ME170b Lecture 6 3/3/23

Experimental Techniques

Last time:

> Normal Distribution

Today:

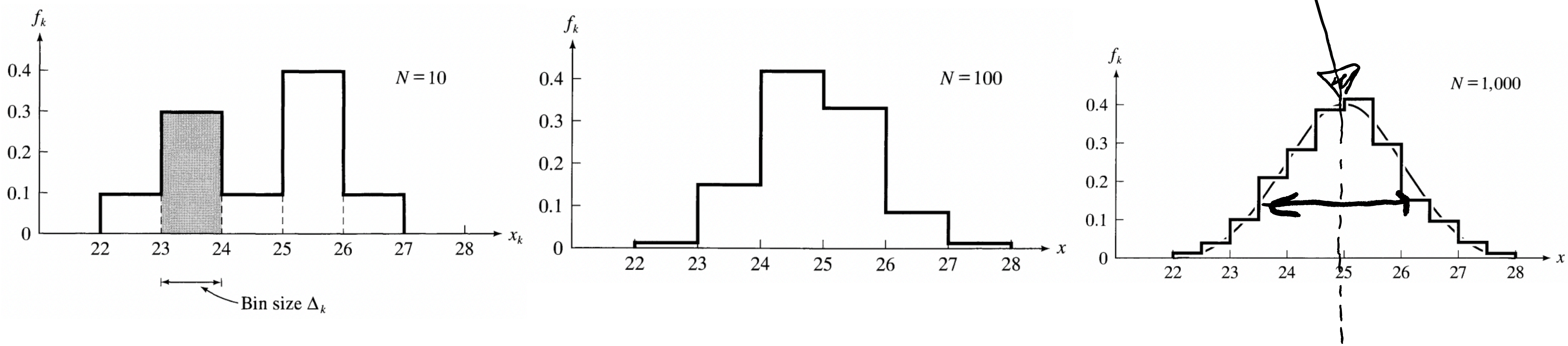
> Ch.6/7/8

> Rejection of data
> Weighted Averages
> Least Squares

Last time: Limiting Distributions

true value, X

σ



Key idea: As $N \to \infty$, the distribution approaches a definite continuous curve.

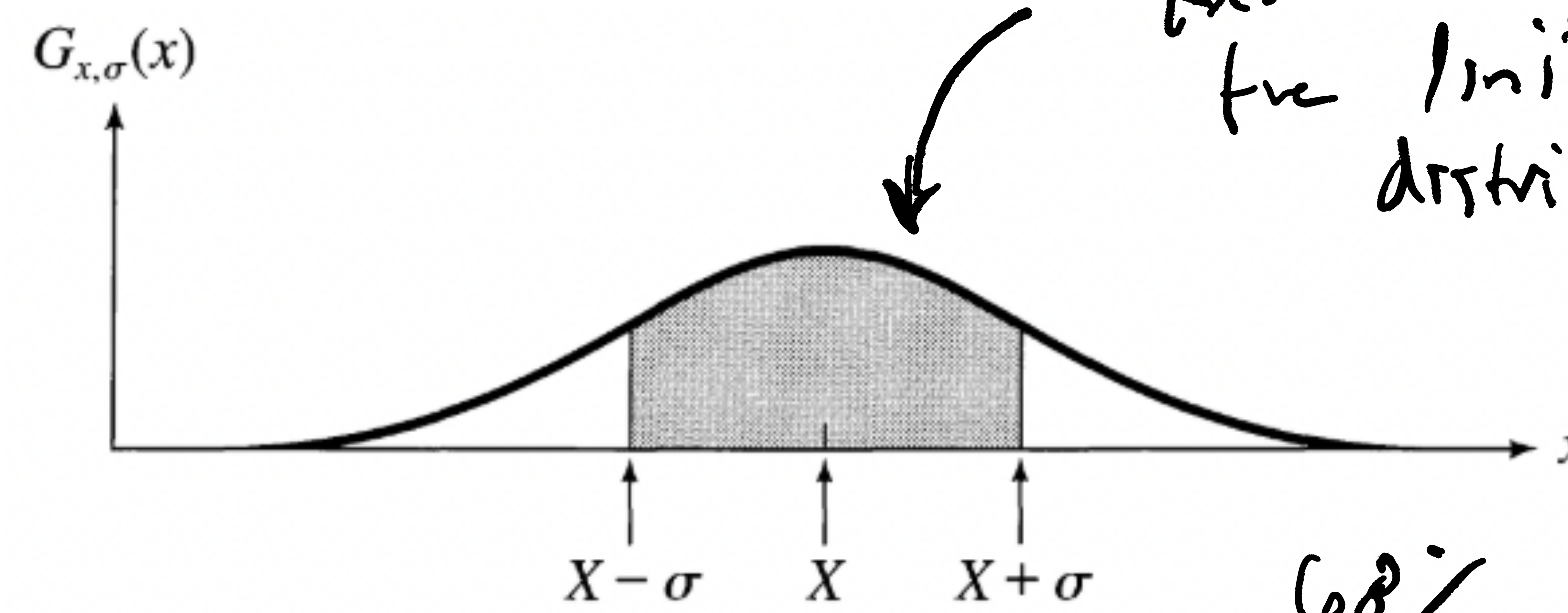
$\longrightarrow$ called "limit distribution"

When is data normal?

The standard deviation as 68% confidence limit

$$Prob(\text{within } \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2}\, dz.$$

if we know the limit distribution



68%

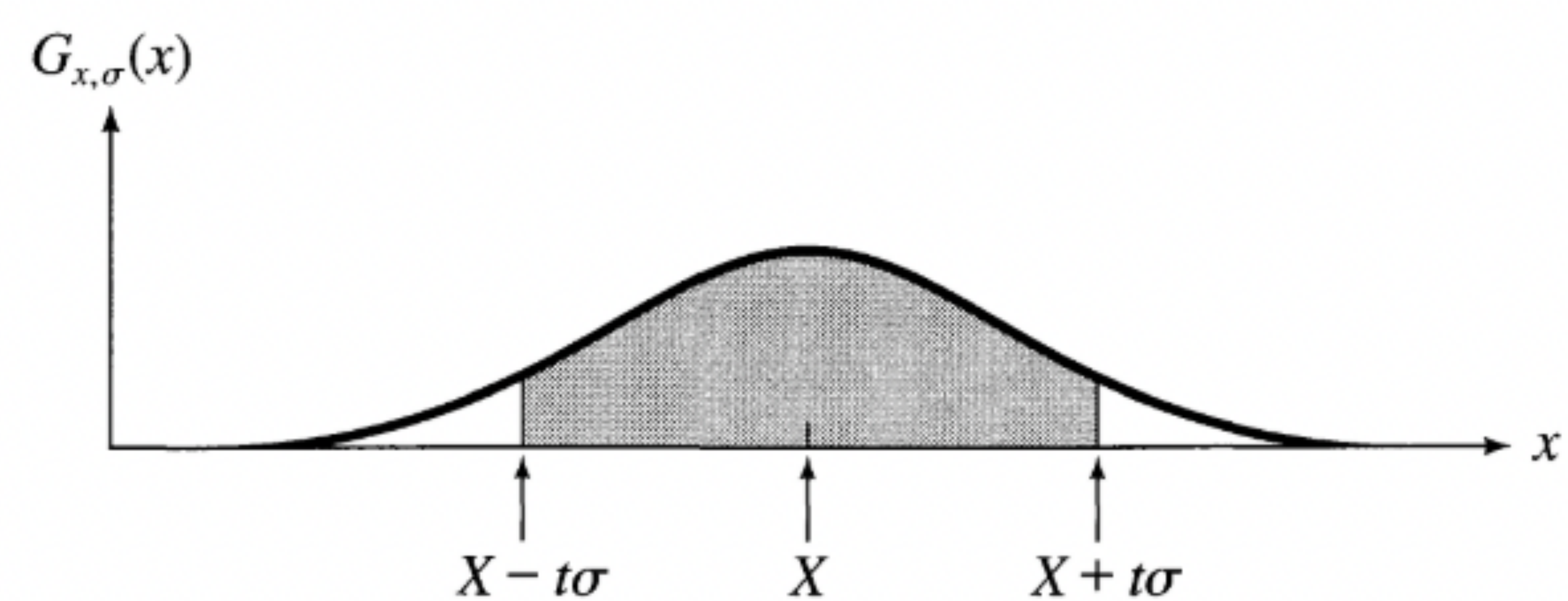
More generally, what is the probability a measurement falls within t*sigma?

$$Prob(\text{within } t\sigma) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz}_{}$$

✓

erf

error function



Gaussian probability table

0.00 0.09

Table A. The percentage probability, $Prob(\text{within } t\sigma) = \int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)\,dx$ as a function of t.

t

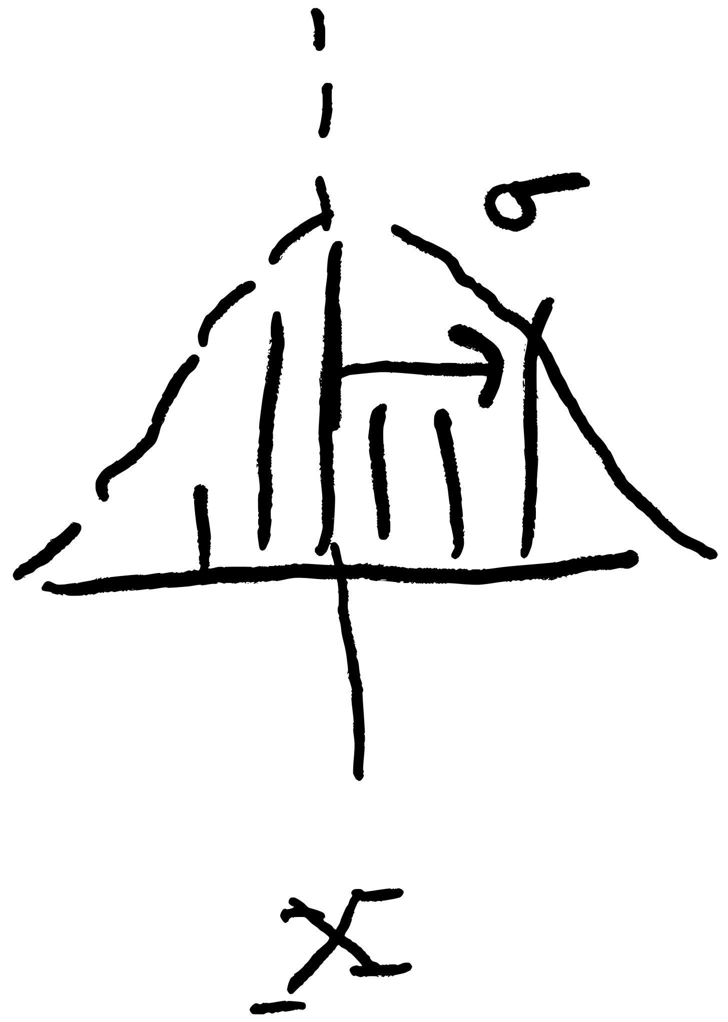
0.1

t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.2	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.8	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.0	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.1	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.2	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.5	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.6	89.04	89.26	89.48	[illegible]	89.90	90.11	90.31	90.51	90.70	90.90
1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.1	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.2	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80
2.3	97.86	97.91	97.97	98.02	98.07	98.12	98.17	98.22	98.27	98.32
2.4	98.36	98.40	98.45	98.49	98.53	98.57	98.61	98.65	98.69	98.72
2.5	98.76	98.79	98.83	98.86	98.89	98.92	98.95	98.98	99.01	99.04
2.6	99.07	99.09	99.12	99.15	99.17	99.20	99.22	99.24	99.26	99.29
2.7	99.31	99.33	99.35	99.37	99.39	99.40	99.42	99.44	99.46	99.47
2.8	99.49	99.50	99.52	99.53	99.55	99.56	99.58	99.59	99.60	99.61
2.9	99.63	99.64	99.65	99.66	99.67	99.68	99.69	99.70	99.71	99.72
3.0	99.73									
3.5	99.95									
4.0	99.994									
4.5	99.9993									
5.0	99.99994									

5.0

87.4 %

Maximum likelihood estimator



→ $x_1, x_2, \ldots, x_N,$ data points

What is the best $\bar{x}$, & σ for our data?

Suppose we know $\bar{x}$, & σ.

center & width parameters.

We can estimate the probability of observing x_1:

$$\text{Prob}(x \text{ between } x_1 \text{ \& } x_1 + dx_1) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x_1-\bar{x})^2/2\sigma^2}\, dx_1$$

$$\boxed{\text{Prob}(x_1) \propto \frac{1}{\sigma}\, e^{-(x_1-\bar{x})^2/2\sigma^2}}$$

We can do the same for all data points:

$$\text{Prob}(x_2) \propto \frac{1}{\sigma}\, e^{-(x_2-\bar{x})^2/2\sigma^2} \qquad \text{Prob}(x_n) \propto \frac{1}{\sigma}\, e^{-(x_n-\bar{x})^2/2\sigma^2}$$

Maximum likelihood estimator

We can estimate the probability of obtaining each of the readings, x_1, x_2 … x_n:

$$\text{Prob}(x_1) \times \text{Prob}(x_2) \times \cdots \times \text{Prob}(x_n)$$

or

$$\boxed{\text{Prob}_{\bar{x},\sigma}(x_1, x_2 \cdots x_n) \propto \frac{1}{\sigma^N} e^{-\sum (x_i - \bar{X})^2 / 2\sigma^2}}$$

In reality we don't know $\bar{X}$, σ.

MLE → Choose $\bar{X}$, & σ to maximize Prob(·)

→ you can iteratively adjust $\bar{X}$ & σ till you maxime the Prob(data).

Maximum likelihood estimator: summary

Given: N observations, x_1, x_2 … x_n
Find: X and Sigma, expected value (mean) and standard deviation of the limiting distributions

The best estimate, maximizes the following probability:

$$Prob_{X,\sigma}(x_1, \ldots, x_N) \propto \frac{1}{\sigma^N} e^{-\Sigma(x_i - X)^2/2\sigma^2}.$$

mle

Maximum likelihood estimates

R2022b

collapse all in page

MATLAB MLE function

Syntax

```
phat = mle(data)
phat = mle(data,Name,Value)
[phat,pci] = mle(__)
```

Rejection of Data - Ch.6

outlier?

3.8, 3.5, 3.9. 3.9, 3.4, 1.8

What's wrong with this data set? Why does it matter? What can we do about it?

> we can check how many std the "outlier" is

> only remove it you can justify its very unlikely real data!

Rejection of Data

3.8, 3.5, 3.9. 3.9, 3.4, 1.8

Let's calculate our an estimate of x with and without the "outlier"

mean +/- SE

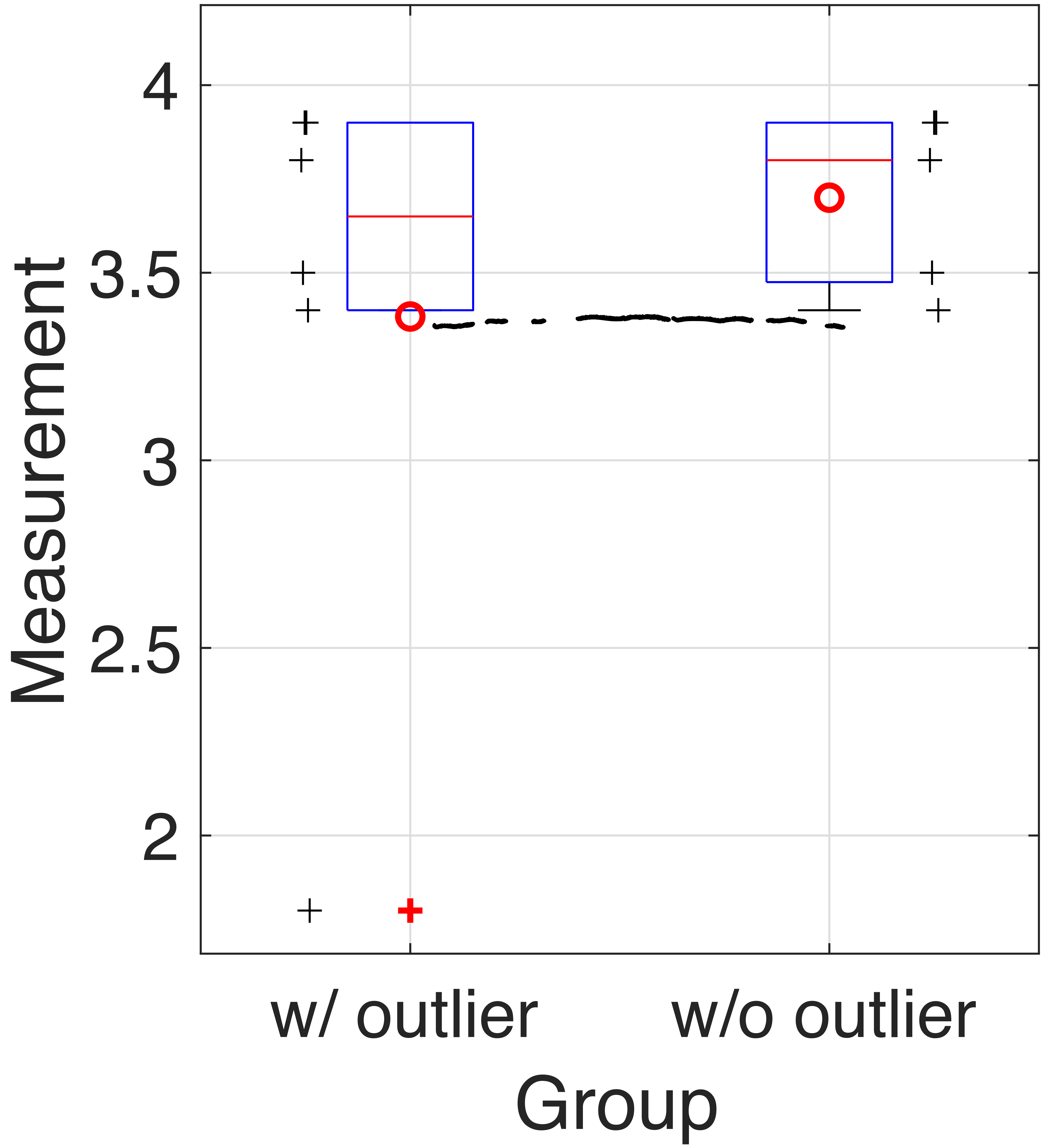
3.4 +/- 0.1 vs 3.70 +/- 0.05

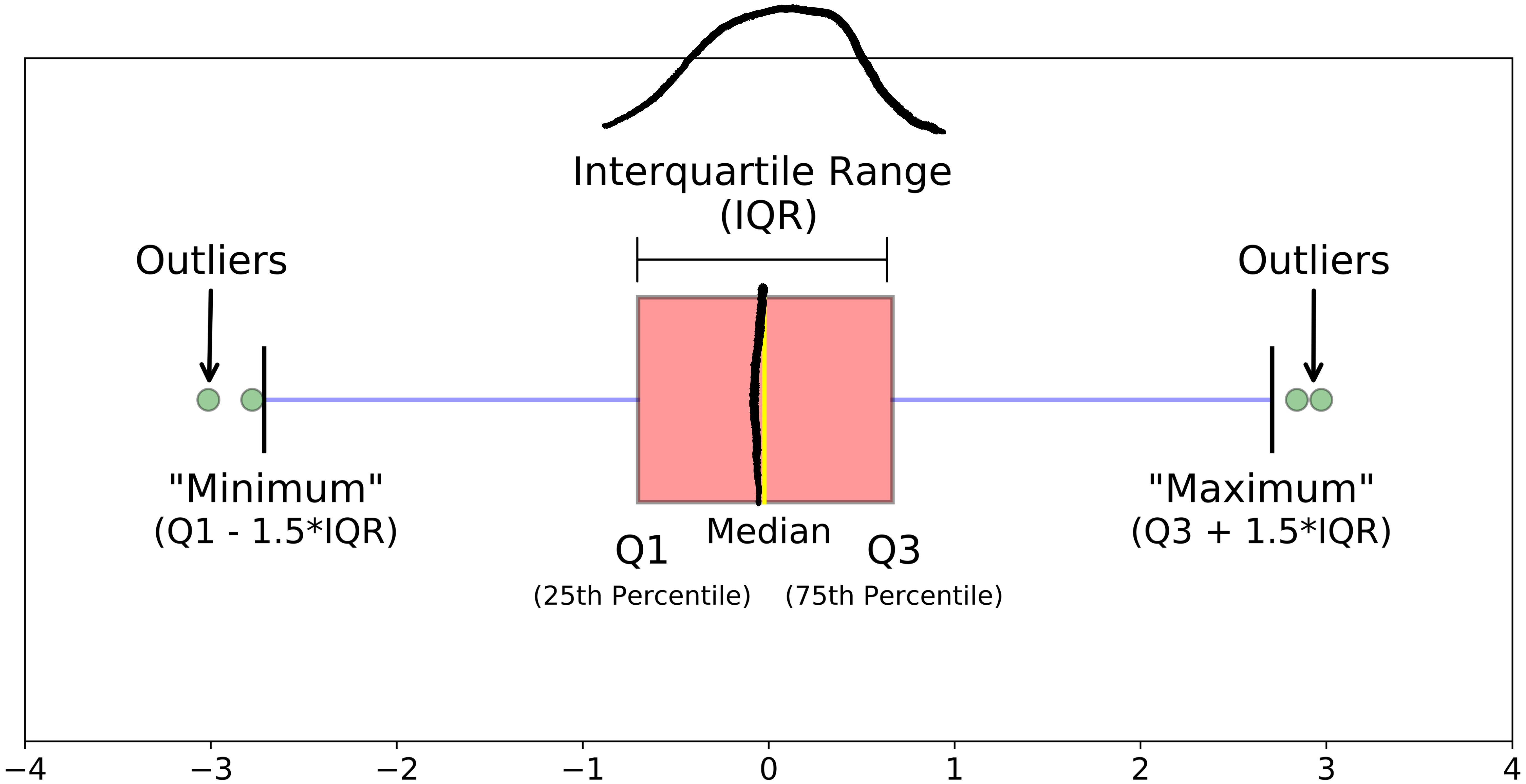
These are significantly different!

* Uncertainty doubles!

Rejection of Data

3.8, 3.5, 3.9. 3.9, 3.4, 1.8





if the outlier is fake, it has consquence on our experimental result.

Rejection of Data

3.8, 3.5, 3.9. 3.9, 3.4, 1.8

When should an experimenter "reject data"?

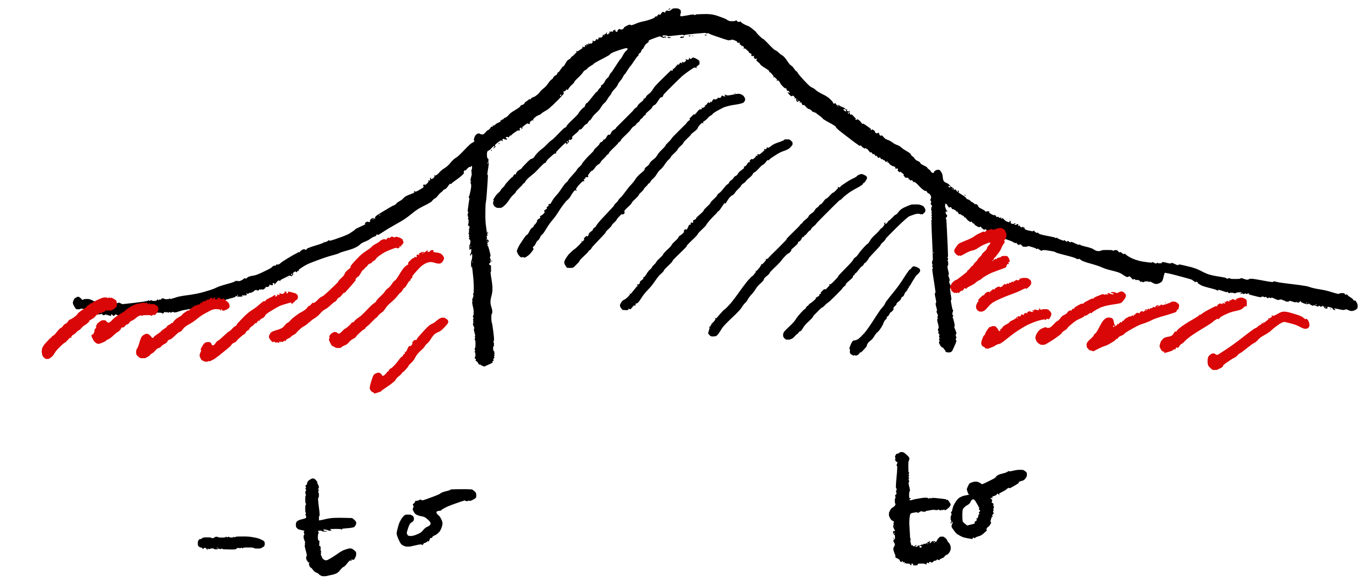
> Controversial topic!

> some say never.

> ultimately it is a subjective decision of experimenter.

Chauvenet's criterion: asses whether one piece of experimental data – an outlier – from a set of observations, is likely to be spurrious (fake).

Chauvenet's criterion



3.8, 3.5, 3.9, 3.9, 3.4, 1.8

Let's calculate the mean std:

$\bar{x} = 3.4\,s$

$\sigma_x = 0.8\,s$

Prob(outside 2σ)

$=$

$1 -$ Prob(within 2σ)

$= 1 - .95 = 5\%$

What's the probability of obtaining the outlier?

$$\text{Prob(within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz$$

$$\underbrace{3.4 - 1.8}_{\bar{x} - x_{sus}} = 1.6/\sigma = 2$$

Chauvenet's criterion

3.8, 3.5, 3.9, 3.9, 3.4, 1.8

What's the probability of obtaining the outlier measurement?

$$Prob(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz$$

3.4 - 1.8 = 1.6 = 2\sigma

$$\begin{aligned} Prob(\text{outside } 2\sigma) &= 1 - Prob(\text{within } 2\sigma) \\ &= 1 - 0.95 \\ &= 0.05. \end{aligned}$$

What does this mean?

5% of measurement being as deviant as xsus

1/20 measurements should that deviant

(expected # of deviant samples on order of 1.8)

$= N \times 0.5 = 6 * 0.05$

$= 0.3$

Chauvenet's criterion: main idea

$$x_1, \ldots, x_N$$

Set a boundary do decide if data is an outlier:
if the expected number of measurements at least as deviant as the suspect measurements is less than $\underline{\underline{1/2}}$, the you should reject the suspect.

$x_1 \ldots\ldots x_N$

$$t_{sus} = \frac{|\bar{x} - x_{sus}|}{\sigma_x}$$ #1

Prob (outside $t_{sus}\sigma$)

n = (expected # of deviants on order of x_{sus})

$= N \times$ Prob(outside $t_{sus}\sigma$)

$n < 1/2 \rightarrow$ reject!

Chauvenet's criterion: what to do if you have an outlier?

> if you decide to reject x_sus
you can recalculate $\bar{x}$, σ_x without
x_{sus}

> also do analysis twice and compare
if the outlier actually effects outcome

Chauvenet's criterion: example

46, 48, 44, 38, 45, 47, 58, 44, 45, 43

$\bar{x} = 45.8$ #1

$\sigma_x = 5.1$

$$t_{sus} = \frac{|\bar{x} - x_{sus}|}{\sigma_x} = \frac{|45.8 - 58|}{5.1}$$ #2

$$= 2.4$$

$$\text{Prob(outside } 2.4\sigma) = 1 - \text{Prob(within } 2.4\sigma)$$

$$= 1 - 0.984$$

$$= 0.016$$

$$n = 10 \times \text{Prob(outside } 2.4\sigma) = \boxed{0.16 < 0.5}$$

> We should consider rejecting that piece.

Discussion: this topic is still contentious

Chavenet's criterion should be as last resort for when you cannot generate more data

Let's think about the issues, what's wrong with 'rejecting data'

→ some scientist believe data should not be removed.

→ reasonable comprise: use Chauvenet's criterion to consider reject. the do analysis twice, with / without outliers.

→ 1/2 boundary is arbitrary.

→ unless you have large number of samples, σ is very uncertain and therefor so is your criterion.

Weigted Averages — CH. 7

How can we combine two or more separate & independent measurements?

$$\text{Student } A: \quad x = x_A \pm \sigma_A$$

$$\text{Student } B: \quad x = x_B \pm \sigma_B$$

> taking the average. X not good.

Before combining measurements must check consistency

$$\text{Student } A: \quad x = x_A \pm \sigma_A$$

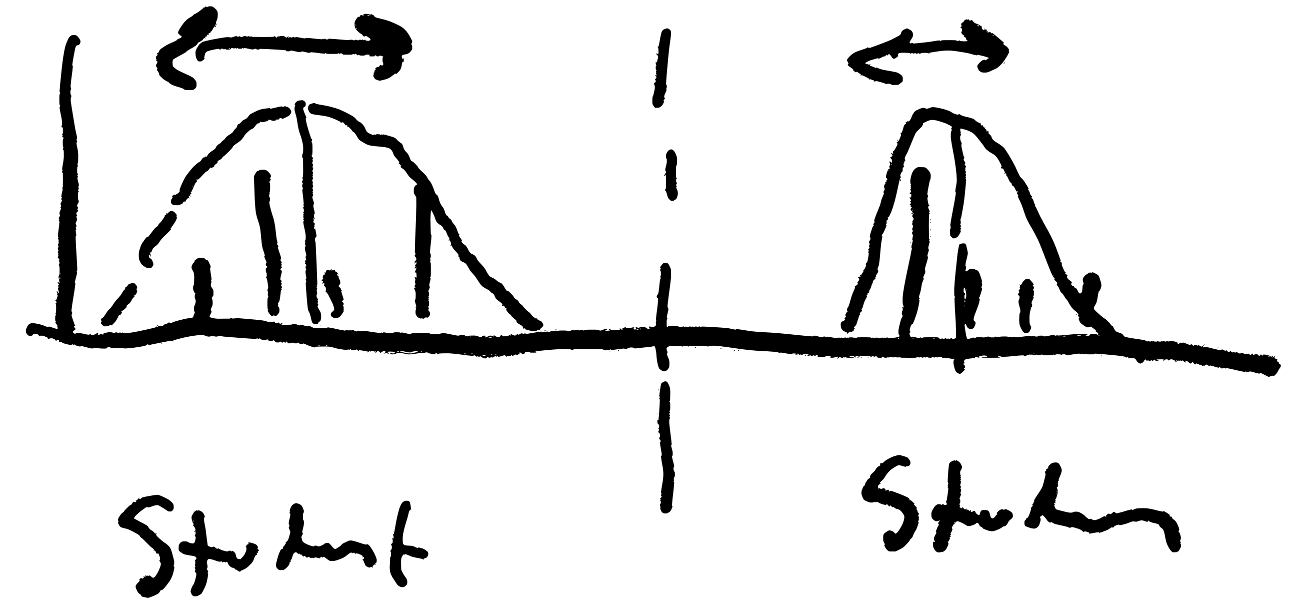
$$\text{Student } B: \quad x = x_B \pm \sigma_B$$

How?

discrepancy : $|x_a - x_b| < \sigma_a$ and σ_b

if not, the measurements are inconsistent!

Naive approach — let's just average?



$$(x_A + x_B)/2$$

Why is this not appropriate?

- unsuitable if the two uncertainties are not equal.
- gives equal importance to both measurements

but if $\sigma_a \ll \sigma_b$, we "trust" x_a more

We can use principles of maximum likelihood to solve this

assume both measurement are Gaussian

→ errors only random

$$\mathrm{Prob}_X(X_A) \propto \frac{1}{\sigma_A} e^{-(x_A - X)^2/2\sigma_A^2}$$

$$\mathrm{Prob}_X(X_b) \propto \frac{1}{\sigma_B} e^{-(x_B - X)^2/2\sigma_B^2}$$

sum of the squares of the deviation from X of two measurements

We don't know X!

$$\mathrm{Prob}_X(X_A, X_B) = \mathrm{Prob}_X(x_A) \cdot \mathrm{Prob}_X(x_B)$$

$$\propto \frac{1}{\sigma_A \sigma_B} e^{-\chi^2/2}$$

$$\chi^2 = \left(\frac{x_A - X}{\sigma_A}\right)^2 + \left(\frac{x_B - X}{\sigma_B}\right)^2$$

We can use principles of maximum likelihood to solve this

ML: our best estimate for the unknown X is the value X that maximizes the prob. of observing both x_A & x_B

$$\max \frac{1}{\sigma_A \sigma_B} e^{-\chi^2/2}$$

u^2

$2uu'$

$$\min \chi^2 = \left(\frac{x_A - X}{\sigma_A}\right)^2 + \left(\frac{x_B - X}{\sigma_B}\right)^2$$

$$\frac{d\chi^2}{dX} = 2\left(\frac{x_A - X}{\sigma_A}\right) + 2\left(\frac{x_B - X}{\sigma_B}\right) = 0$$

$$(\text{best estimate of } X) = \left(\frac{x_A}{\sigma_A^2} + \frac{x_B}{\sigma_B^2}\right) \Big/ \left(\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}\right)$$

We can use principles of maximum likelihood to solve this

$$(\text{best estimate of } X) = \left(\frac{x_A}{\sigma_A^2} + \frac{x_B}{\sigma_B^2}\right) \bigg/ \left(\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}\right)$$

$$X = x_{wav} = \frac{w_A x_A + w_B x_B}{w_A + w_B}$$

$$w_A = \frac{1}{\sigma_A^2} \qquad w_B = \frac{1}{\sigma_B^2}$$

analogy: very similar formula for the center of gravity of two bodies, where w_A & w_B are weights & x_A & x_B are positions

Quick Check 7.1. Workers from two laboratories report the lifetime of a certain particle as 10.0 ± 0.5 and 12 ± 1, both in nanoseconds. If they decide to combine the two results, what will be their respective weights as given by (7.8) and their weighted average as given by (7.9)?

Weights : 4 ; 1

weighted average : 10.4 ns

Easily generalizes for N measurements

$$x_1 \pm \sigma_1 \, , \quad x_2 \pm \sigma_2 \quad \cdots \quad x_n \pm \sigma_n$$

$$x_{wav} = \frac{\sum w_i x_i}{\sum w_i}$$

$$w_i = \frac{1}{\sigma_i^2}$$

Uncertainty of the weighted average?

we can error propogation to

determine uncertain of x_{wav}

$$\sigma_{wav} = \frac{1}{\sqrt{\sum w_i}}$$

Uncertainty of the weighted average?

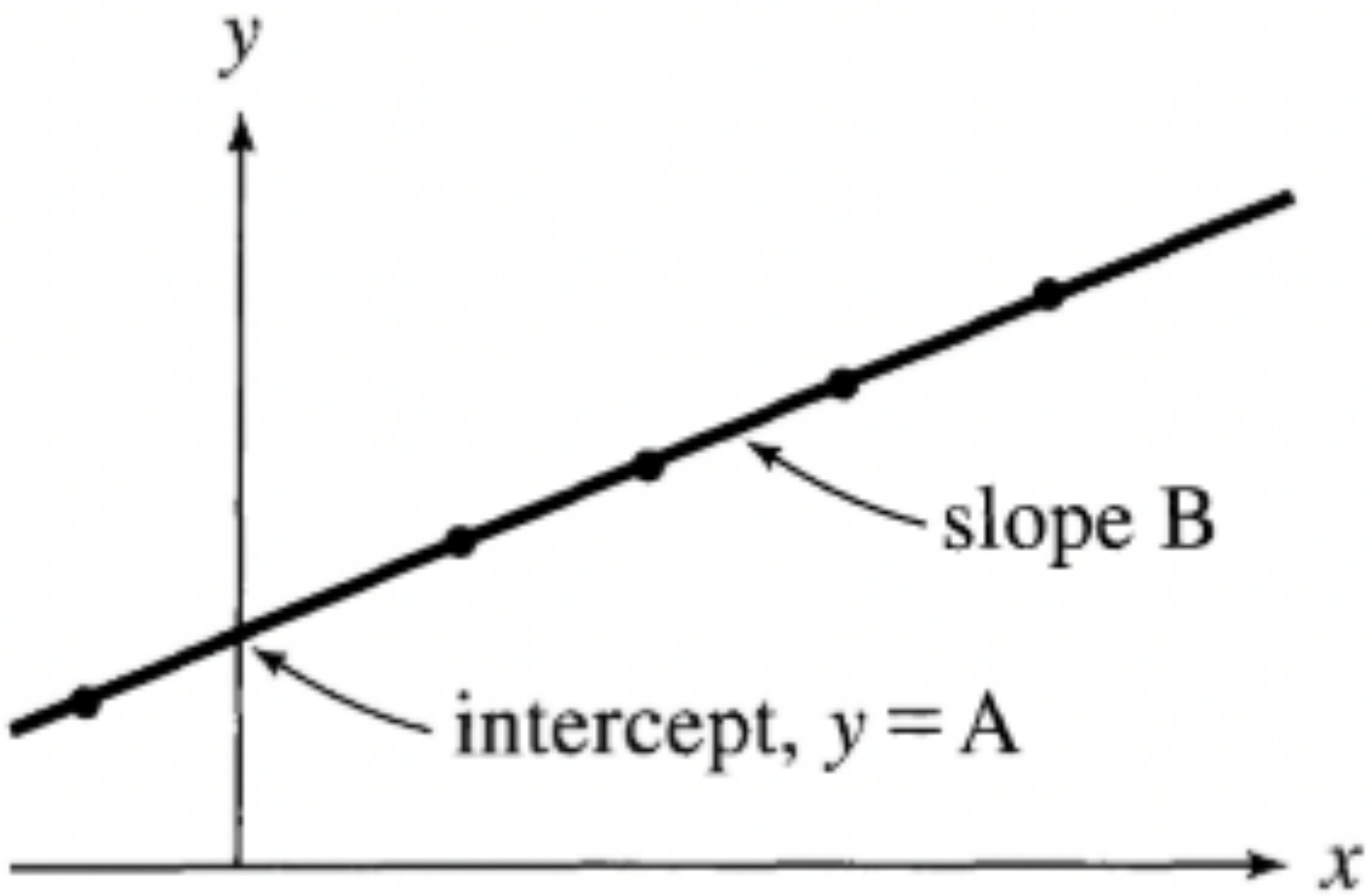
Least-Squares — Ch.8

$$\underline{y = A + Bx}$$

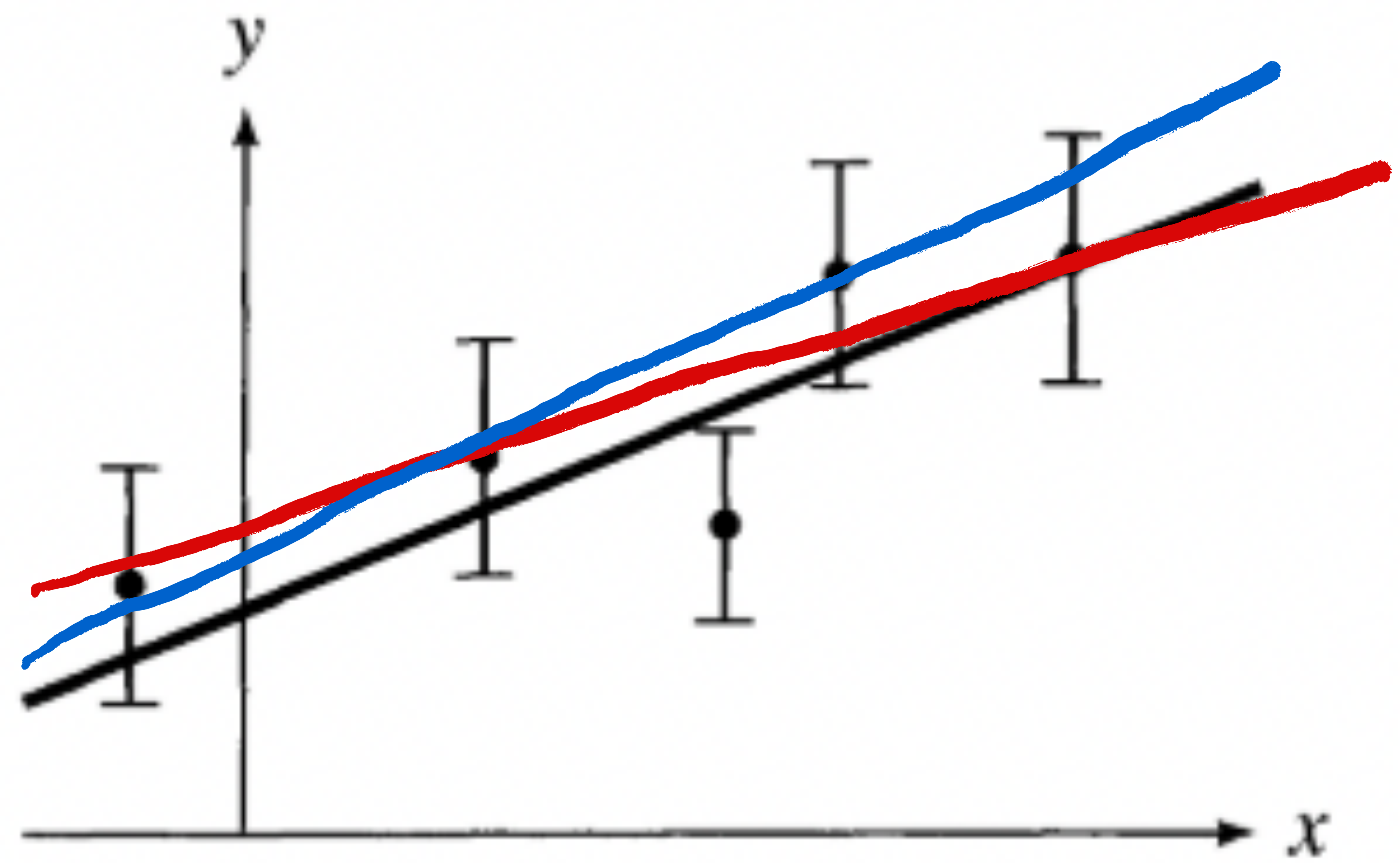
the most common & interesting types of experiment: we measure several values of two quantities to investige there relationship.

linear most important.

The least squares problem: how to find best fit?



No uncertainty –
relationship is clear



Uncertainty, we need
technique to find
"best" line

ML: we are trying to find parameters (A, B)
that best explain the data (maximizes
the probability of our observations)

What is the purpose?

$$y = A + Bx$$

① We want estimates for A & B

② A determination of whether the data $(x_i \ldots, y_i \ldots)$ a really linear

" how well does the data fit our model? Ch. 9

How to estimate A and B?

$$(x_1, y_1), \ldots, (x_N, y_N)$$

How to estimate A and B?

How to estimate A and B?

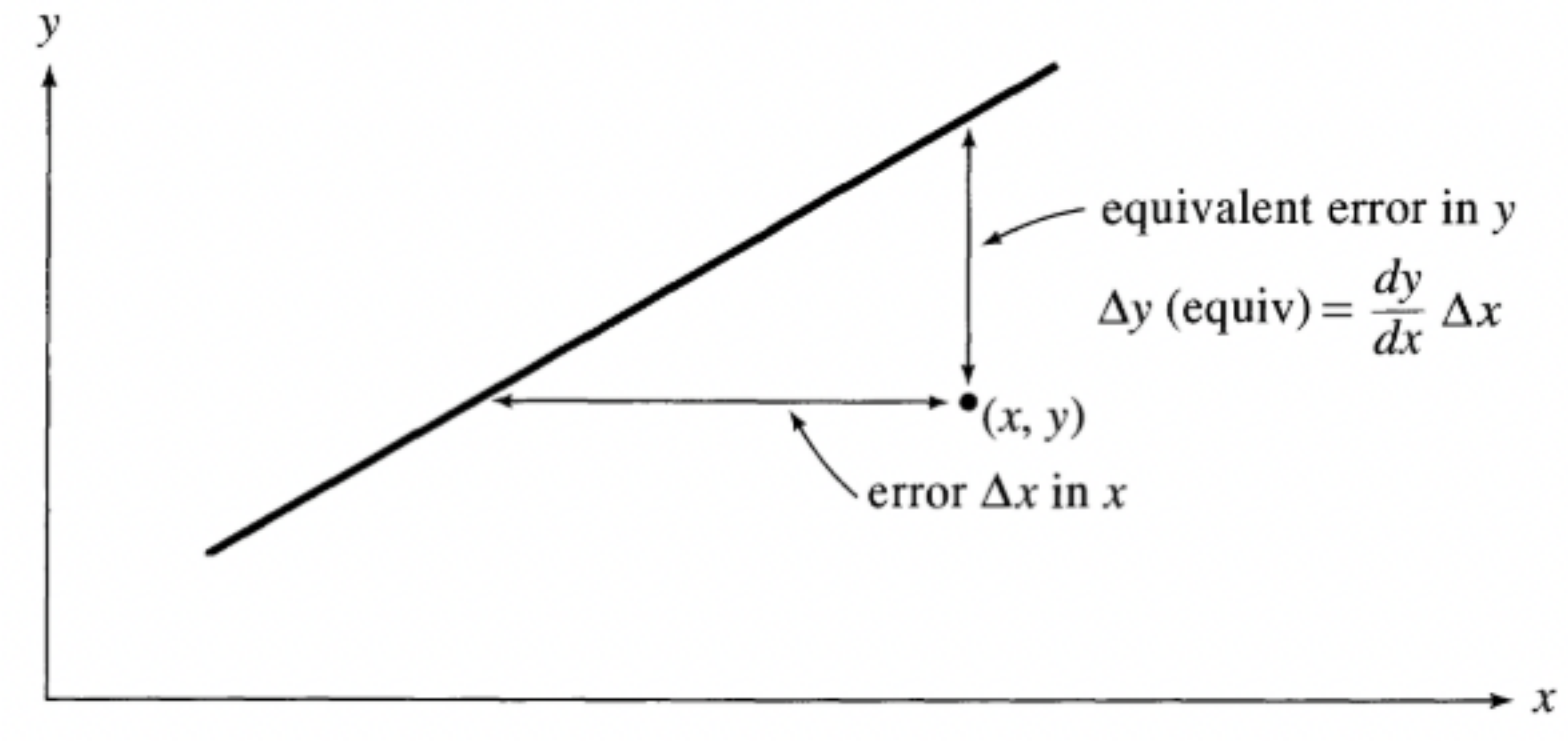
How to estimate uncertainty in y?

How to estimate uncertainty in A and B?

Some caveats

What if both x and y have uncertainties

Assume error in x only



What if both x and y have uncertainties

We can use least squares to fit nonlinear curves!

General case when least squares can fit