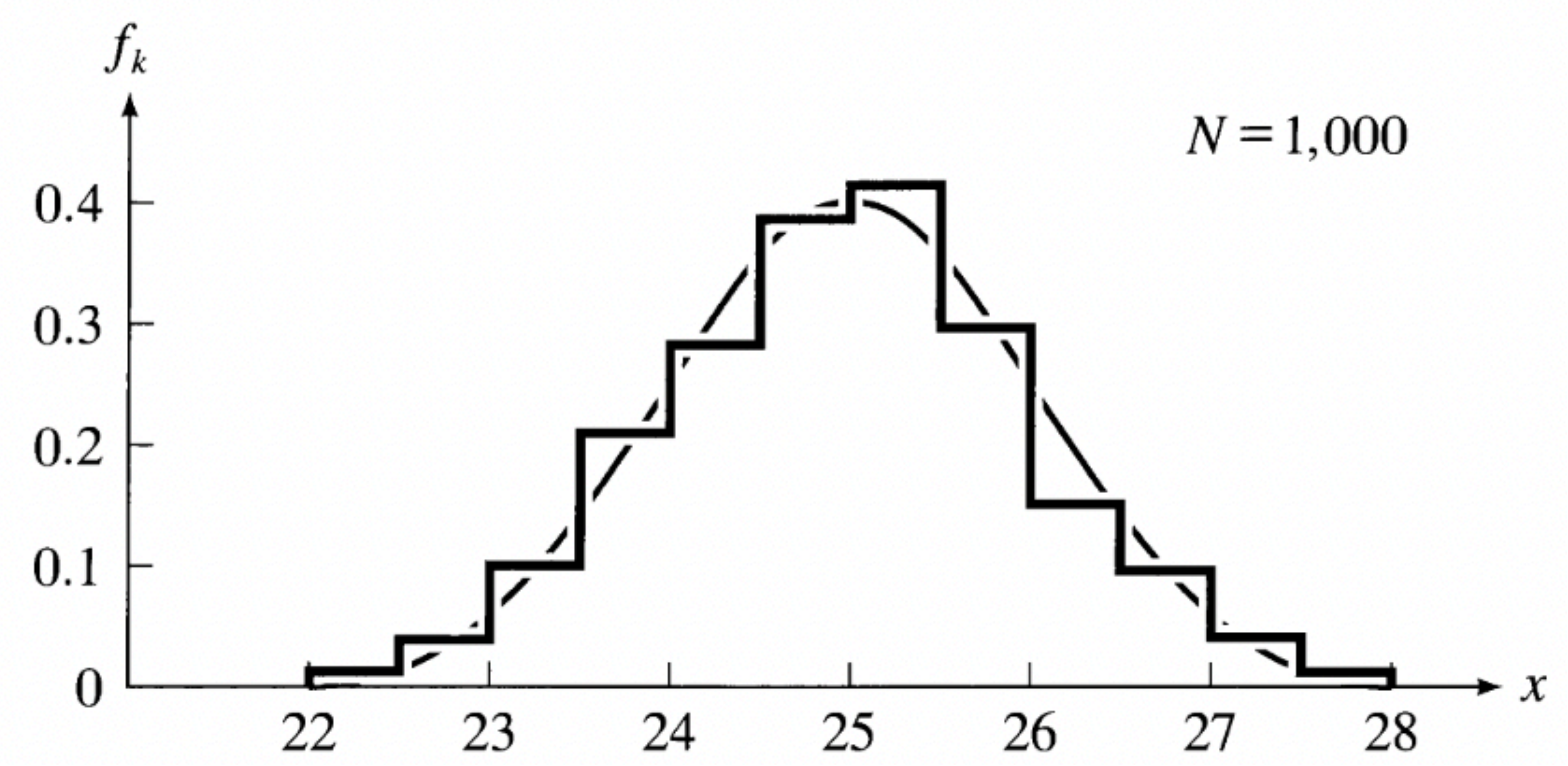
> Finish Ch.5 - Normal Distributions

Limiting Distributions



Key Idea: As N-> infinity, the distribution approaches a definite, continuous curve — this curve is called the “limiting distribution”

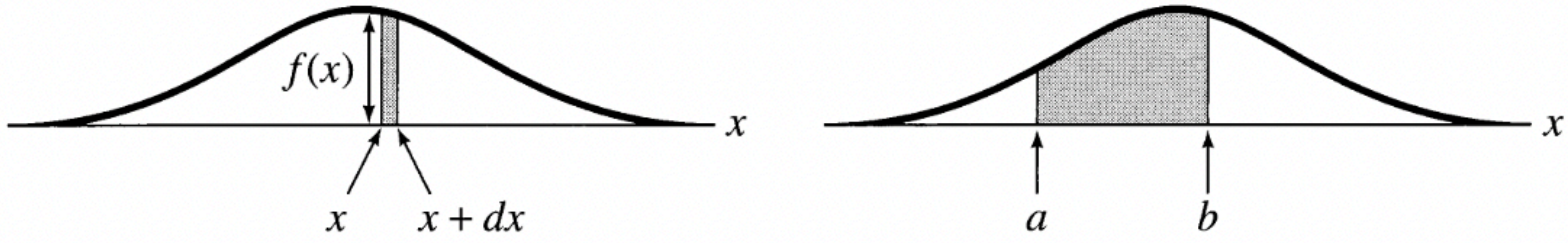
Does every measurement have a limiting distribution?



Limiting Distribution is theoretical construct — can never be measured exactly!

Short answer: yes, under most conditions

Math of limiting distribution

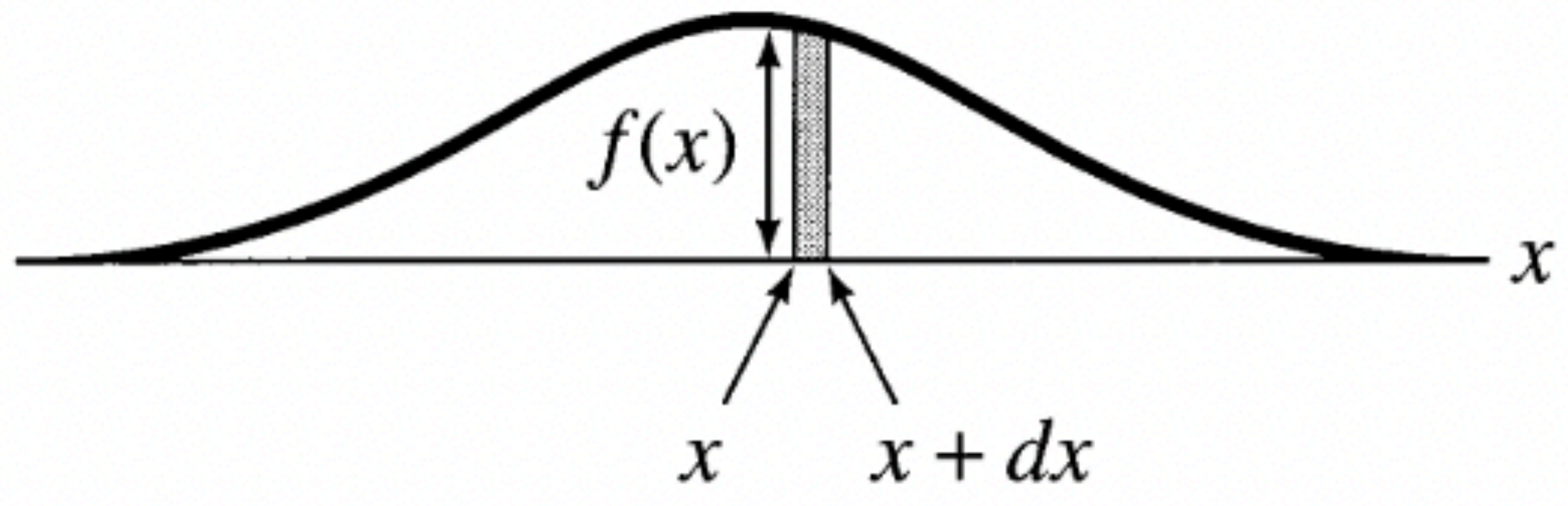


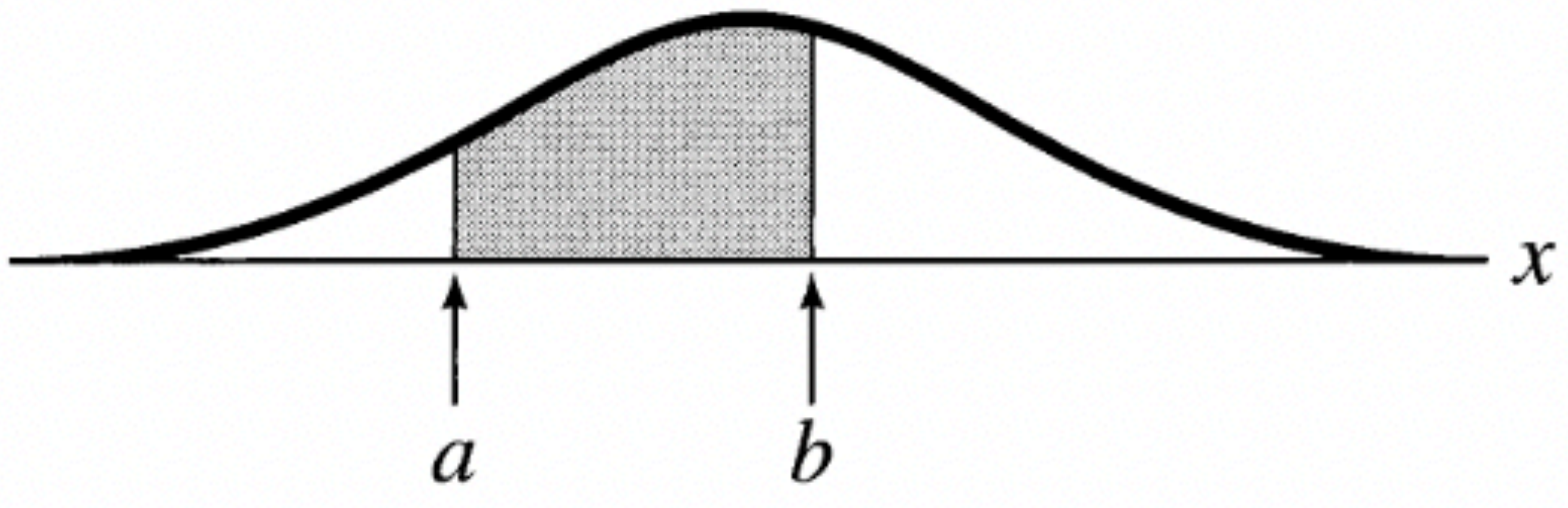
$f(x)\,dx$ = fraction of measurements that fall between x and $x + dx$.

$\int_a^b f(x)\,dx$ = fraction of measurements that fall between $x = a$ and $x = b$.

this only works well if you have a *large* number of measurement — e.g., you have a good approximation of the limiting distribution

What is the interpretation?





$f(x)\,dx$ = fraction of measurements that fall between x and $x + dx$.

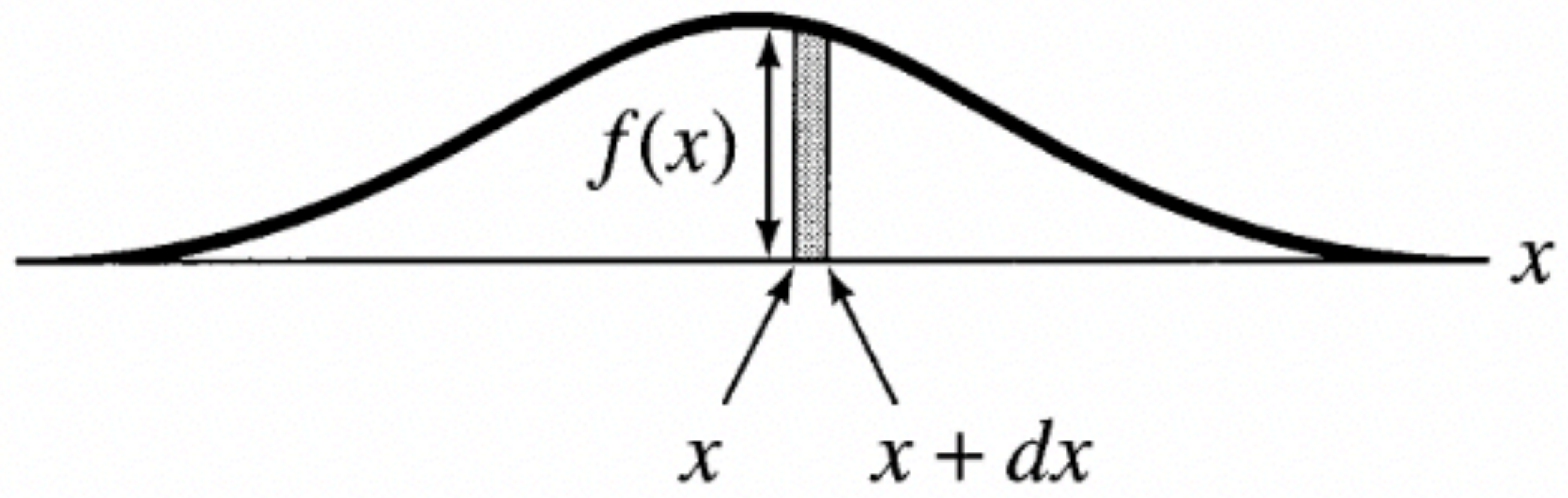
$\int_a^b f(x)\,dx$ = fraction of measurements that fall between $x = a$ and $x = b$.

the probability that any one measurement falls within x and x + dx

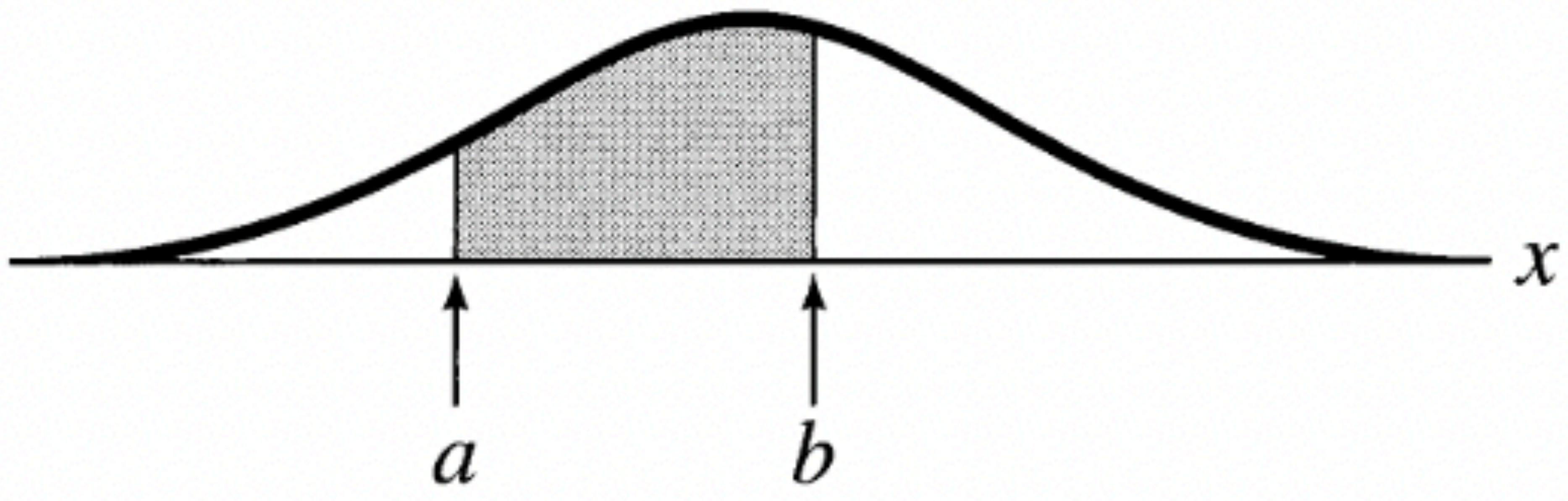
the probability that any one measurement falls within a and b

why?

What is the interpretation?



$f(x)\,dx$ = fraction of measurements that fall between x and $x + dx$.

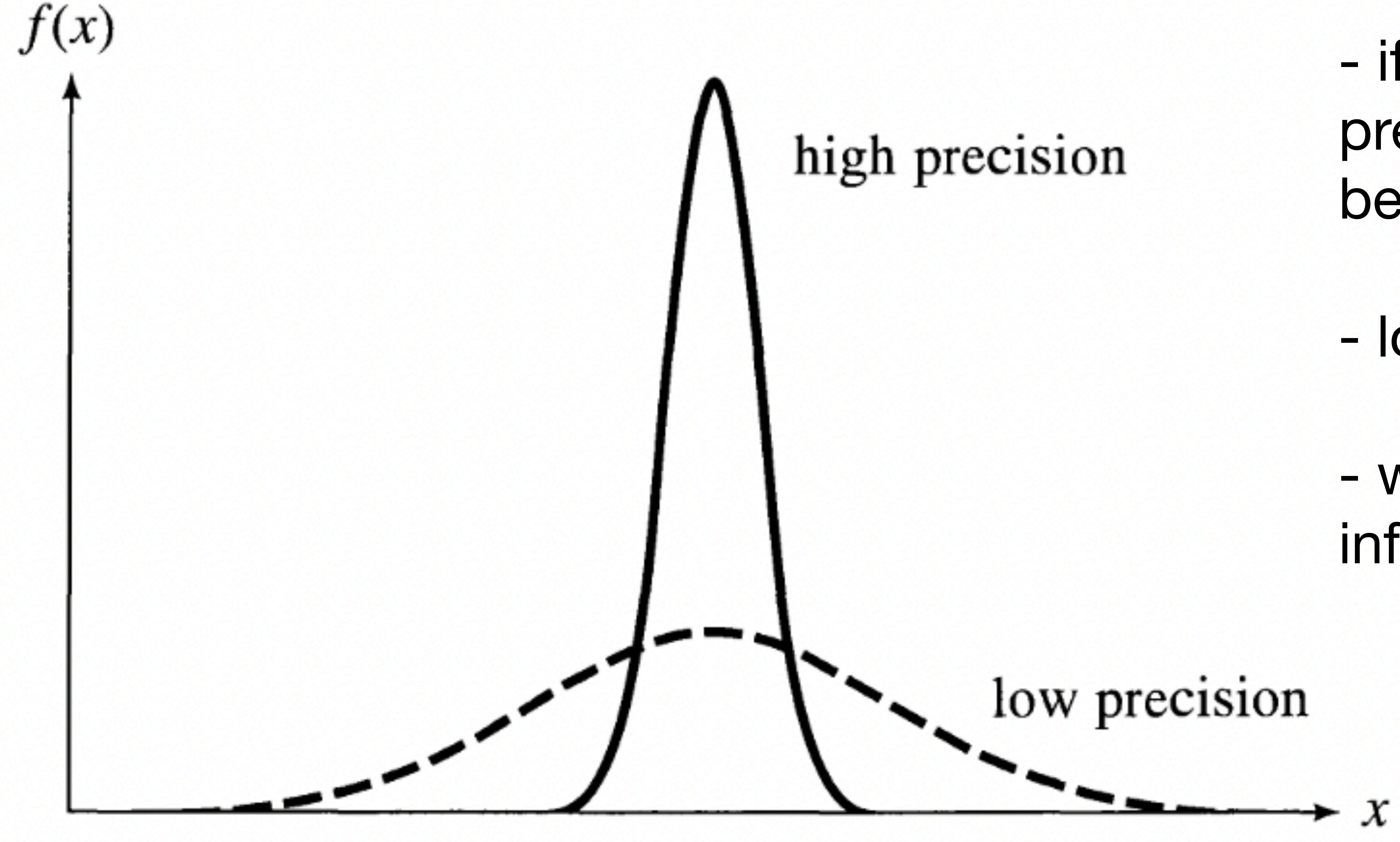


$\int_a^b f(x)\,dx$ = fraction of measurements that fall between $x = a$ and $x = b$.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

f(x) is know as the probability density function (PDF)

The limiting distribution (PDF) tells us a lot!



- if the measurement is precise, the distribution will be narrow

- low precision — long tails

- we can “pull” important info from the distribution

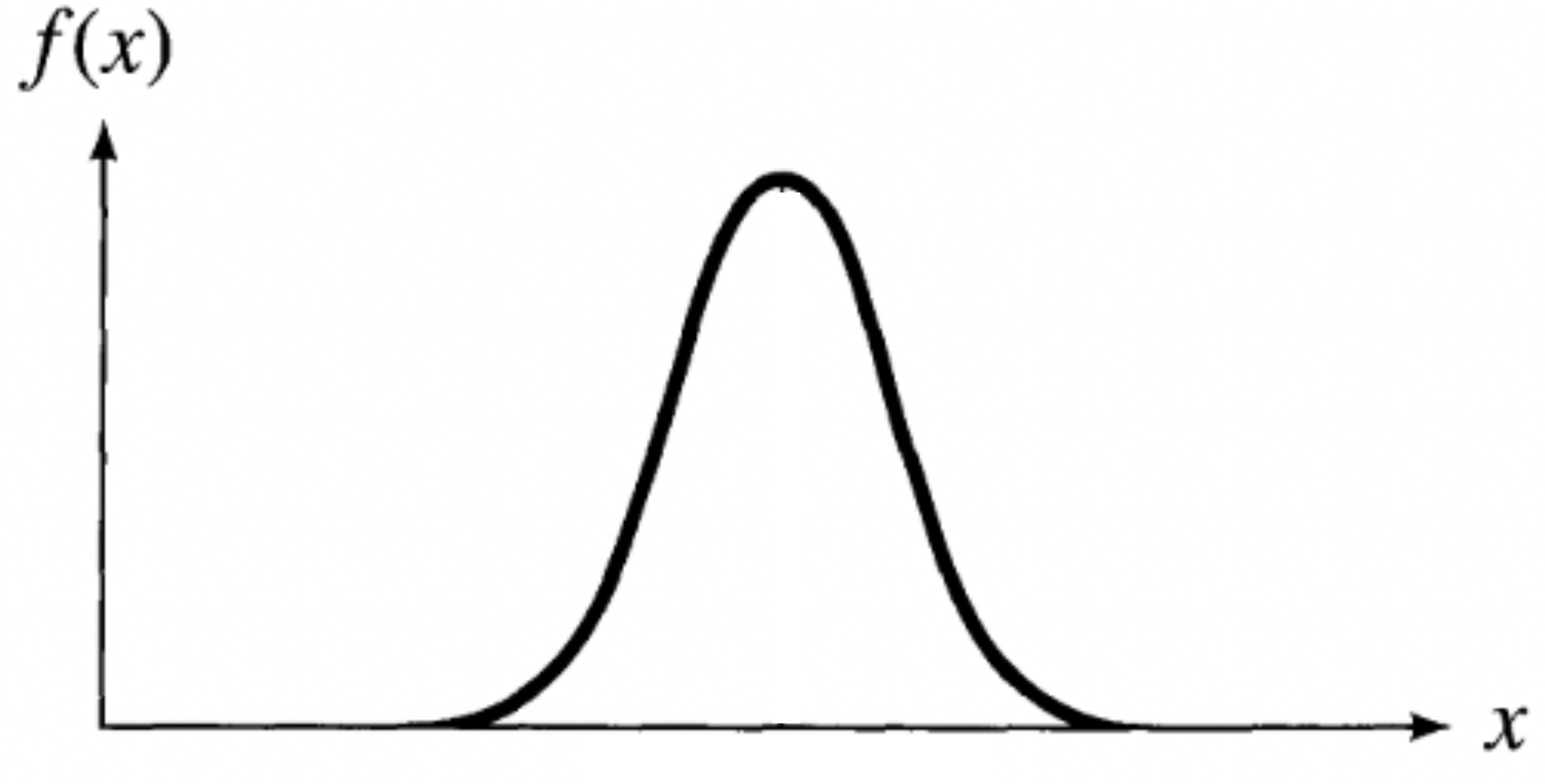
We can get mean and variance directly from PDF

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx \qquad \sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) dx$$

Remember: these give expected mean / standard deviation after infinite measurements

These are true regardless of f(x) (e.g., doesn’t have to be normal)!

The Normal Distribution — most important ‘limiting distribution’



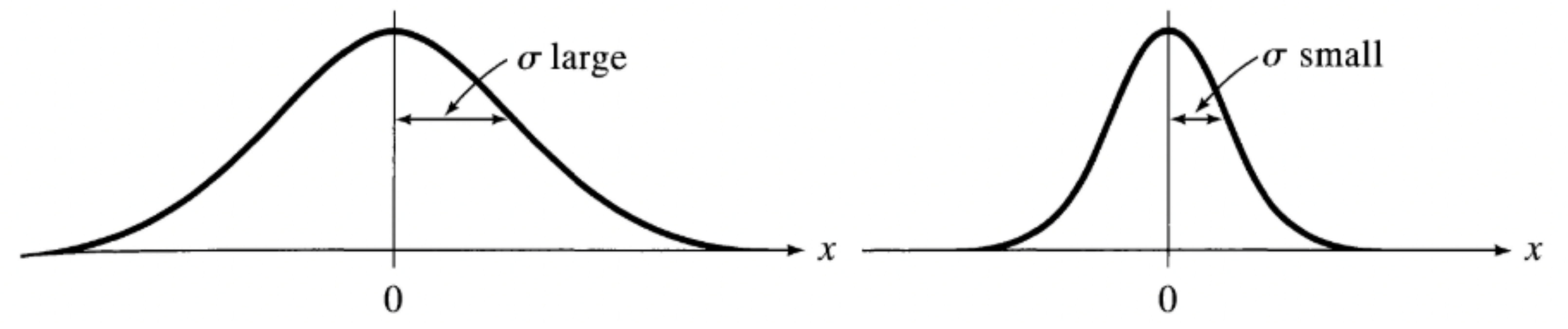
If a measurement is subject to many small sources of random error and negligible systematic error, the limiting distribution will be the bell shaped normal curve

What is the ‘true value’ of x?

What do we mean by “true value”?

“True Value” is an ‘idealize’ quantity (like a mathematician’s point)
the value that one approaches as increasing measurements are made

The Normal Distribution — most important 'limiting distribution'



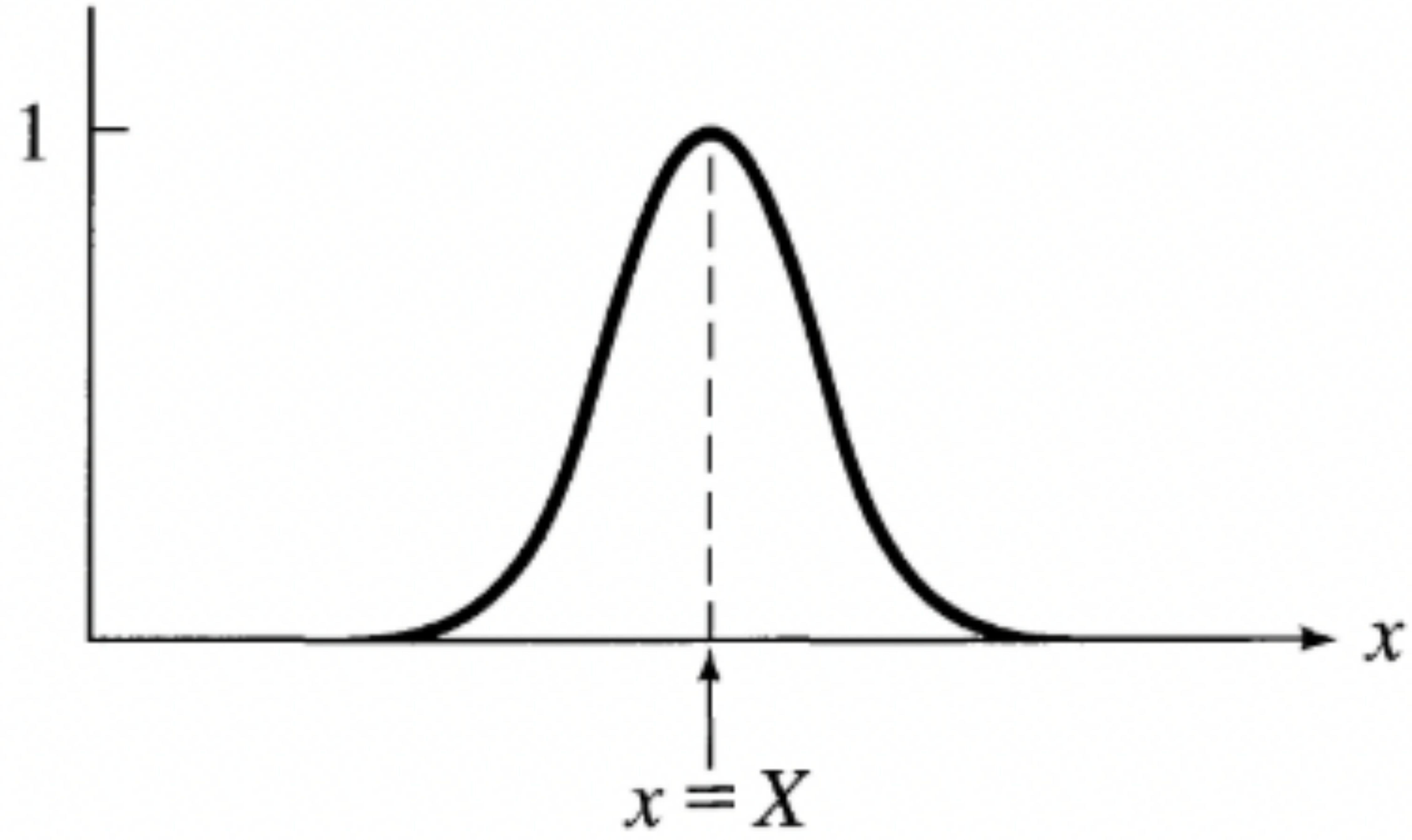
Gauss's Function

$$e^{-x^2/2\sigma^2}$$

'sigma' - width parameter

- Gauss's Function is symmetric about x=0
- tends towards zeros as x increases or decreases
- 'sigma' determines how fast/slow curves tends to zero

The Normal Distribution — most important 'limiting distribution'



Gauss's Function - shifted

$$e^{-(x-X)^2/2\sigma^2}$$

We can replace 'x' with 'x - X' to center Gauss's curve on non zero 'x'

The Normal Distribution — most important 'limiting distribution'

All 'limiting distributions' should be normalized such that: $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

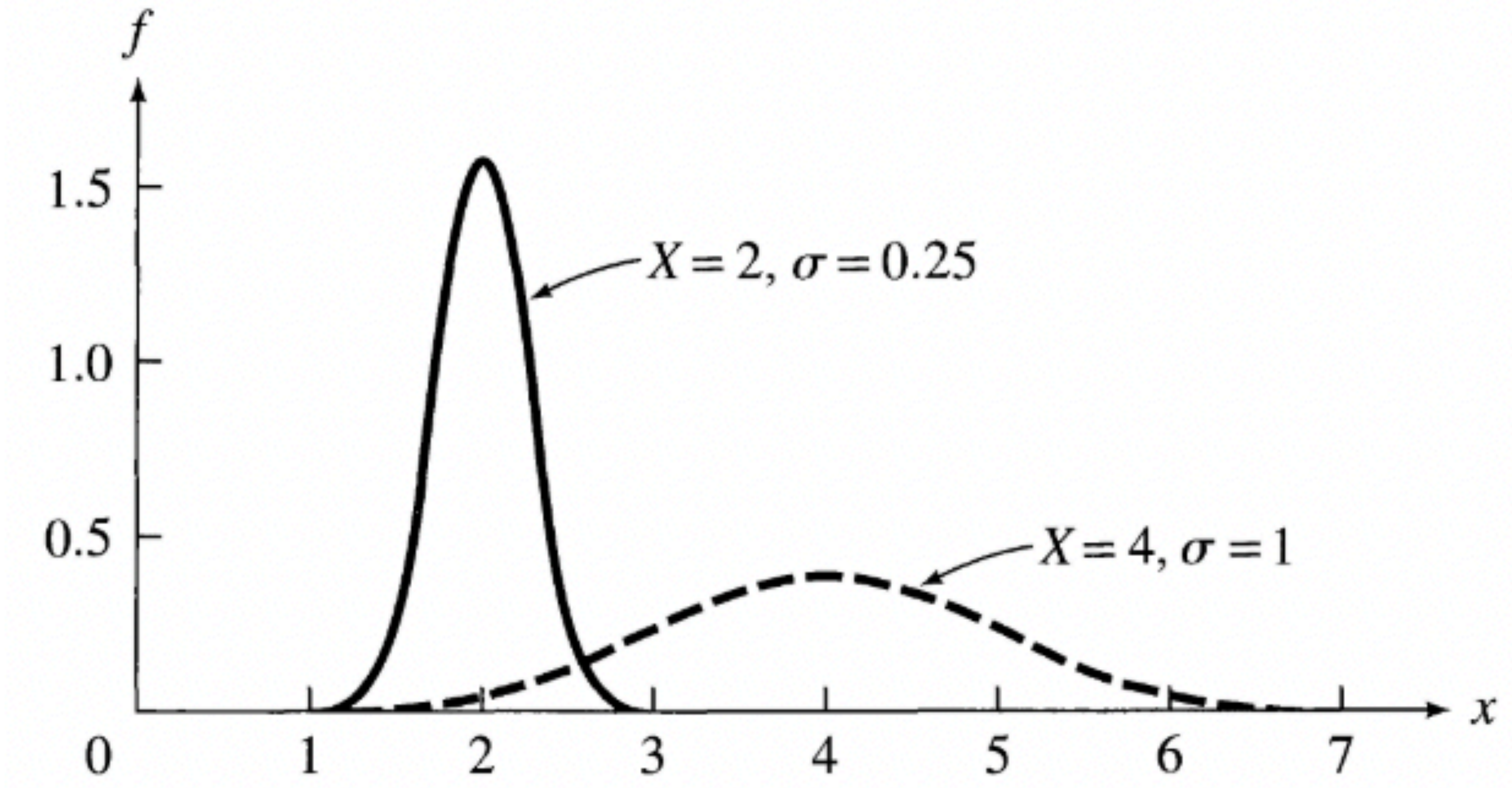
This means: $$f(x) = Ne^{-(x-X)^2/2\sigma^2}$$

With normalization factor chosen as: $$N = \frac{1}{\sigma\sqrt{2\pi}}$$

This is the 'Gaussian Distribution': $$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{(x-X)^2/2\sigma^2}$$

parameters: X, sigma — center, and width

The Normal Distribution — most important ‘limiting distribution’



Two Gaussian distributions with different ‘centers’ and ‘widths’. Tall, narrow distributions (sharp peaked) correspond to more precise measurements (since measurements fall closer together!) while broad distributions correspond to low precise measurements (measurement fall farther away from each other)

The Normal Distribution — ‘expected value’ or ‘average’

$$\bar{x} = \int_{-\infty}^{\infty} x \, G_{X,\sigma}(x) \, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \, e^{-(x-X)^2/2\sigma^2} \, dx$$

If we make the change of variables $y = x - X$, then $dx = dy$ and $x = y + X$. Thus,

$$\bar{x} = \frac{1}{\sigma\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} y \, e^{-y^2/2\sigma^2} \, dy + X \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} \, dy \right)$$

$=0$

$$= \frac{1}{\sigma\sqrt{2\pi}}$$

$$\bar{x} = X$$

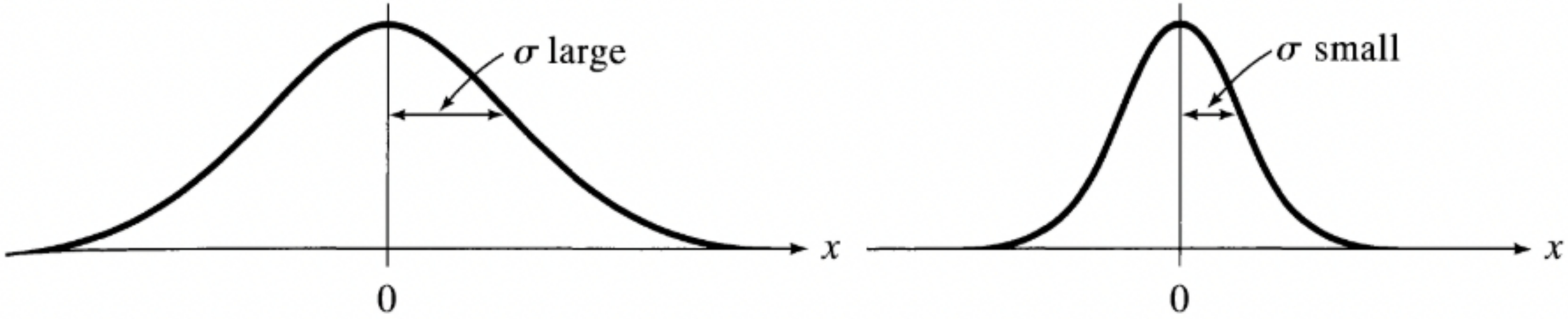
This shows that the average is exactly the ‘center’ parameter

The Normal Distribution — 'standard deviation' is the 'width'

$$\sigma_x^{\ 2} = \int_{-\infty}^{\infty} (x - \bar{x})^2 G_{X,\sigma}(x)\, dx.$$

This integral is evaluated easily. We replace $\bar{x}$ by X, make the substitutions $x - X = y$ and $y/\sigma = z$, and finally integrate by parts to obtain the result (see Problem 5.16)

$$\sigma_x^{\ 2} = \sigma^2$$



The standard deviation as 68% confidence limit

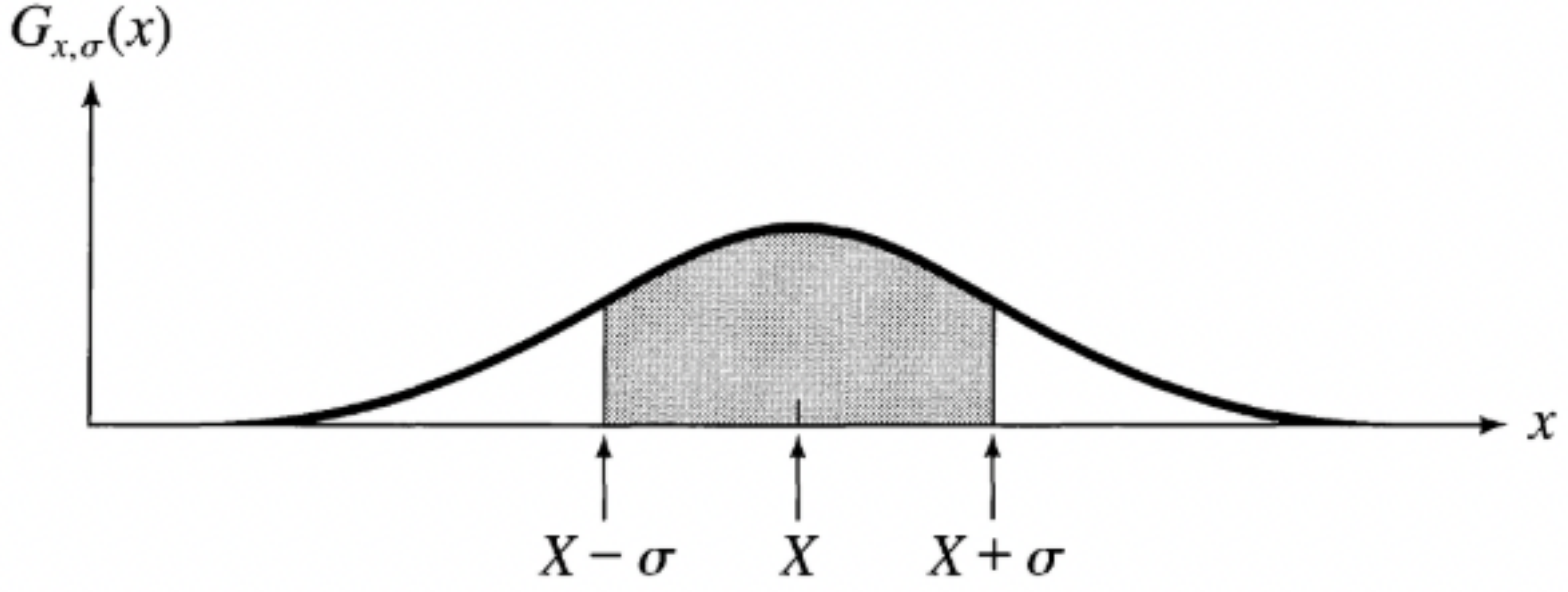
$$\int_a^b f(x)\,dx$$

the probability that any measurement falls within a <= x <= b, with any limiting distribution f(x)

What is the probability that a measurement falls within one standard deviation if the f(x) is Gaussian?

$$Prob(\text{within } \sigma) = \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{X-\sigma}^{X+\sigma} e^{-(x-X)^2/2\sigma^2}\,dx$$



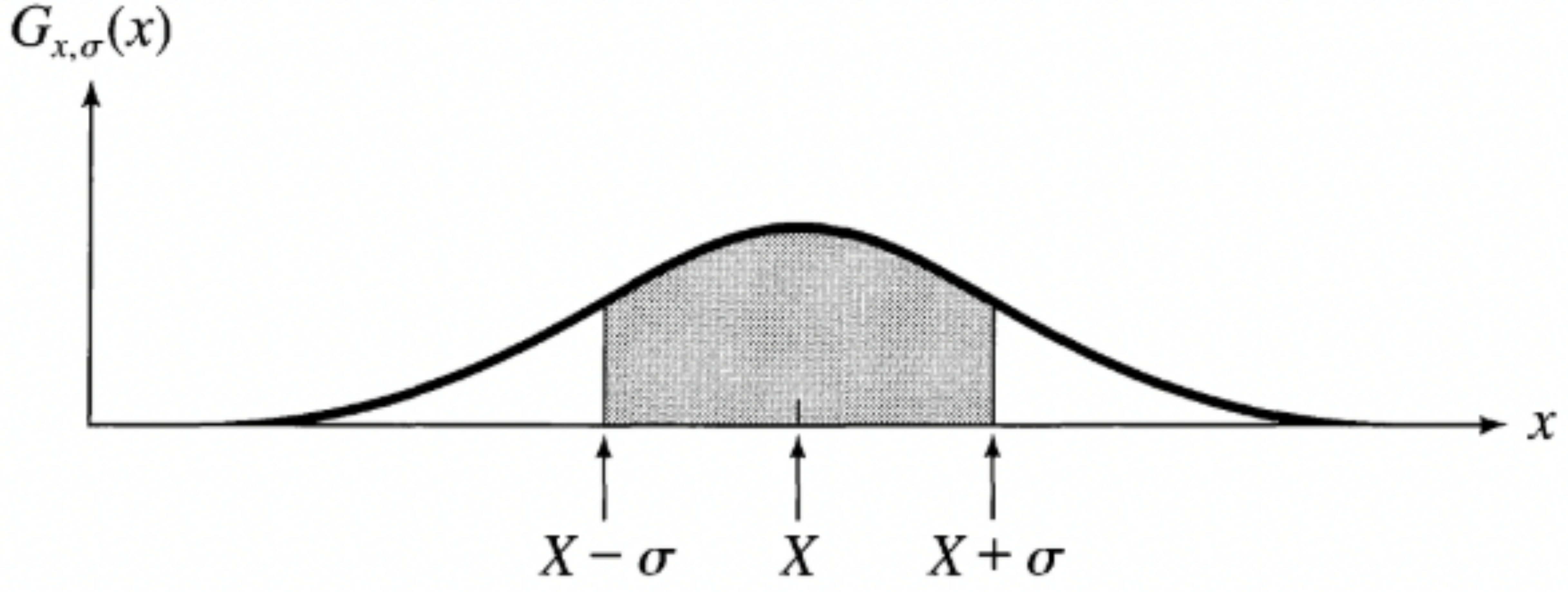
The standard deviation as 68% confidence limit

$$Prob(\text{within } \sigma) = \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{X-\sigma}^{X+\sigma} e^{-(x-X)^2/2\sigma^2}\,dx$$

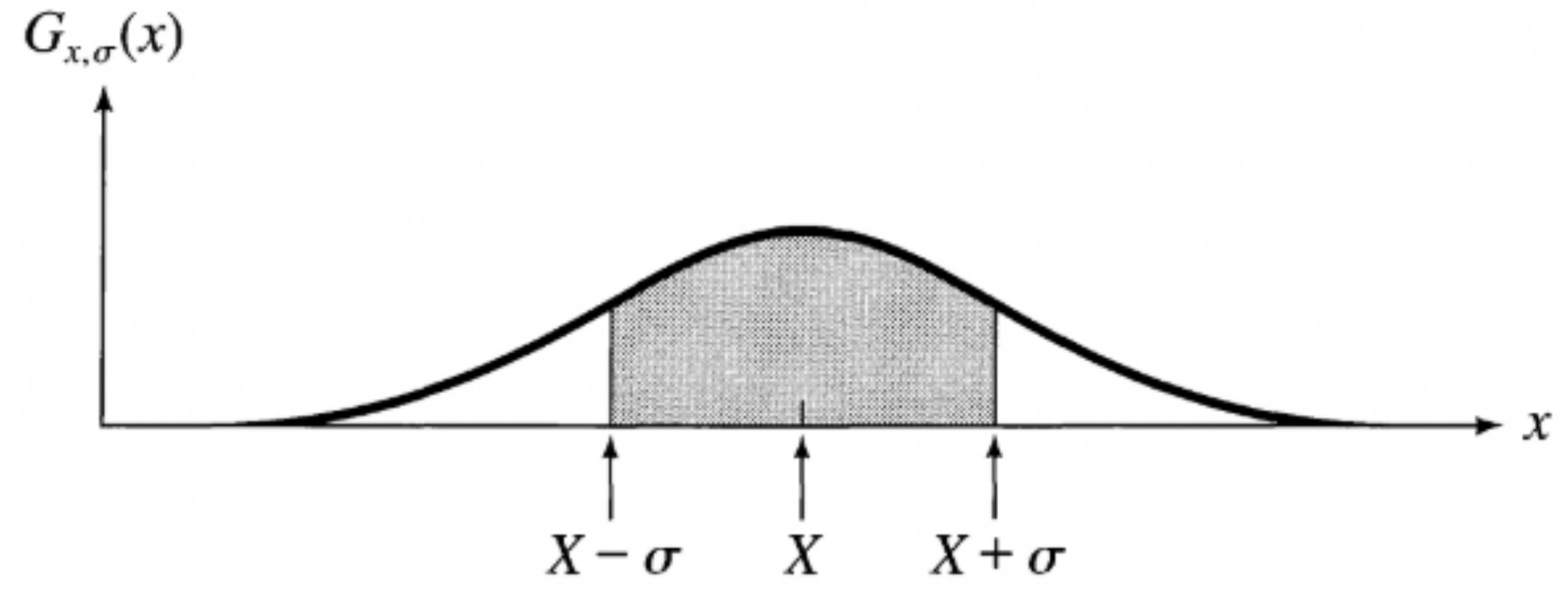
substituting $(x - X)/\sigma = z$.

$$Prob(\text{within } \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2}\,dz.$$



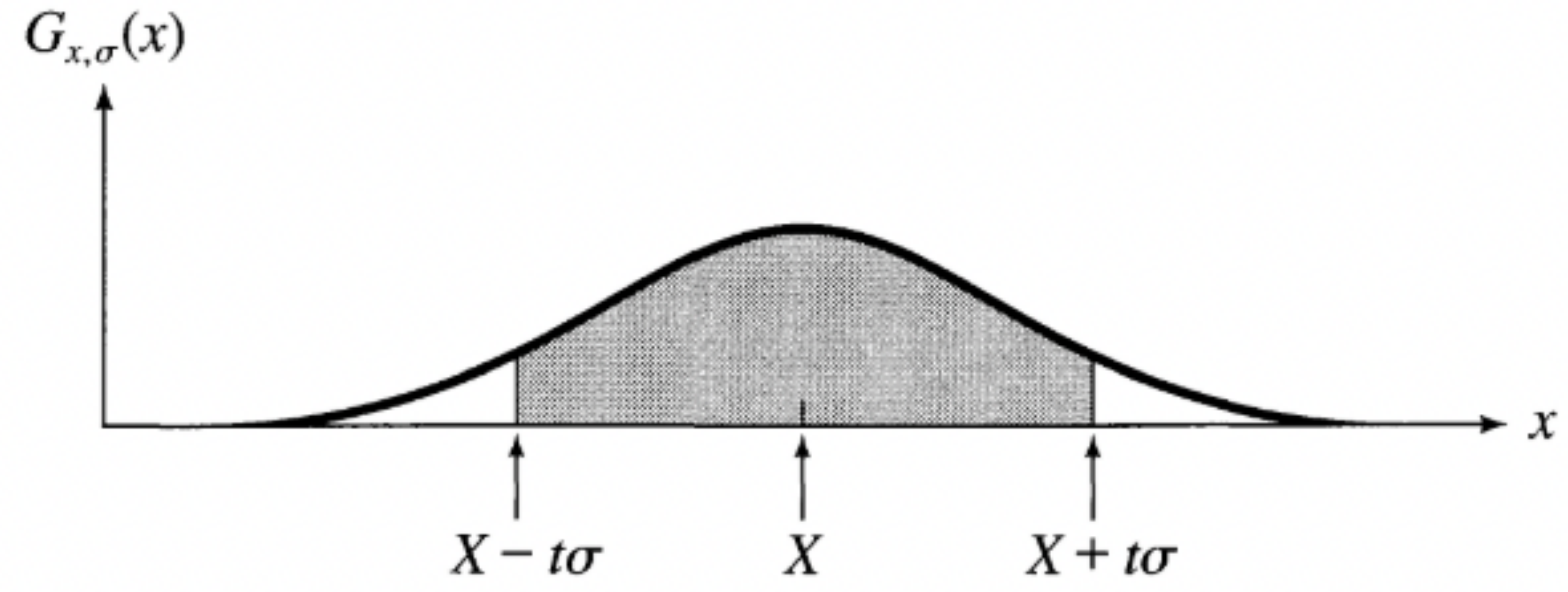
The standard deviation as 68% confidence limit

$$Prob(\text{within } \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2}\, dz.$$



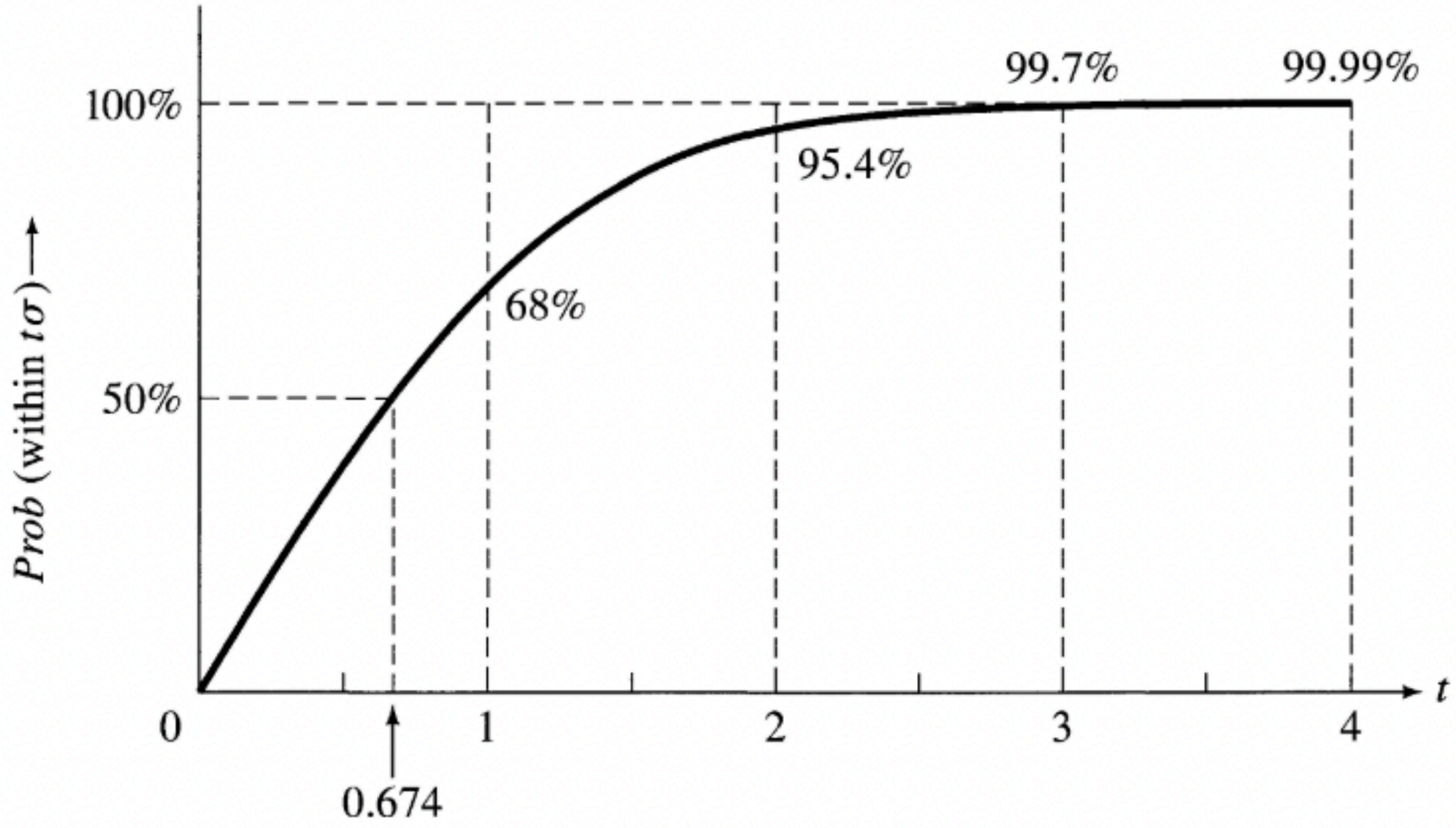
More generally, what is the probability a measurement falls within t*sigma?

$$Prob(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz$$



The error function

$$Prob(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz$$



this is an important integral in mathematical physics, called the 'error function' or 'normal error integer'

t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
Prob (%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Summary of what we’ve discussed so far:

> 'limiting distribution’ is the distribution is infinite measurements were taken

> we call this ‘limiting distribution’ f(x)

> if f(x) is known (or approximated) we can directly calculate mean and standard deviation from f(x) alone

> if the distribution is normal, than the mean x corresponds to the ‘true value’ (center) of the distribution

Main problem: we never actual know f(x), and in practice only have a finite number of measurements and our problem is to find the best estimate based on these!

Maximum likelihood estimator

$x_1, x_2, \ldots, x_N,$ data points

Suppose we <u>know</u> the 'center' and 'width' parameters of a Gaussian that describes our finite set of data points

We can estimate the probability of observing x_1 given our Gaussian parameters :

$$Prob(x \text{ between } x_1 \text{ and } x_1 + dx_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_1 - X)^2/2\sigma^2} \, dx_1.$$

$$Prob(x_1) \propto \frac{1}{\sigma} e^{-(x_1 - X)^2/2\sigma^2}$$

We can do the same for x_2 … x_n:

$$Prob(x_2) \propto \frac{1}{\sigma} e^{-(x_2 - X)^2/2\sigma^2}$$

$$Prob(x_N) \propto \frac{1}{\sigma} e^{-(x_N - X)^2/2\sigma^2}$$

Maximum likelihood estimator

We can estimate the probability of obtaining each of the readings, x_1, x_2 … x_n:

$$Prob_{X,\sigma}(x_1, \ldots, x_N) = Prob(x_1) \times Prob(x_2) \times \ldots$$

or

$$Prob_{X,\sigma}(x_1, \ldots, x_N) \propto \frac{1}{\sigma^N} e^{-\Sigma(x_i - X)^2/2\sigma^2}$$

In reality, the Gaussian parameters X and sigma can not be known!

By iteratively adjusting X and sigma to maximize the probability of observing the data we can get a good estimate of X and sigma from our data points!

Maximum likelihood estimator: summary

Given: N observations, x_1, x_2 … x_n
Find: X and Sigma, expected value (mean) and standard deviation of the limiting distributions

The best estimate, maximizes the following probability:

$$Prob_{X,\sigma}(x_1, \ldots, x_N) \propto \frac{1}{\sigma^N} e^{-\Sigma(x_i - X)^2/2\sigma^2}.$$

mle

Maximum likelihood estimates

R2022b

collapse all in page

MATLAB MLE function

Syntax

```
phat = mle(data)
phat = mle(data,Name,Value)
[phat,pci] = mle(__)
```

Justification of mean as the best estimate

$$Prob_{X,\sigma}(x_1, \ldots, x_N) \propto \frac{1}{\sigma^N} e^{-\Sigma(x_i - X)^2/2\sigma^2}.$$

When is this maximum?

when sum term is minimum!

$$\sum_{i=1}^{N} (x_i - X)^2/\sigma^2$$

When is this minimum?

differentiate with respect to x, set to zero:

$$\sum_{i=1}^{N} (x_i - X) = 0 \quad \rightarrow \quad \text{(best estimate for } X) = \frac{\sum x_i}{N}.$$

This proves that the mean is the best estimate if the limiting distribution is Gaussian!

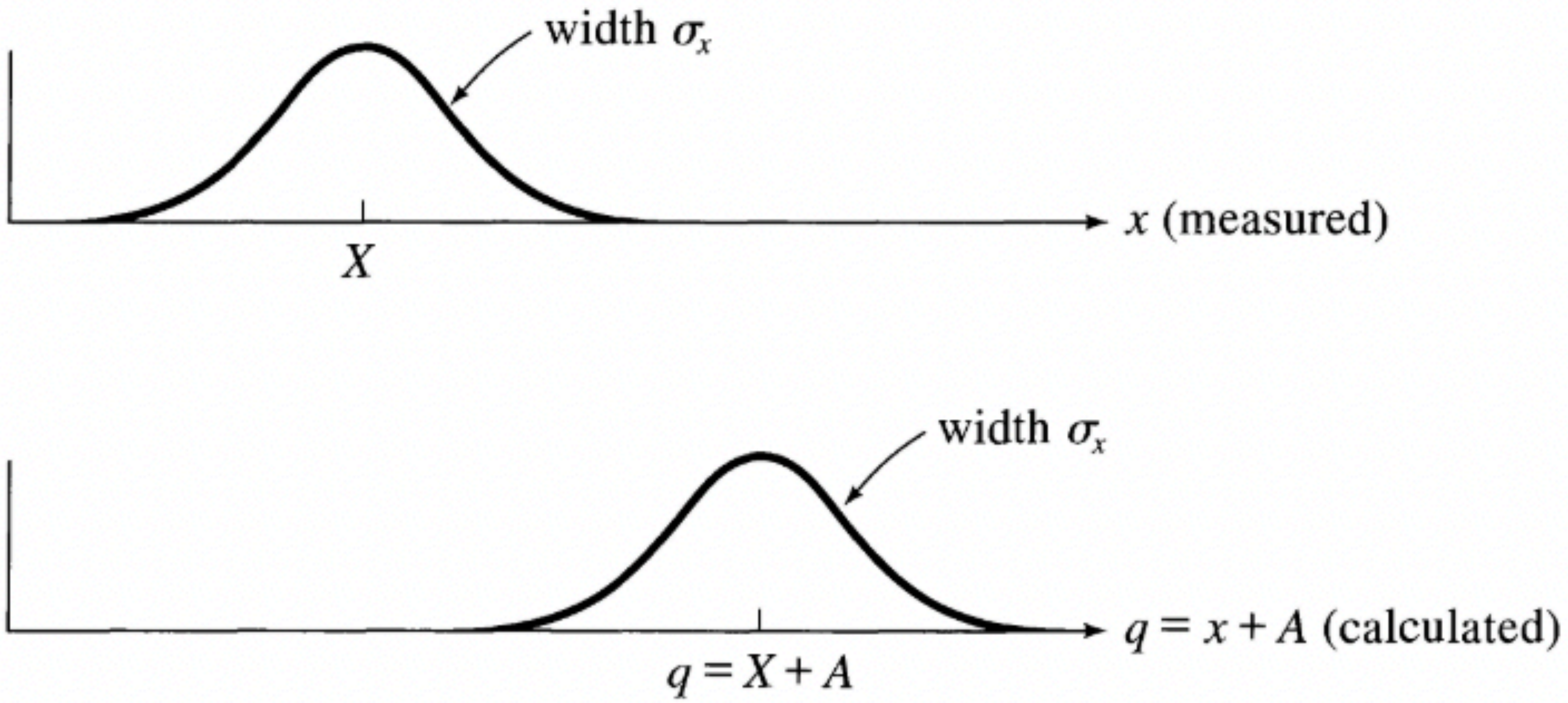
Justification of mean as the best estimate

We can use same arguments for sigma:

$$\text{(best estimate for } \sigma) = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}.$$

Let’s revisit our previous uncertainty estimate with our new framework

$$q = x + A$$ A is a fixed number with no uncertainty

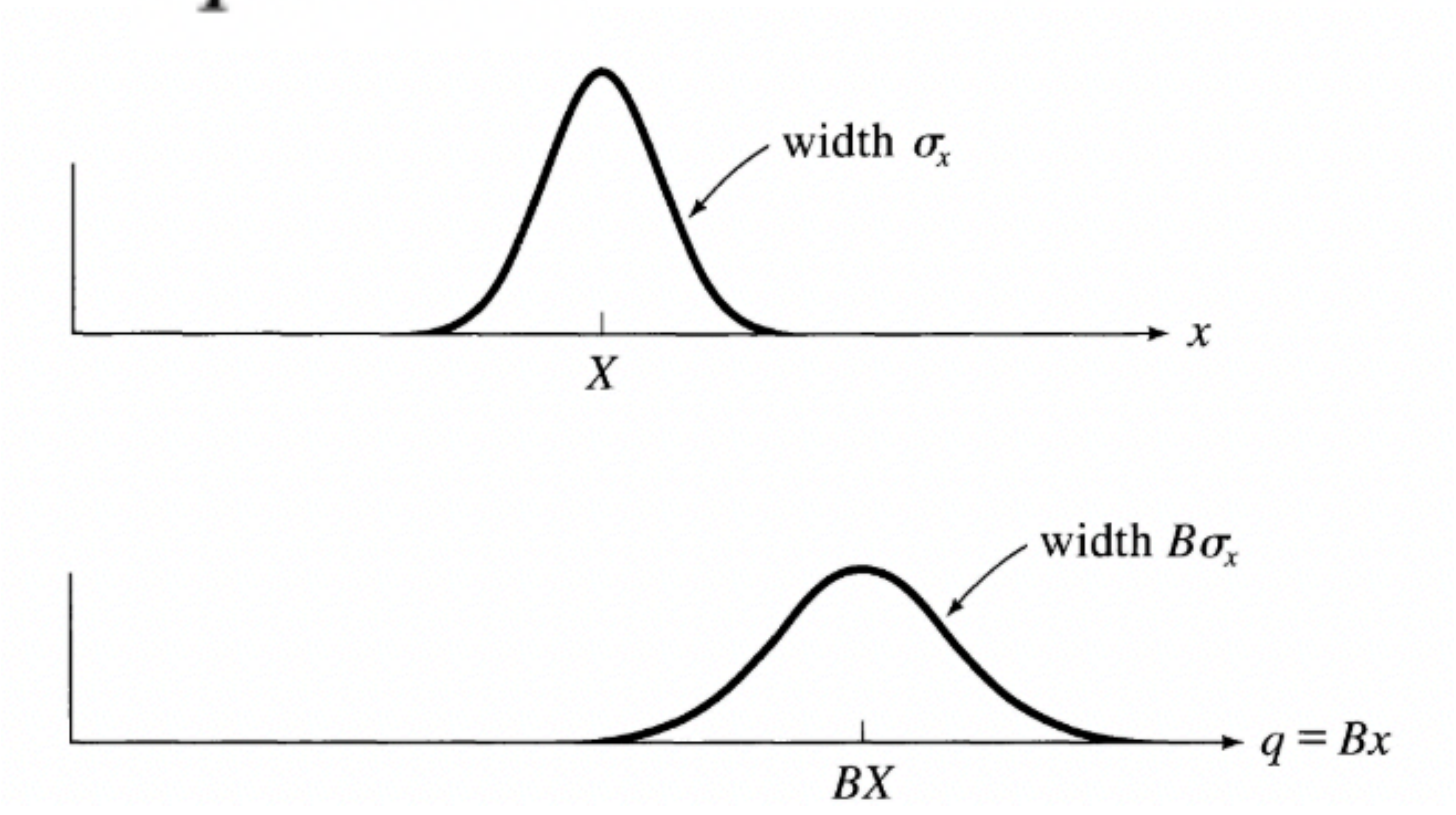


x is our measurement, but q is our experimental outcome, e.g., we need an uncertainty measure of q from x

width (sigma) doesn’t change!

Let’s revisit our previous uncertainty estimate with our new framework

$$q = Bx \qquad \text{where } B \text{ is a fixed number}$$

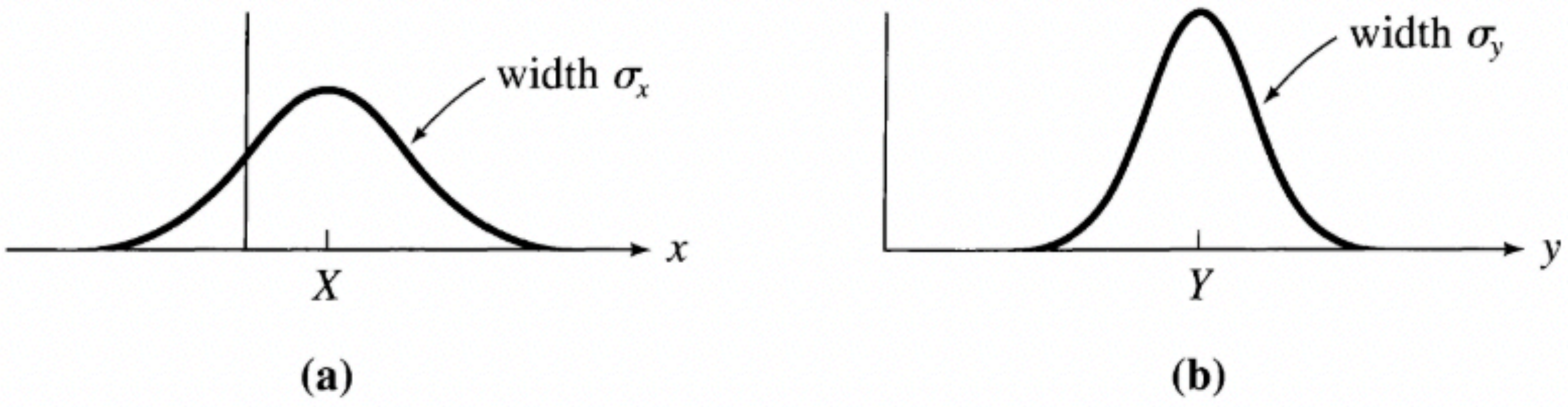


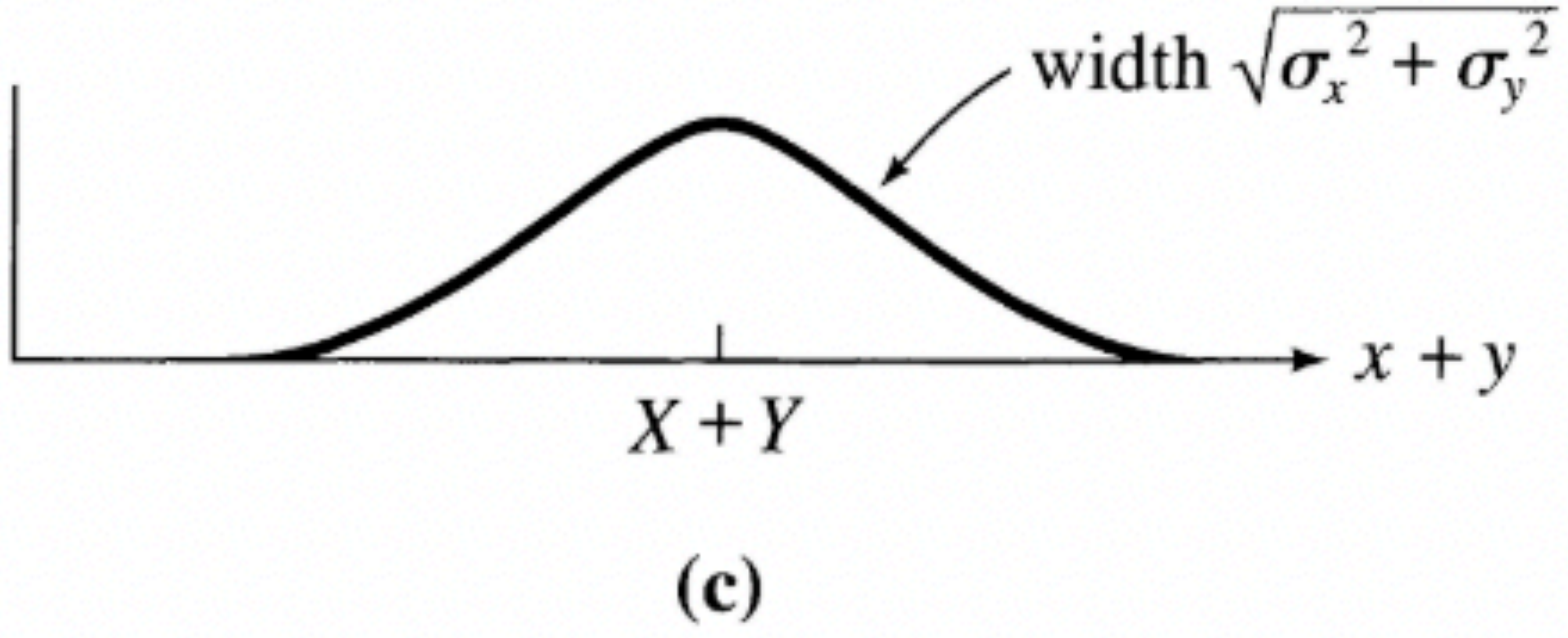
new sigma after B is B*sigma!

Let’s revisit our previous uncertainty estimate with our new framework

$$q = x + y.$$

both x and y have their own sigmas





new width = $\sqrt{\sigma_x{}^2 + \sigma_y{}^2}$

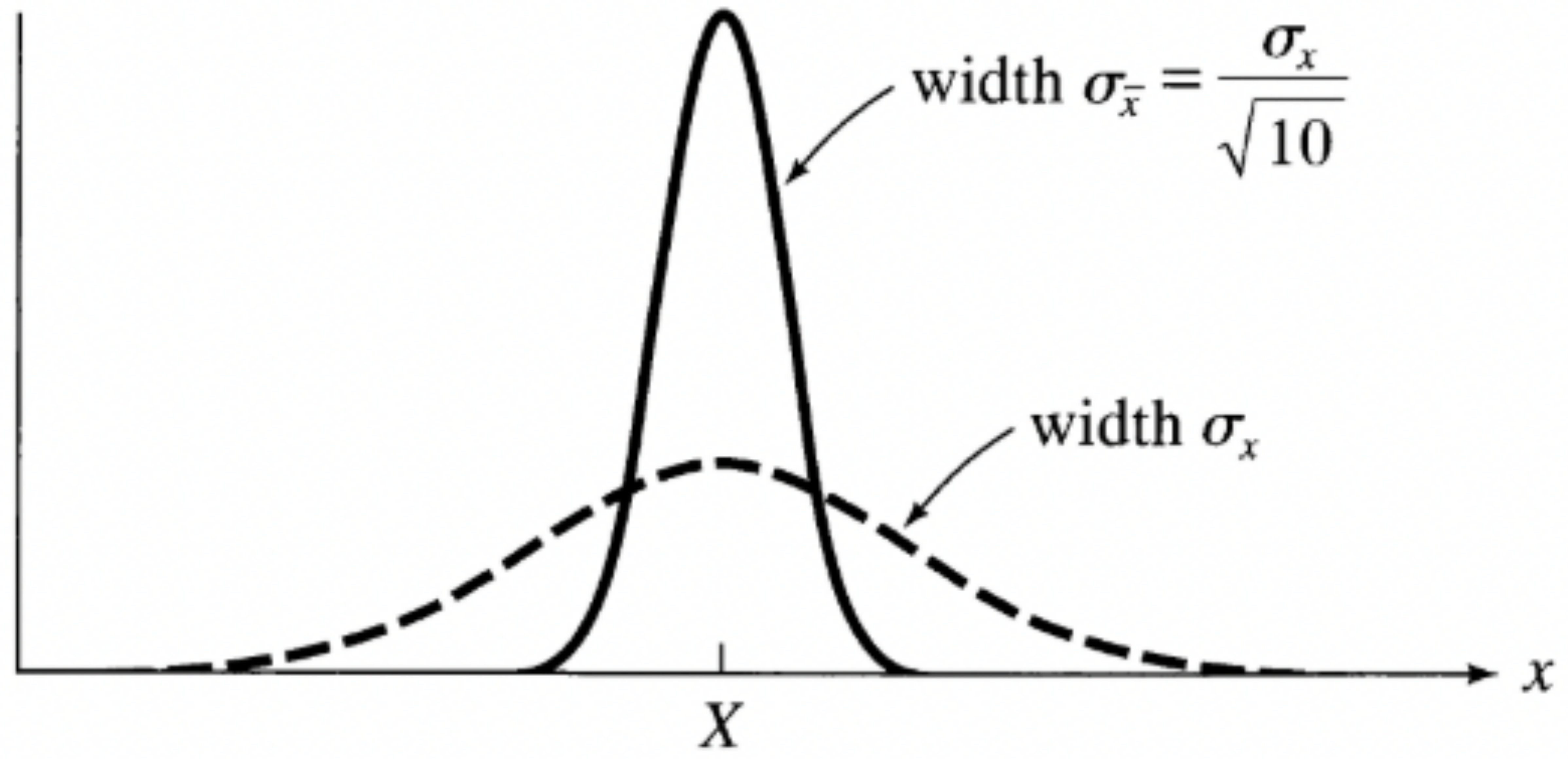
proof in book — this proves <u>addition quadrature</u> is correct

Standard Deviation of the Mean

$$\sigma_{\bar{x}} = \sigma_x/\sqrt{N}$$

recall that the SDM is best estimate of uncertainty from N measurements

This can be proved directly (in the book). Take away:



Summary

If we measure a quantity x many times, the mean of the measurements corresponds to our best estimate, and the standard deviation of the mean a measure of our uncertainty

$$(\text{value of } x) = \bar{x} \pm \sigma_{\bar{x}},$$

This statement means: we expect 68% of measurements, take in the same way, to fall within our estimated value

Using the Gaussian framework, we can now calculate probabilities directly. You can use this to determine if a ‘discrepancy’ is significant or not. Roughly, this is how ‘p-values’ or significance is calculated is practice.