ME170b Lecture 3 1/27/23

Experimental Techniques

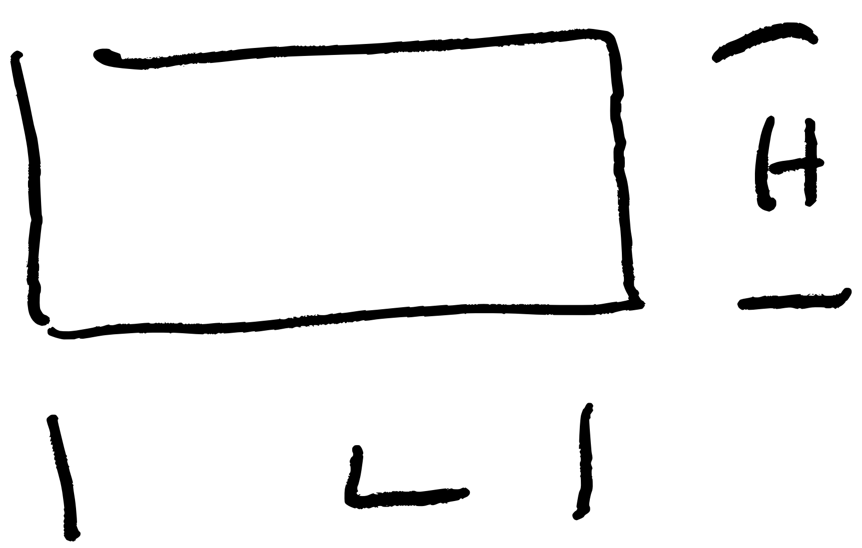
Last time:
> Standard Form $x = x_{best} \pm \delta x$
> Discrepancy
> Fractional Uncertainty
> Graphical Methods
> Difference and multiplication

Today:
> Uncertainty in measurements Review
> Square root rule
> Revisit sum/difference prod/quotient
> Independence
> uncertainty in functions
> General Forumula

Propagation of Uncertainties

→ uncertainties 'piling' up

Most physical quantities cannot be measured by single measurement.



$A = L \cdot H$

$v = \frac{d}{t}$

When measurements require two steps, uncertainties also requires two steps

1. estimate each uncert.
2. **determined** how uncert. propagated

We already discussed basic propagation

Uncertainty in a Difference
(Provisional Rule)

If two quantities x and y are measured with uncertainties δx and δy, and if the measured values x and y are used to calculate the difference $q = x - y$, the *uncertainty in* q is the *sum of the uncertainties in* x *and* y:

$$\delta q \approx \delta x + \delta y. \tag{2.18}$$

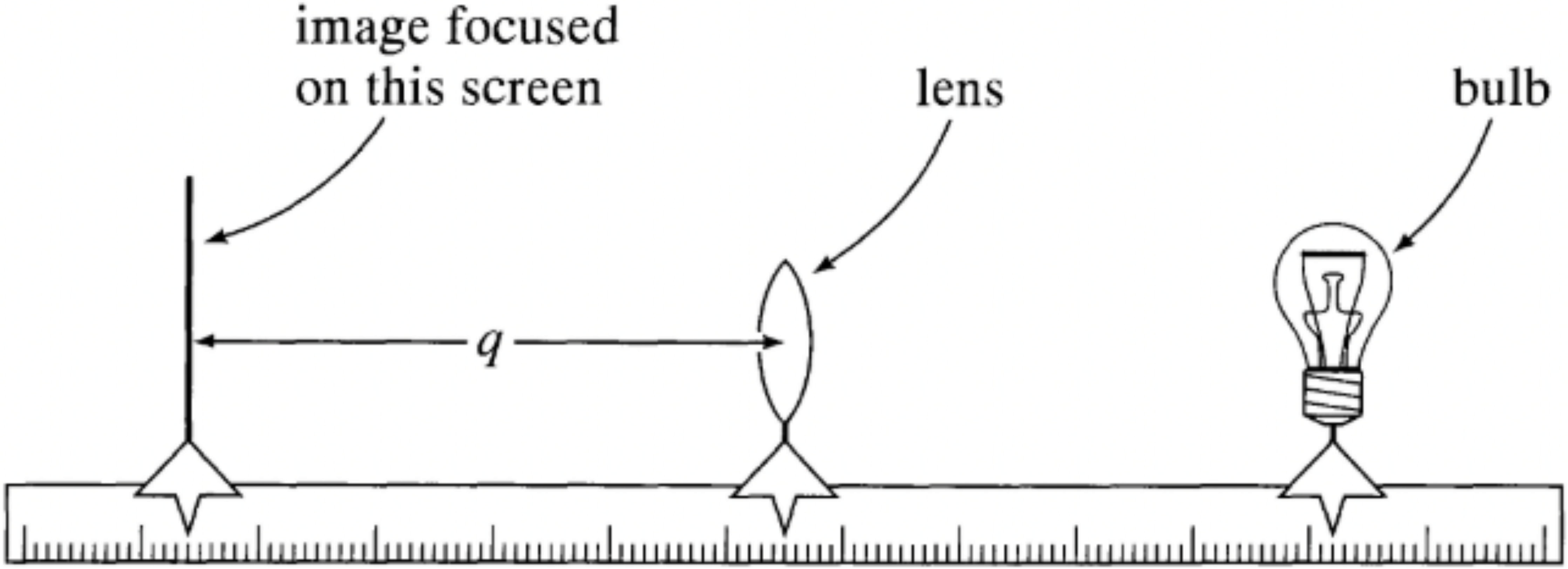
Uncertainty in a Product
(Provisional Rule)

If two quantities x and y have been measured with small fractional uncertainties $\delta x/|x_{\text{best}}|$ and $\delta y/|y_{\text{best}}|$, and if the measured values of x and y are used to calculate the product $q = xy$, then the *fractional uncertainty in* q *is the sum of the fractional uncertainties in* x *and* y,

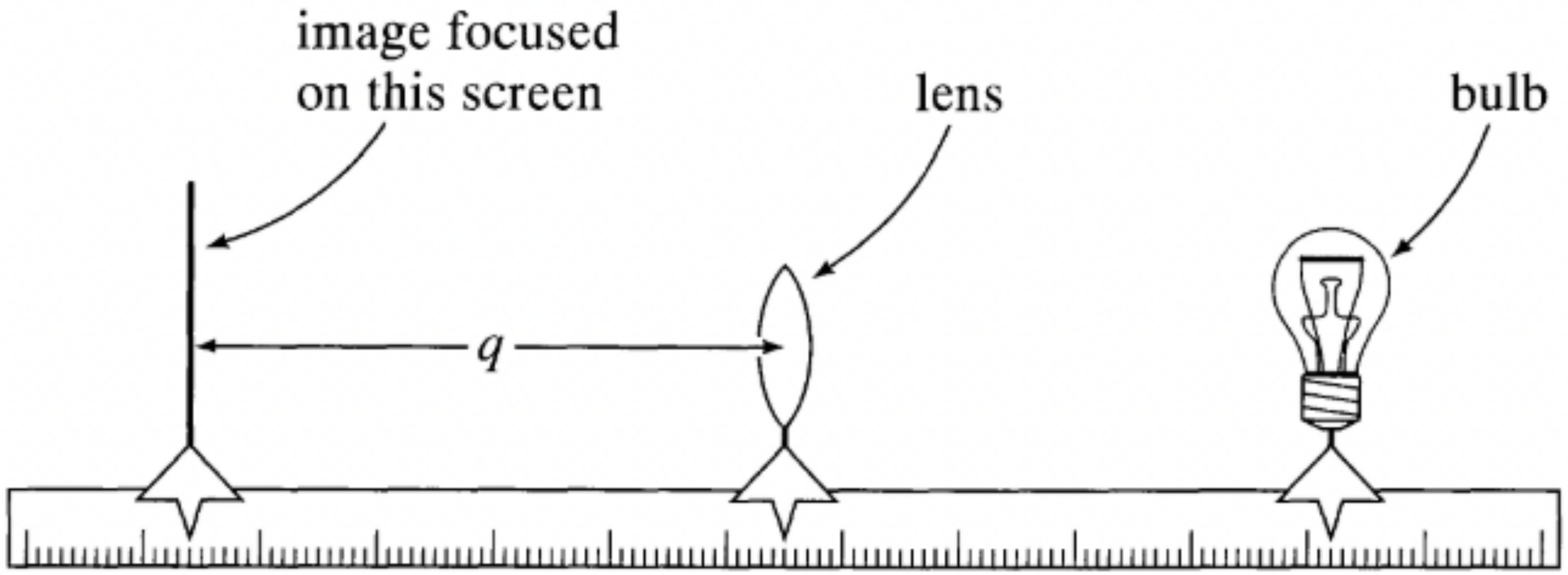
$$\frac{\delta q}{|q_{\text{best}}|} \approx \frac{\delta x}{|x_{\text{best}}|} + \frac{\delta y}{|y_{\text{best}}|}. \tag{2.28}$$

What about for more complicated situations?

First, review uncertainty in direct measurements



Recall how to find uncertainty in direct measurements



What are some challenges in determining uncertainty?

- where is center of lens
- could be a range of distance with image focused
- challenge is neither point is clearly defined
    - problem of definition

What're are the techniques we can use?

- repeated measures
- digital devices only specify sig. figs.

$$x = \underbrace{x_{best}}\ \pm\ \underbrace{\delta_x}$$

Counting experiments and uncertainty

A demographer want to know the average births at a given hospital

<u>H: The average births at Hospital Y is equal to the average births in city X.</u>

What experiment should we do?

→ count births within a fixed window

Demographer completes experiment & counts
14 birth in 1 week.

Key idea: uncertainty has a different interpretation
Uncertainties is not in observations, rather how well
the observation matches the true value

Counting experiments - $\boxed{\text{the square root rule}}$

Poisson Process — counting process

↳ Poisson Distribution

CH. 11

$$(\text{Average number of events in time } T) = \nu \pm \sqrt{\nu}$$

Quick Check 3.1. **(a)** To check the activity of a radioactive sample, an inspector places the sample in a liquid scintillation counter to count the number of decays in a two-minute interval and obtains 33 counts. What should he report as the number of decays produced by the sample in two minutes? **(b)** Suppose, instead, he had monitored the same sample for 50 minutes and obtained 907 counts. What would be his answer for the number of decays in 50 minutes? **(c)** Find the percent uncertainties in these two measurements, and comment on the usefulness of counting for a longer period as in part (b).

a. 33 ± 6 , b. 910 ± 30 , c. 18% , 3%

Review for uncertainty propagation for Difference/Addition

Key Idea: you can use the highest and lowest probable values to estimate new uncertainty

$x = x_{best} \pm \delta x$

$y = y_{best} \pm \delta y$

$q = x + y$

δq ?

$q_{high} = x_{best} + y_{best} + \delta x + \delta y$

$q_{low} = x_{best} + y_{best} - (\delta x + \delta y)$

$\delta q \simeq \delta x + \delta y$

Easy to show this also hold for Subtraction

Review for uncertainty propagation for Difference/Addition

Uncertainty in Sums and Differences
(Provisional Rule)

If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values used to compute

$$q = x + \cdots + z - (u + \cdots + w),$$

then the uncertainty in the computed value of q is the sum,

$$\delta q \approx \delta x + \cdots + \delta z + \delta u + \cdots + \delta w, \tag{3.4}$$

of all the original uncertainties.

Same idea applies to products/quotients, but fractional form is used

$x = x_{best} \pm \delta x$ $\qquad$ $y = y_{best} \pm \delta y$

$z = x / y$

$\delta z = ?$

$$\frac{\delta x}{|x_{best}|}$$ fraction uncertainty.

$$\frac{\delta x}{|x|}$$

$$z = \frac{x_{best}}{y_{best}} \left(\frac{1 \pm \frac{\delta x}{|x|}}{1 \pm \frac{\delta y}{|y|}} \right)$$

Problem: How to extreme values of second term.

Same idea applies to products/quotients, but fractional form is used

largest value $$\frac{x_{best}}{y_{best}} \underbrace{\left(\frac{1 + \frac{\delta x}{|x|}}{1 - \frac{\delta y}{|y|}} \right)} = q_{high}$$

$$\frac{1+a}{1-b} \qquad a \ll 1 \;,\; b \ll 1$$

<u>Binomial Theorem:</u> $$\frac{1}{1-b} \approx 1+b$$

$$\therefore \; (1+a)(1+b) = 1 + a + b + \cancel{ab} \to 0$$

$$q_{high} = \frac{x_{best}}{y_{best}} \left(1 + \frac{\delta x}{|x|} + \frac{\delta y}{|y|} \right)$$

Special Cases: Multiplication with a constant

$$q = B\,x$$

no uncertainty

$$\delta q = ?$$

$$q = B\,(x_{best} \pm \delta x) = B\,x_{best} \pm \underbrace{|B|\,\delta x}_{\delta q}$$

Special Cases: Powers

$T = \frac{1}{2} m v^2$ we measure $v \longrightarrow$ what's uncertainty of v^2

$$z = x^n$$

$$\frac{\delta z}{|z|} = n \frac{\delta x}{|x|}$$

Independent Uncertainties in Sums

Summary so far: sub/add → uncertainties add

multi/div → fractional uncertainties add

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* In reality these are really conservative estimates.

* If the uncertainties are independent & random, we can come up with a better estimate.

Let's explore why the original formulation is conservative

$$q = x_{best} + y_{best} \pm (\delta x + \delta y)$$

When would actual value of q = the extreme value?

Only occurs if we underestimate x by the full δx & we underestimate y by the full δy.

If x & y are independent, & error is random then 50% that an underestimate in one is accompanied by an overestimate in y.

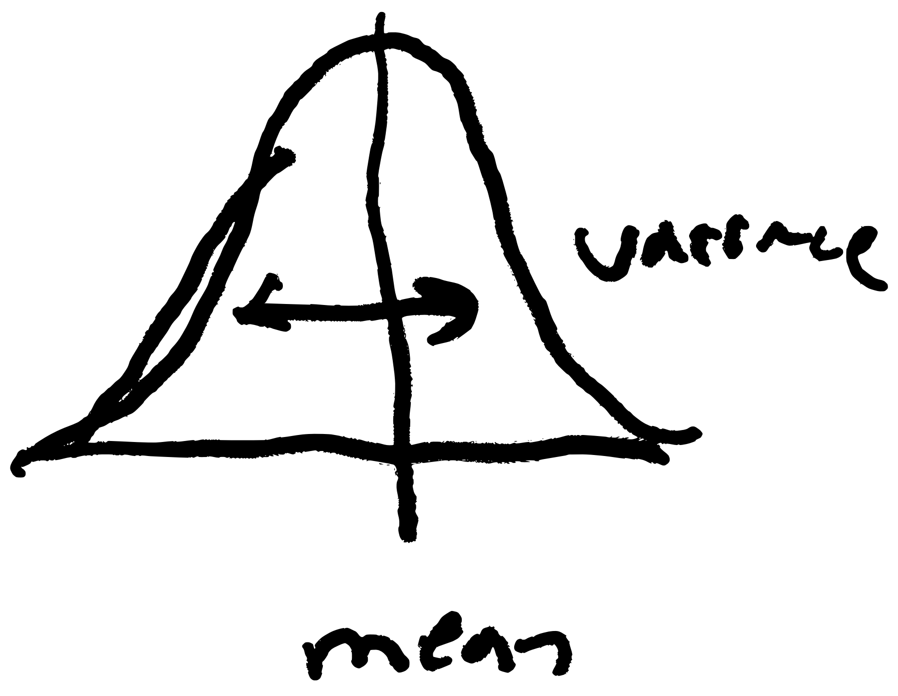
* point super unlikely that the extreme is true!

What should be done?

→ Really depends on statistical laws governing the error in measurement

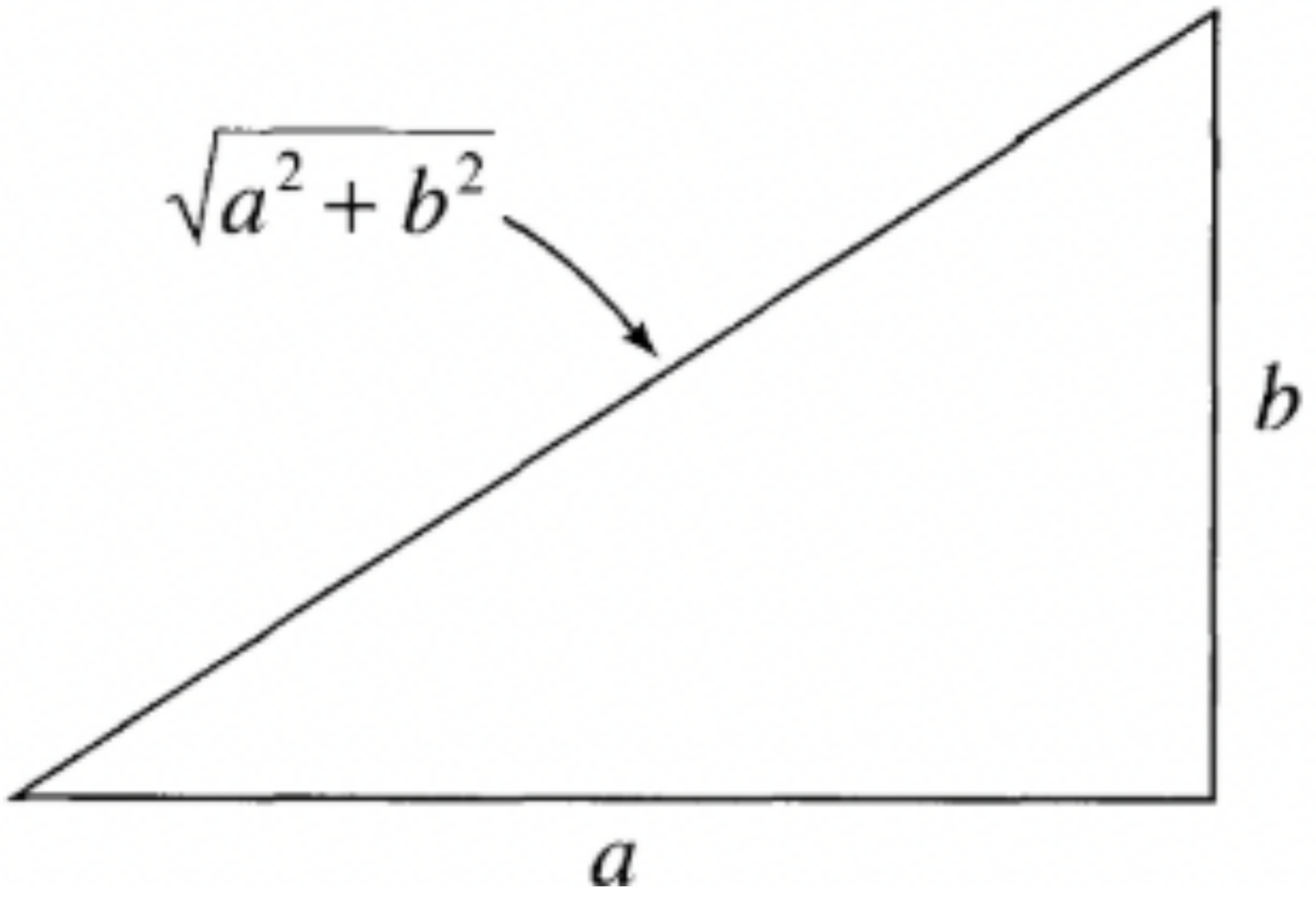
$\underline{\underline{\text{CH. 5}}}$

if you can assume independence & that the uncertainties are random & gaussian:



$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

added in quadrature



$$\sqrt{a^2 + b^2} < a + b$$

if you can satisfy the assumptions
you can be more certain

Summary for new uncertainty estimates

Uncertainty in Sums and Differences

Suppose that $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$ and the measured values used to compute

$$q = x + \cdots + z - (u + \cdots + w).$$

If the uncertainties in $x, \ldots, w$ are known to be *independent and random,* then the uncertainty in q is the quadratic sum

$$\delta q = \sqrt{(\delta x)^2 + \cdots + (\delta z)^2 + (\delta u)^2 + \cdots + (\delta w)^2}$$

of the original uncertainties. In any case, δq is never larger than their ordinary sum,

$$\delta q \leqslant \delta x + \cdots + \delta z + \delta u + \cdots + \delta w.$$

Uncertainties in Products and Quotients

Suppose that $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w}.$$

If the uncertainties in $x, \ldots, w$ are *independent and random,* then the fractional uncertainty in q is the sum in quadrature of the original fractional uncertainties,

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \cdots + \left(\frac{\delta z}{z}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \cdots + \left(\frac{\delta w}{w}\right)^2}.$$

In any case, it is never larger than their ordinary sum,

$$\frac{\delta q}{|q|} \leqslant \frac{\delta x}{|x|} + \cdots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \cdots + \frac{\delta w}{|w|}.$$

Summary for new uncertainty estimates

Sum & differences

$$q = x + \cdots + z - (u + \cdots w)$$

$$\delta q = \sqrt{(\delta x)^2 \cdots\cdots (\delta w)^2}$$

$$\delta q \leq \delta x + \cdots\cdots + \delta w$$

Products & Quotients

$$q = \frac{x \cdot y \cdot \cdots\cdots w}{u \times \cdots\cdots v}$$

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 \cdots\cdots \left(\frac{\delta w}{w}\right)^2}$$

$$\frac{\delta q}{|q|} \leq \frac{\delta x}{|x|} + \cdots \frac{\delta w}{|w|}$$

Quick Check 3.6. Suppose you measure three numbers as follows:

$$x = 200 \pm 2, \quad y = 50 \pm 2, \quad z = 20 \pm 1,$$

where the three uncertainties are independent and random. What would you give for the values of $q = x + y - z$ and $r = xy/z$ with their uncertainties?

$q = 230 \pm 3$, $r = 500 \pm 6.5\% = 500 \pm 30$

Arbitrary Functions of One Variable

Ex. Find refractive index n of glass by measuring critical angle θ

$$n = \frac{1}{\sin(\theta)}$$

What is the uncertainty of n?

Arbitrary Functions of One Variable

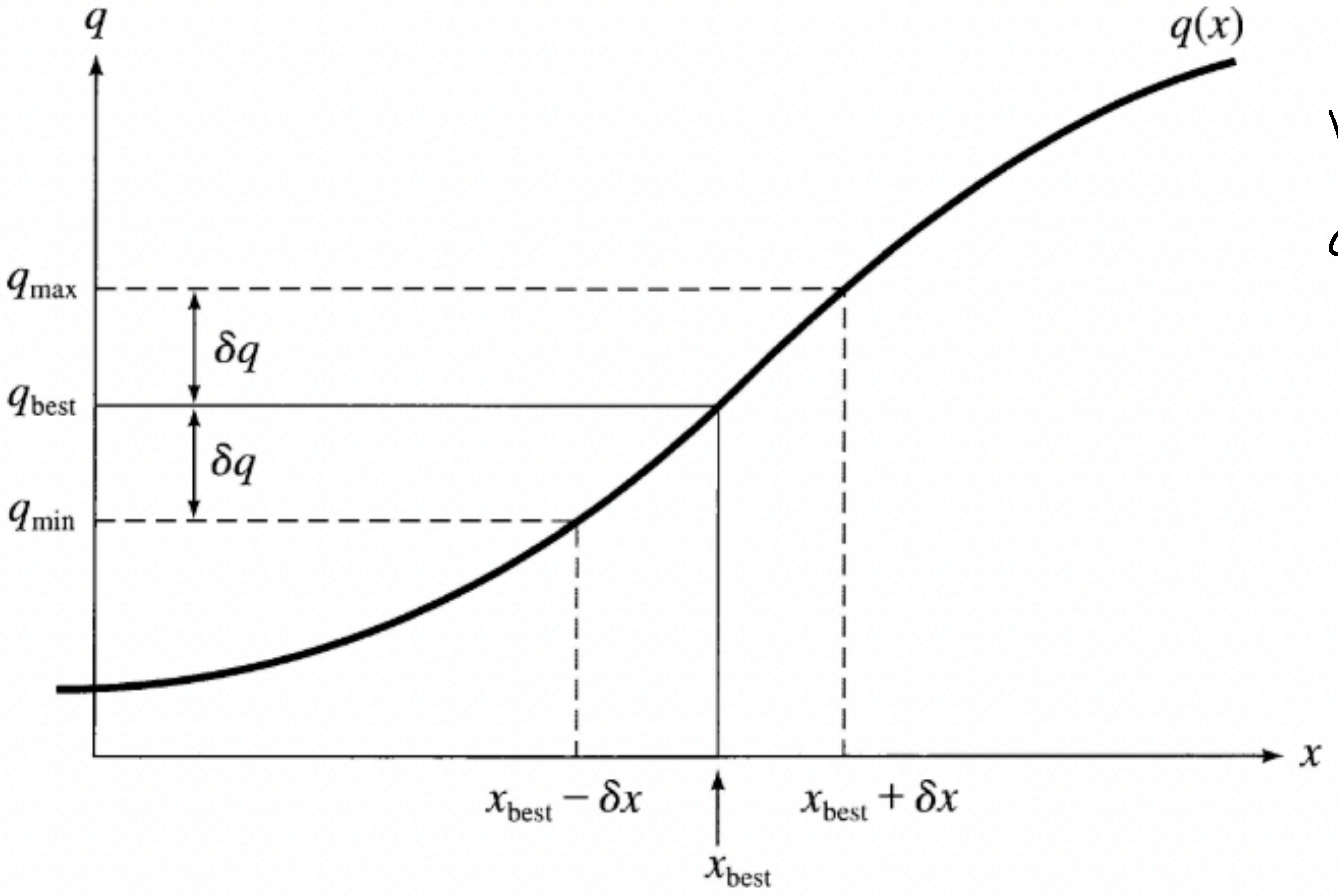
$x = x_{best} \pm \delta x$, and we want

$q(x)$ what is δq

How should we approach this?

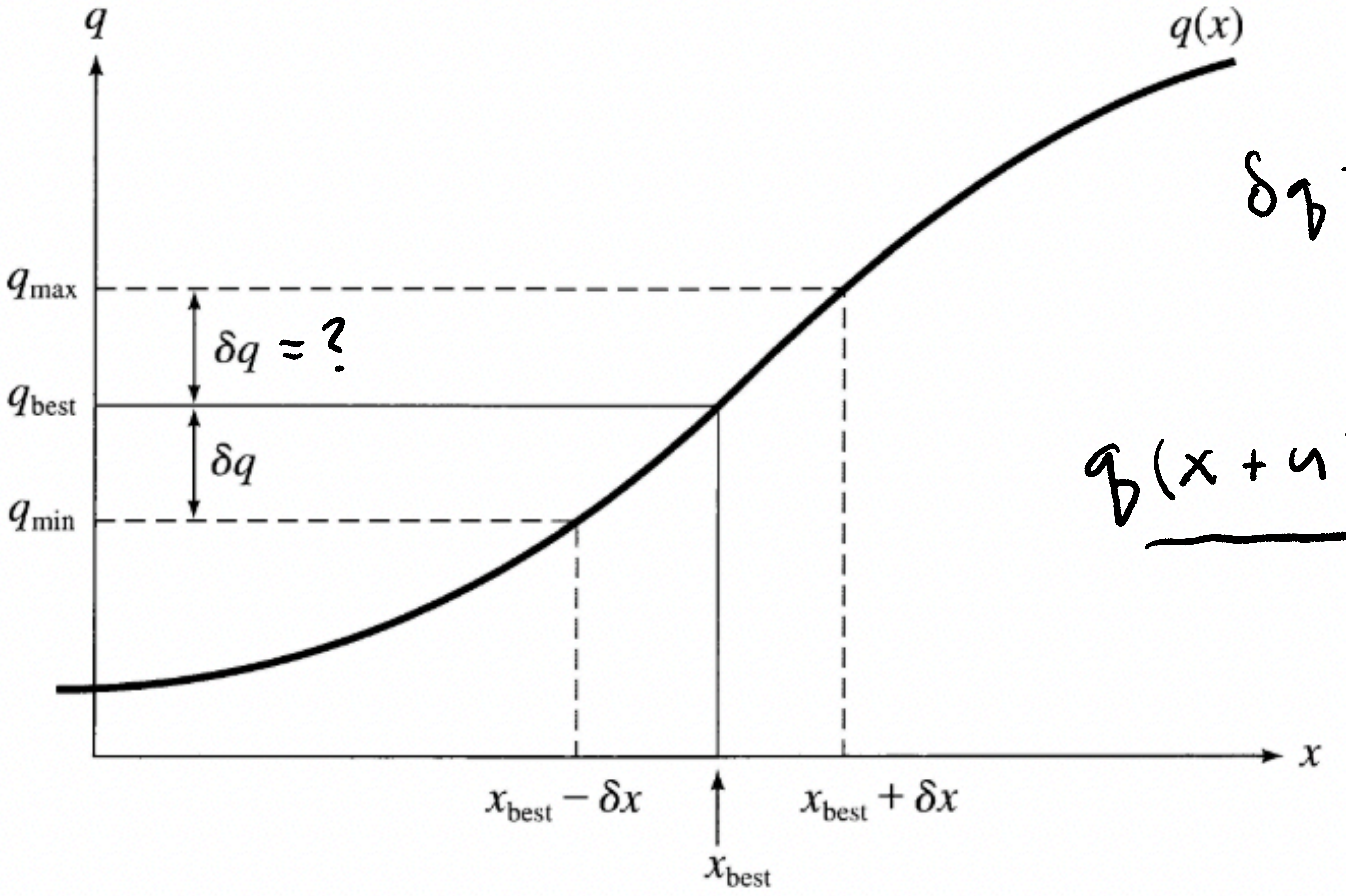
→ The usual way → find the extreme values

Arbitrary Functions of One Variable



We can just graph it!

Arbitrary Functions of One Variable



$$\delta q = q(x_{best} + \delta x) - q(x_{best})$$

$$\frac{q(x+h) - q(x)}{h} = \frac{dq}{dx}$$

$$\frac{df}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Arbitrary Functions of One Variable

Uncertainty in Any Function of One Variable

If x is measured with uncertainty δx and is used to calculate the function $q(x)$, then the uncertainty δq is

$$\delta q = \left| \frac{dq}{dx} \right| \delta x.$$

Arbitrary Functions of One Variable

$q(x)$

$$\boxed{\delta q = \left| \frac{dq}{dx} \right| \delta x}$$

Quick Check 3.7. Suppose you measure x as 3.0 ± 0.1 and then calculate $q = e^x$. What is your answer, with its uncertainty?

$$\boxed{q = 20 \pm 2}$$

$$q_{best} = 20.08\ldots$$

$$\delta q = \left|\frac{dq}{dx}\right| \delta x$$

$$\delta q = e^x\Big|_x \delta x$$

$$\delta q = 20 \cdot 0.1$$

$$\delta q = 2$$

Special Case — power

$$q(x) = x^n$$

$$\delta q = \left| \frac{dq}{dx} \right| \delta x = \left| n x^{n-1} \right| \delta x$$

$$\frac{\delta q}{|q|} = \frac{|n| \, \delta x}{|x|}$$

Special Case — power

Uncertainty in a Power

If x is measured with uncertainty δx and is used to calculate the power $q = x^n$ (where n is a fixed, known number), then the fractional uncertainty in q is $|n|$ times that in x,

$$\frac{\delta q}{|q|} = |n| \frac{\delta x}{|x|}.$$

Special Case — power

Putting Everything Together: Propagation Step-by-Step

* Any calculation can be broken down into a sequence, each involving only one of the following

1. Sum & differences
2. products & quotients
3. computation of a function of one variable

Putting Everything Together: Propagation Step-by-Step

$$q = x(y - z \sin u)$$

measure x, y, z, u

1. uncertainty in $\sin(u)$
2. uncertainty in product $z \cdot \sin u$
3. " " difference $y - z \cdot \sin u$
4. " " product $x(y - z \sin u)$

Quick Check 3.9. Suppose you measure three numbers as follows:

$$x = 200 \pm 2, \quad y = 50 \pm 2, \quad z = 40 \pm 2,$$

where the three uncertainties are independent and random. Use step-by-step propagation to find the quantity $q = x/(y - z)$ with its uncertainty. [redacted]

$$\boxed{q = 20 \pm 6}$$

$$y - z \Rightarrow \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\left(\frac{x}{y-z}\right) \Rightarrow \sqrt{\left(\frac{2}{200}\right)^2 + \left(\frac{\sqrt{8}}{10}\right)^2}$$

$$= \sqrt{\frac{4}{40000} + \frac{8}{100}} = 0.283$$

$$= 5.66$$

$$\simeq 6.$$

$$\frac{\delta x}{|x|} \cdot |x| = \delta x$$

General Formula for Error Propagation

$x = x_{best} \pm \delta x$ $\qquad$ $y = y_{best} \pm \delta y$ $\qquad$ $z = z_{best} \pm \delta z$

$$q(x, y, z) = \frac{x+y}{x+z}$$

What happens if we do this step by step?

→ misses the possibility that errors in den. (from x) could cancel errors in the num. (from x)

overestimate $x \rightarrow x+y \uparrow$

overestimate $x \rightarrow x+z \uparrow$

$$\frac{x+y}{x+z} \Rightarrow$$ cancel uncertainty

∴ step-by-step no good for some situation.

General Formula for Error Propagation

General Formula for Error Propagation

$$z_{best} = z(x_{best}, y_{best})$$

$$z(x+u, y+v) \approx z(x, y) + \frac{\partial z}{\partial x} u + \frac{\partial z}{\partial y} v$$

$$\vdots$$

$$z = z(x_{best}, y_{best}) \underbrace{\pm \left(\left| \frac{\partial z}{\partial x} \right| \delta x + \left| \frac{\partial z}{\partial y} \right| \delta y \right)}_{\text{the absolute max/min}}$$

General Formula for Error Propagation

Uncertainty in a Function of Several Variables

Suppose that $x, \ldots, z$ are measured with uncertainties $\delta x, \ldots, \delta z$ and the measured values are used to compute the function $q(x, \ldots, z)$. If the uncertainties in $x, \ldots, z$ are independent and random, then the uncertainty in q is

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\,\delta x\right)^2 + \cdots + \left(\frac{\partial q}{\partial z}\,\delta z\right)^2}.$$

In any case, it is never larger than the ordinary sum

$$\delta q \leqslant \left|\frac{\partial q}{\partial x}\right| \delta x + \cdots + \left|\frac{\partial q}{\partial z}\right| \delta z.$$

General Formula for Error Propagation

$$q(x, \cdots, z)$$

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \cdots + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

$$\delta q \leq \left|\frac{\partial q}{\partial x}\right| \delta x + \cdots + \left|\frac{\partial q}{\partial z}\right| \delta z$$