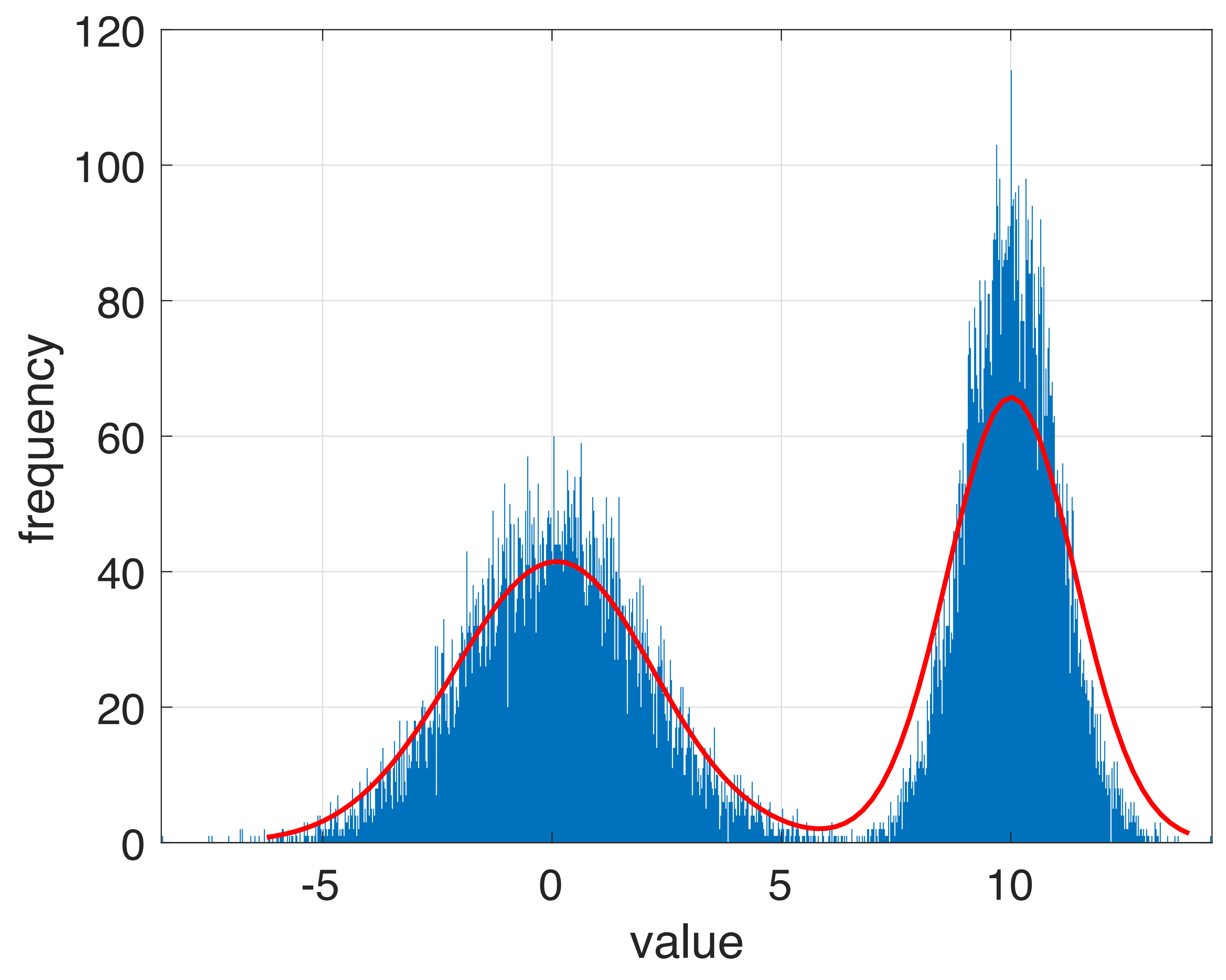
ME170b Lecture 5

2/16/24

Experimental Techniques

Last time:



Today:

> Finish Ch.5 - Normal Distributions

What we know so far:

uncertainty.

Standard form

$$x = x_{best} \pm \delta x$$

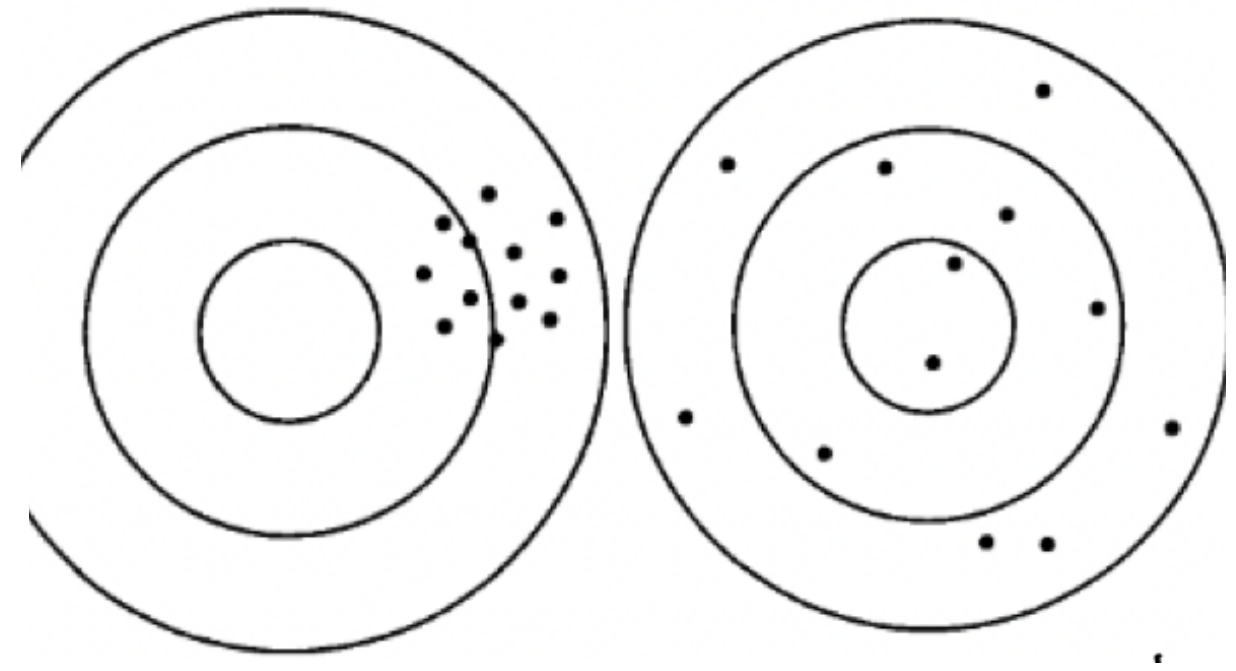
$$q = f(x_1, \dots, z)$$

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \cdots}$$

random & independent

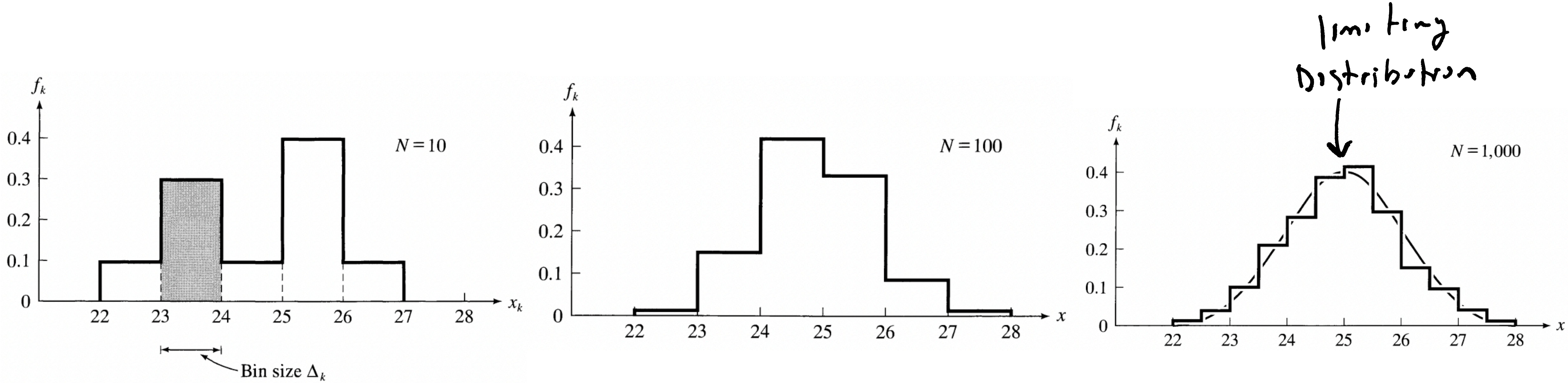
For repeated measures:

$$\boxed{x = \bar{x} \pm \sigma_{\bar{x}}}$$



$$\sigma_x = \sqrt{\frac{1}{N-1}\sum (x_i - \bar{x})^2} \qquad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

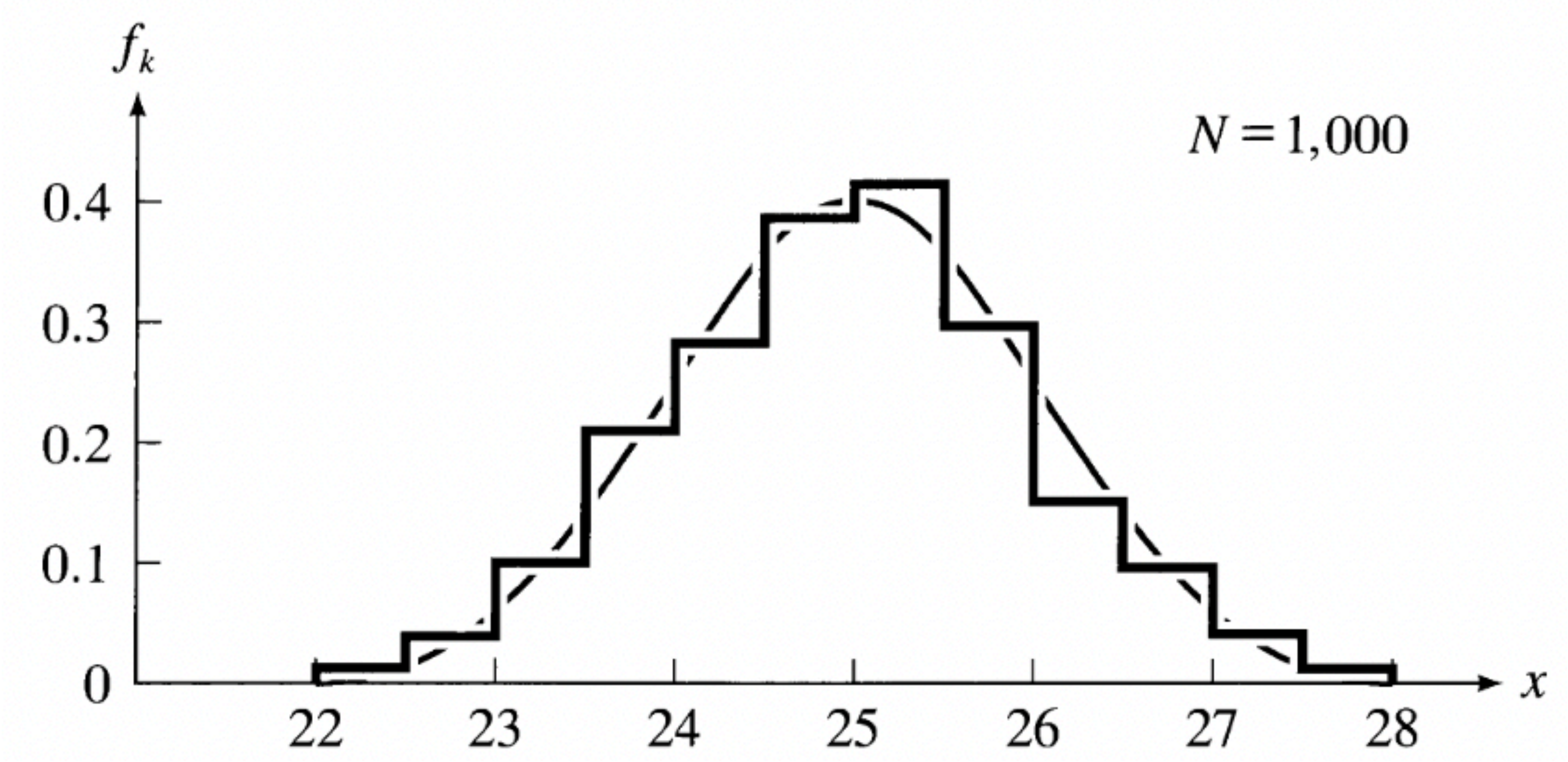
Limiting Distributions



Key Idea: As $N \to \infty$ the distribution approaches a continuous curve

limiting distribution.

Does every measurement have a limiting distribution?



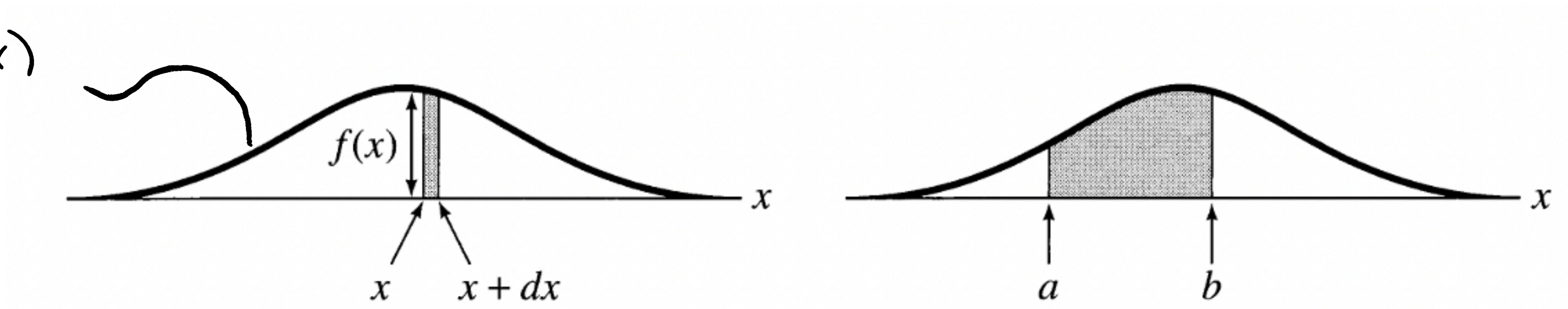
Limit Distribution is theoretical contruct

→ never measured exactly

Short answer, yes under most conditions

Math of limiting distribution

$f(x)$

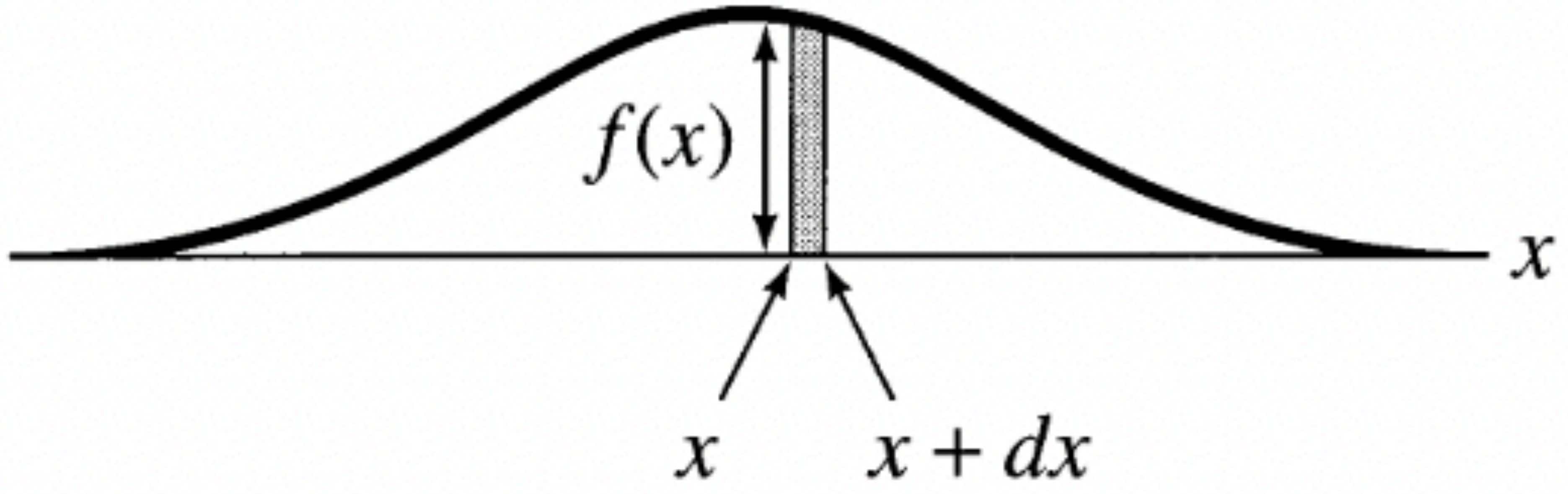


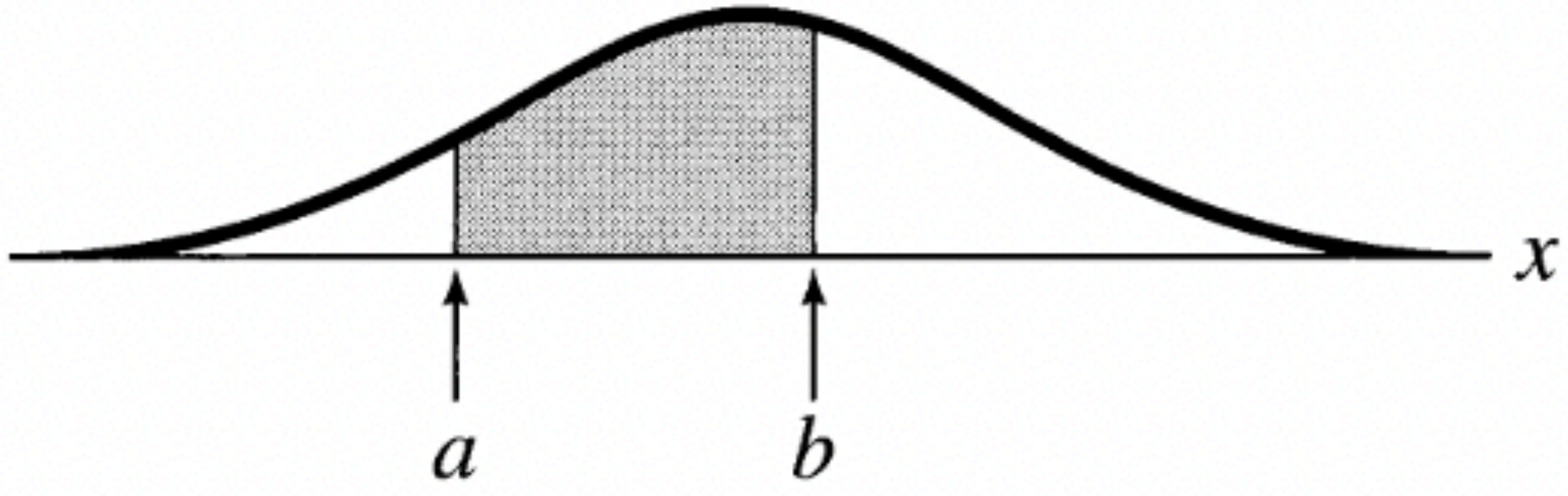
$f(x)\,dx$ = fraction of measurements that fall between $x + dx$

$\int_a^b f(x)\,dx$: fraction of measurements that fall between $x = a$; $x = b$

this only works well if you have a large number of measurements → approximate the limiting distribution

What is the interpretation?



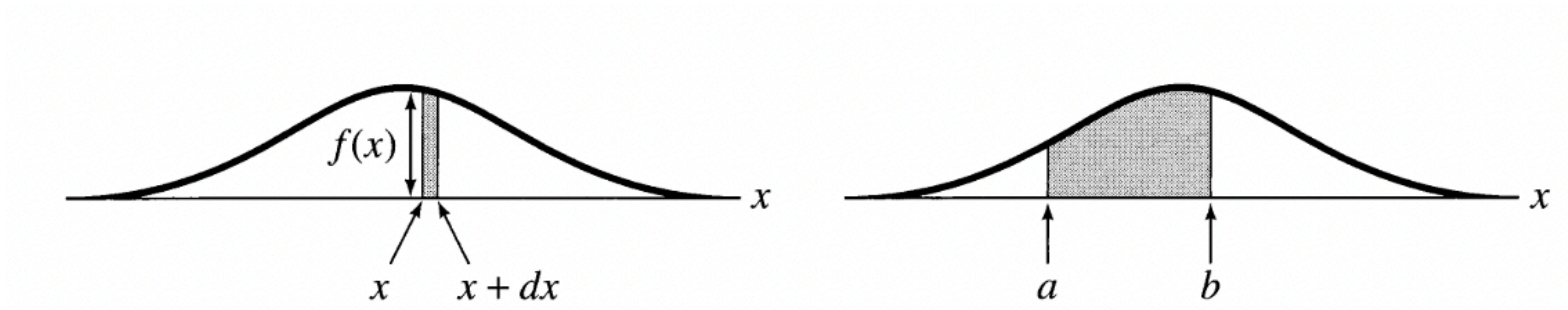


$f(x)\,dx$ = fraction of measurements that fall between x and $x + dx$.

$\int_a^b f(x)\,dx$ = fraction of measurements that fall between $x = a$ and $x = b$.

Why is this true?

What is the interpretation?



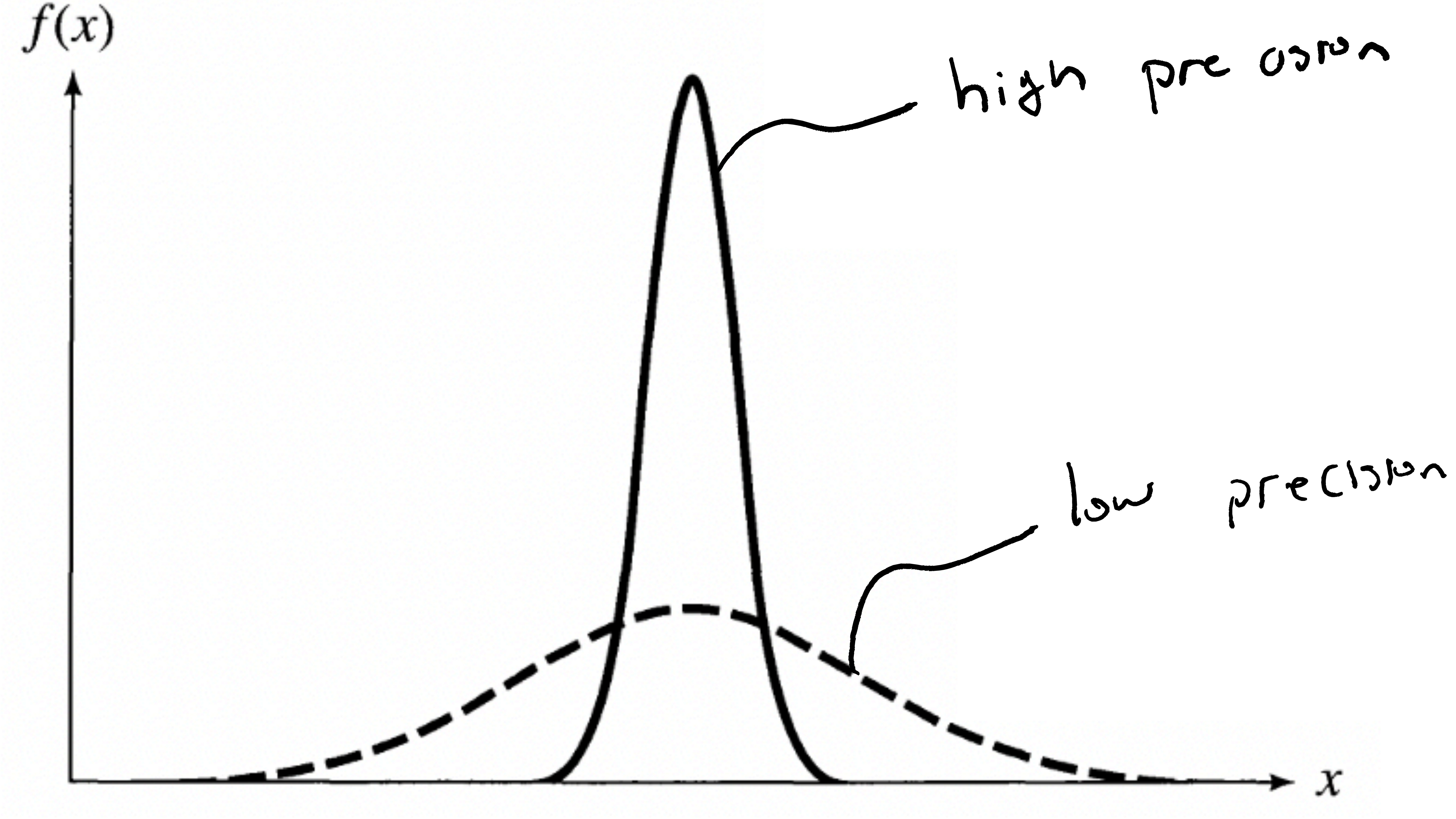
$f(x)\,dx$ = fraction of measurements that fall between x and $x + dx$.

$\int_a^b f(x)\,dx$ = fraction of measurements that fall between $x = a$ and $x = b$.

$f(x)$: probability Density function (PDF)

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

The limiting distribution (PDF) tells us a lot!



Diameter a single pebble

* if the measurement in precise
  ↳ narrow

* in precise
  ↳ long tails

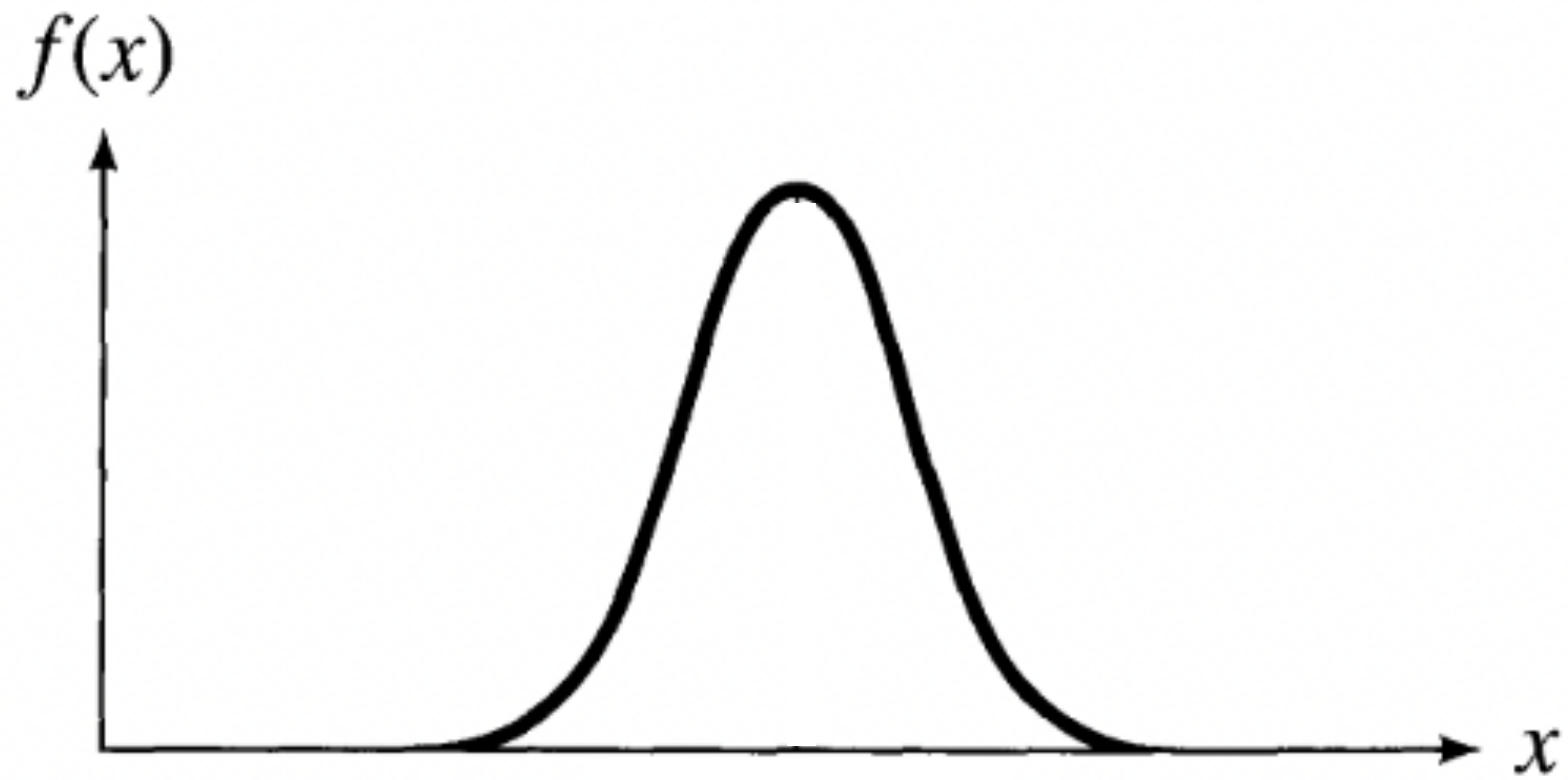
We can get mean and variance directly from PDF

$$\bar{x} = \int_{-\infty}^{\infty} x \, f(x) \, dx$$ "expected value"

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) \, dx$$

this is true regardless of $f(x)$!

The Normal Distribution — most important ‘limiting distribution’



If a measurement is subject to many small sources of <u>random error</u> and negligible systematic error, the limiting distribution will be the bell shaped normal curve

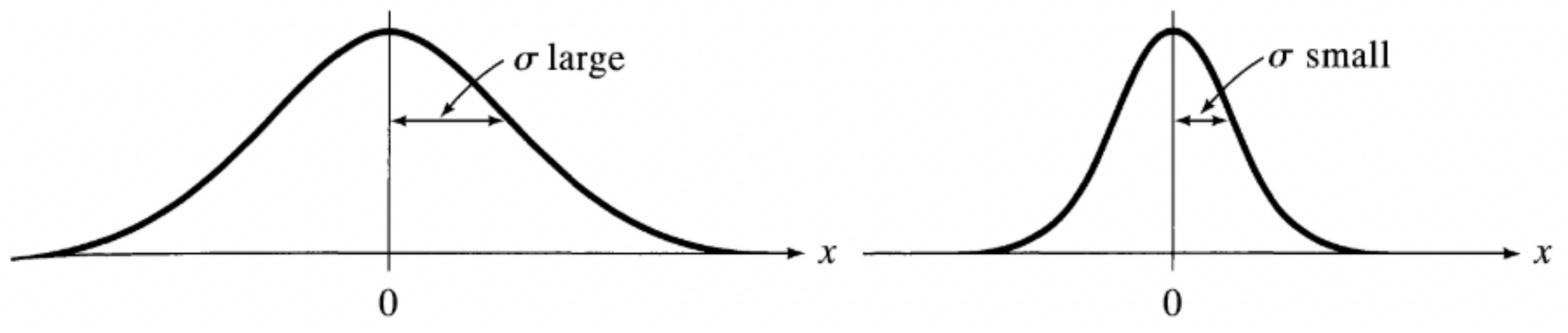
What is the "true value" of x?

What do mean by "true value"

"True value" is an ideal

→ is the value that you approach in the limit

The Normal Distribution — most important 'limiting distribution'



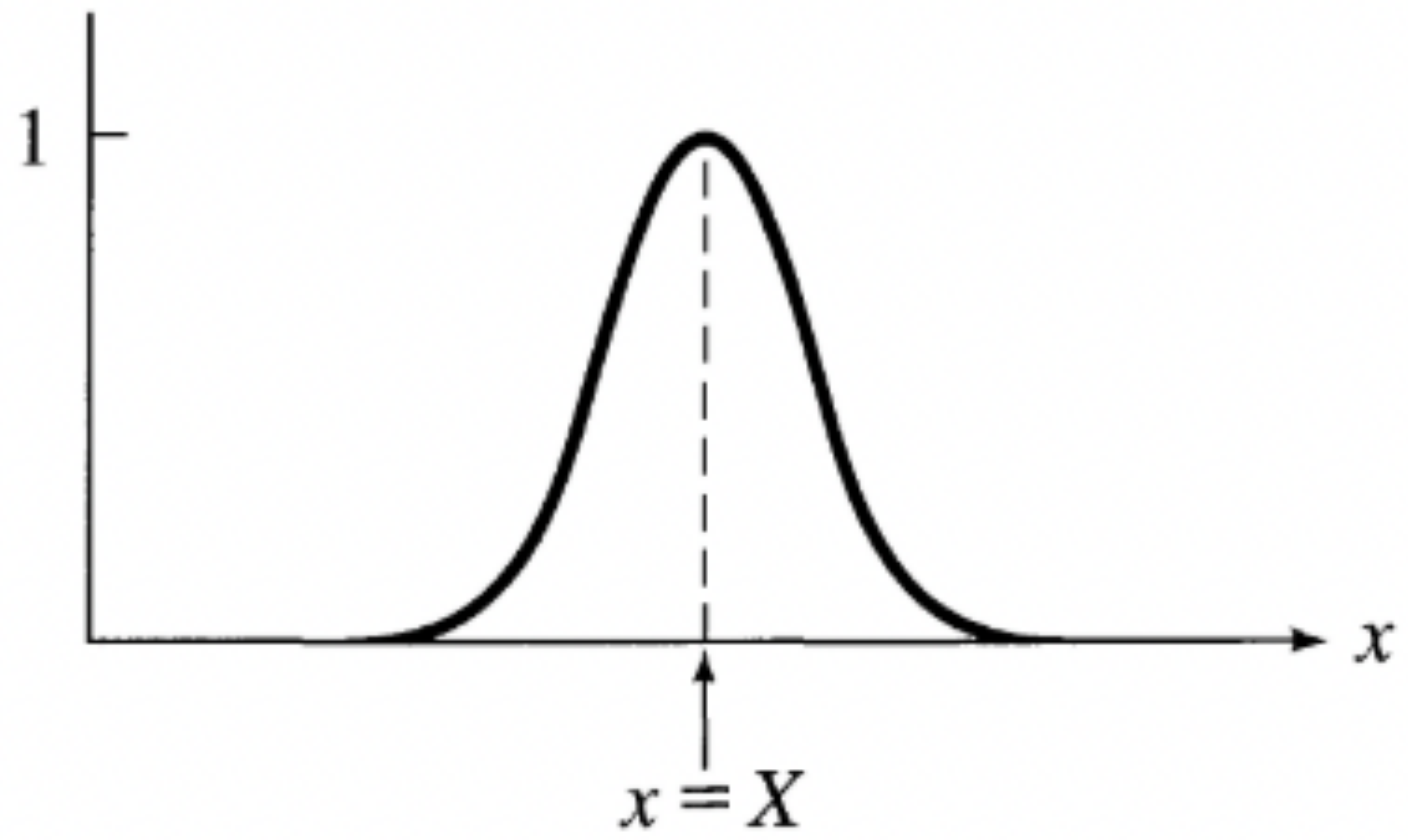
Gauss's Function

$$e^{-x^2/2\sigma^2}$$

σ : width parameter

* Gauss's Function is symmetric about $x=0$
* tends to zero as $x \to \pm\infty$
* σ determines how fast/slow curves $\to 0$

The Normal Distribution — most important 'limiting distribution'



Gauss's Function - shifted

$$e^{-(x-X)^2/2\sigma^2}$$

X : center parameter

The Normal Distribution — most important 'limiting distribution'

All 'limiting distributions' should be normalized such that: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

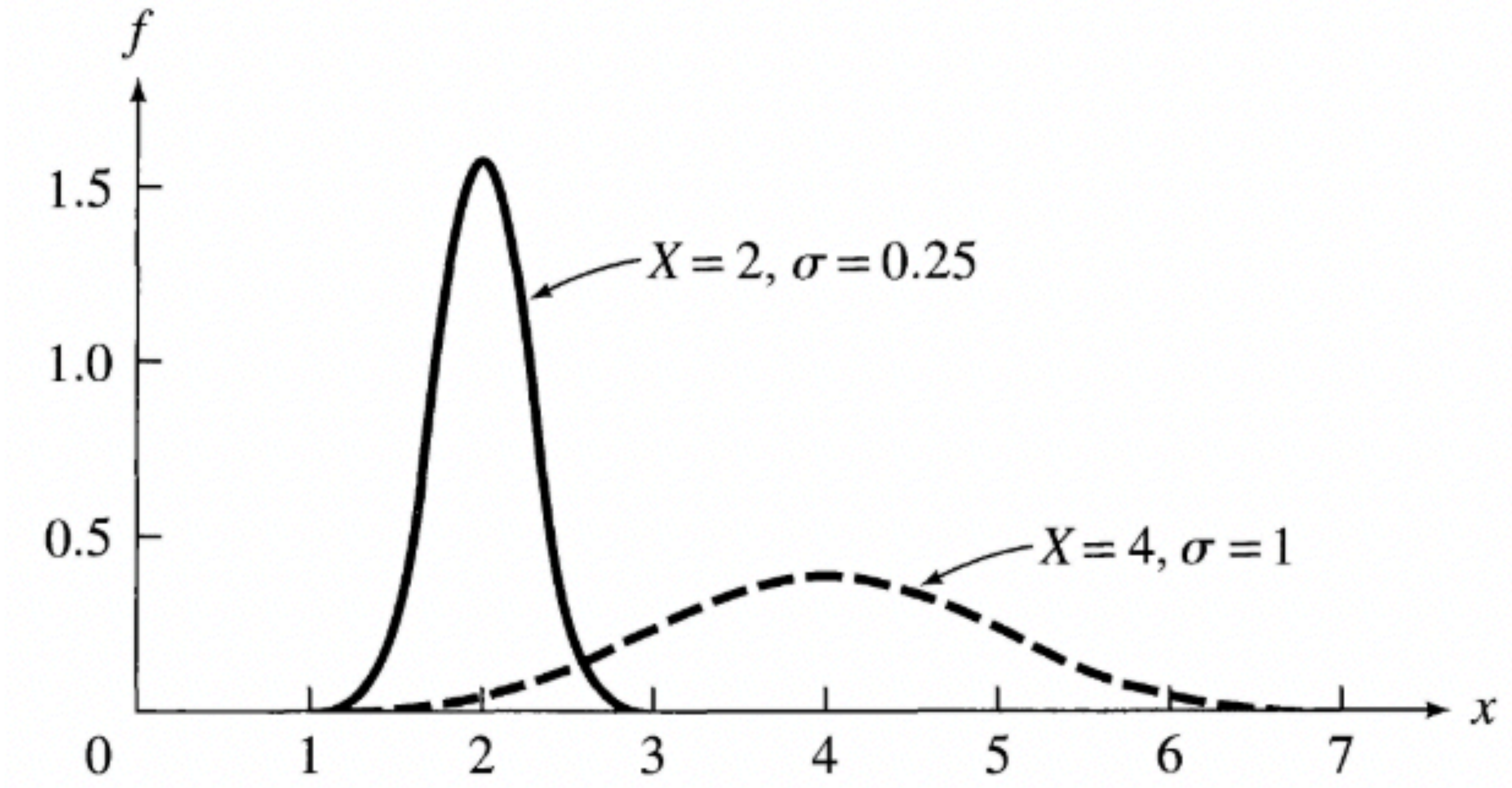
This means: $f(x) = N e^{-(x-X)^2/2\sigma^2}$

With normalization factor chosen as: $N = \dfrac{1}{\sigma\sqrt{2\pi}}$

This is the 'Gaussian Distribution': $G_{X,\sigma}(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$

parameters: X, sigma — center, and width

The Normal Distribution — most important 'limiting distribution'



Two Gaussian distributions with different 'centers' and 'widths'. Tall, narrow distributions (sharp peaked) correspond to more precise measurements (since measurements fall closer together!) while broad distributions correspond to low precise measurements (measurement fall farther away from each other)

The Normal Distribution — ‘expected value’ or ‘average’

$$\bar{x} = \int_{-\infty}^{\infty} x\, G_{X,\sigma}(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-(x-X)^2/2\sigma^2}\, dx$$

If we make change of variable $y = x - X$

$dx = dy$

$x = y + X$

$$\bar{x} = \frac{1}{\sigma\sqrt{2\pi}} \left(\underbrace{\int_{-\infty}^{\infty} y\, e^{-y^2/2\sigma^2}\, dy}_{=0} + X \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, dy}_{=\sigma\sqrt{2\pi}} \right)$$

$$\bar{x} = X$$

The Normal Distribution — ‘expected value’ or ‘average’

$$\bar{x} = \int_{-\infty}^{\infty} x\, G_{X,\sigma}(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-(x-X)^2/2\sigma^2}\, dx$$

If we make the change of variables $y = x - X$, then $dx = dy$ and $x = y + X$. Thus,

$$\bar{x} = \frac{1}{\sigma\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} y\, e^{-y^2/2\sigma^2}\, dy + X \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, dy \right)$$

The Normal Distribution — ‘expected value’ or ‘average’

$$\bar{x} = \int_{-\infty}^{\infty} x\, G_{X,\sigma}(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-(x-X)^2/2\sigma^2}\, dx$$

If we make the change of variables $y = x - X$, then $dx = dy$ and $x = y + X$. Thus,

$$\bar{x} = \frac{1}{\sigma\sqrt{2\pi}} \left(\underbrace{\int_{-\infty}^{\infty} y\, e^{-y^2/2\sigma^2}\, dy}_{=0} + X \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, dy}_{= \frac{\quad}{\sigma\sqrt{2\pi}}} \right)$$

$$\bar{x} = X$$

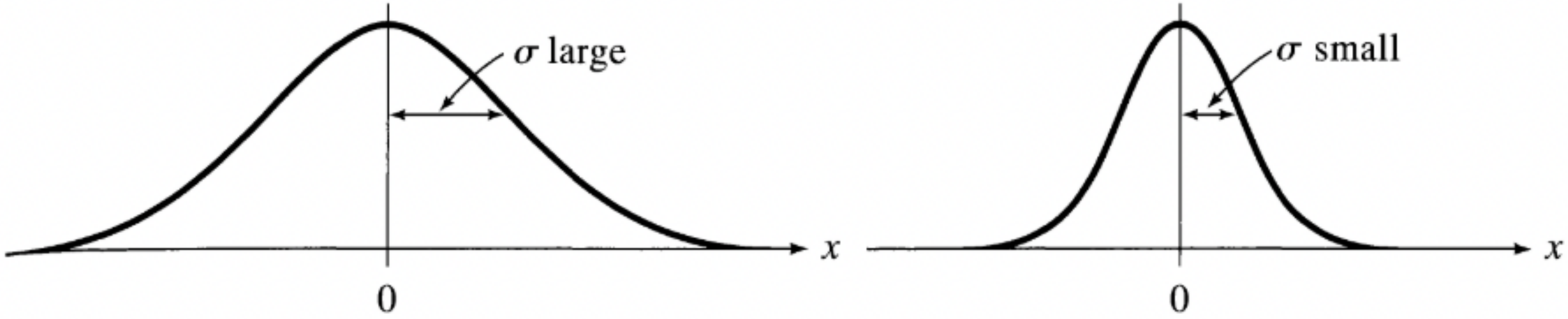
<u>This shows that the average is exactly the ‘center’ parameter</u>

The Normal Distribution — 'standard deviation' is the 'width'

$$\sigma_x{}^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 G_{X,\sigma}(x)\, dx.$$

This integral is evaluated easily. We replace $\bar{x}$ by X, make the substitutions $x - X = y$ and $y/\sigma = z$, and finally integrate by parts to obtain the result (see Problem 5.16)

$$\sigma_x{}^2 = \sigma^2$$



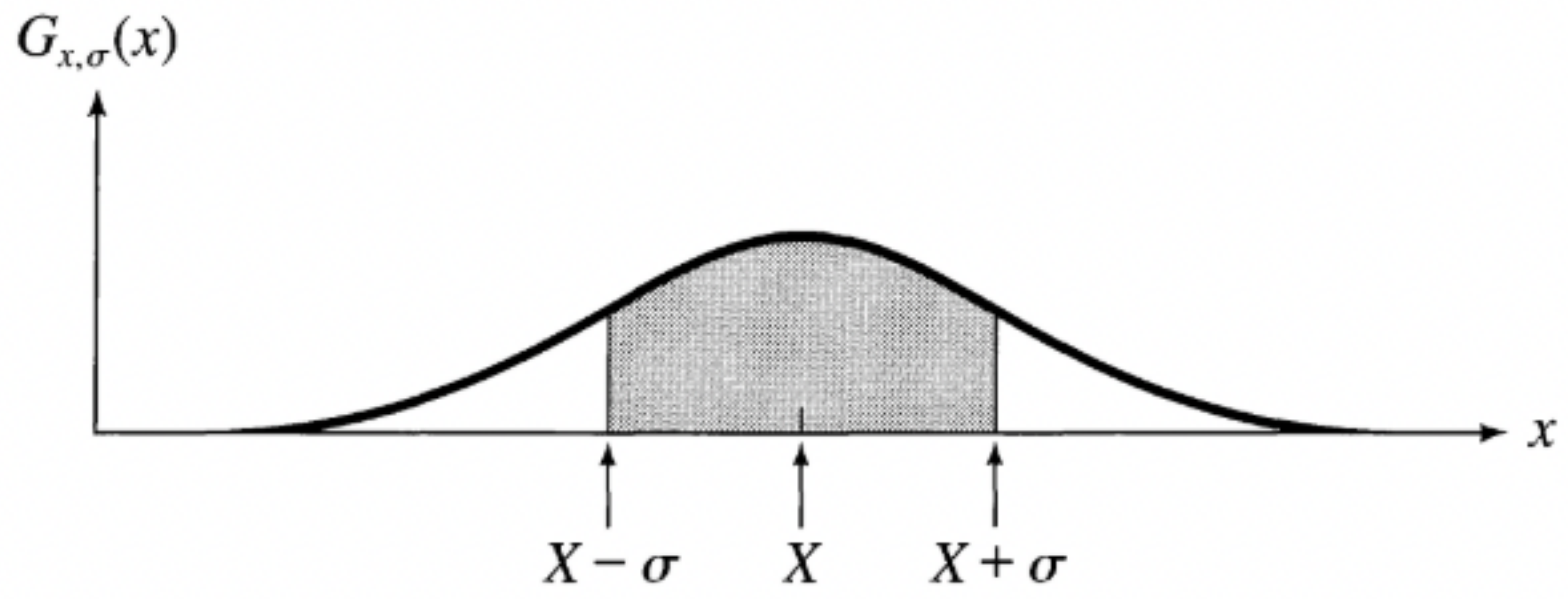
The standard deviation as 68% confidence limit

$$\int_a^b f(x)\,dx$$

the probability that any measurement falls within a <= x <= b, with any limiting distribution f(x)

What is the probability that a measurement falls within one standard deviation if the f(x) is Gaussian?

$$\text{Prob (within } \sigma) = \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}(x)\,dx$$



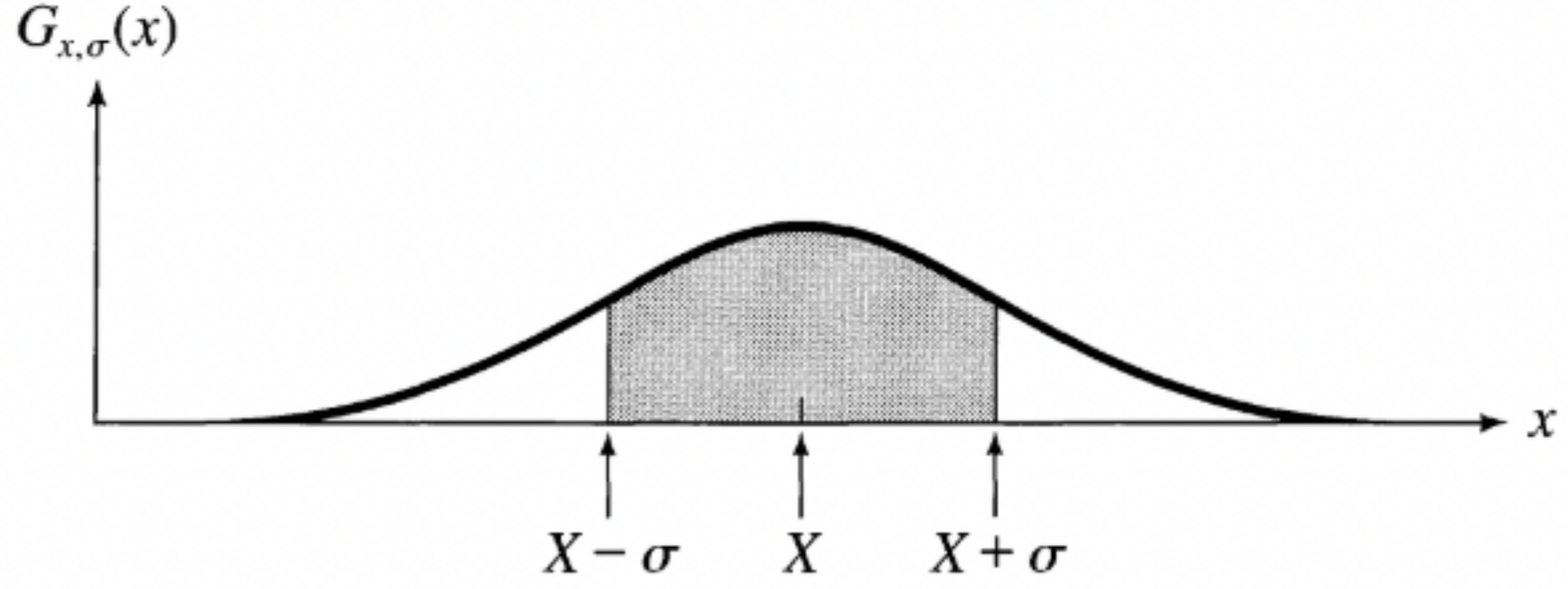
The standard deviation as 68% confidence limit

$$Prob(\text{within } \sigma) = \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{X-\sigma}^{X+\sigma} e^{-(x-X)^2/2\sigma^2}\,dx$$

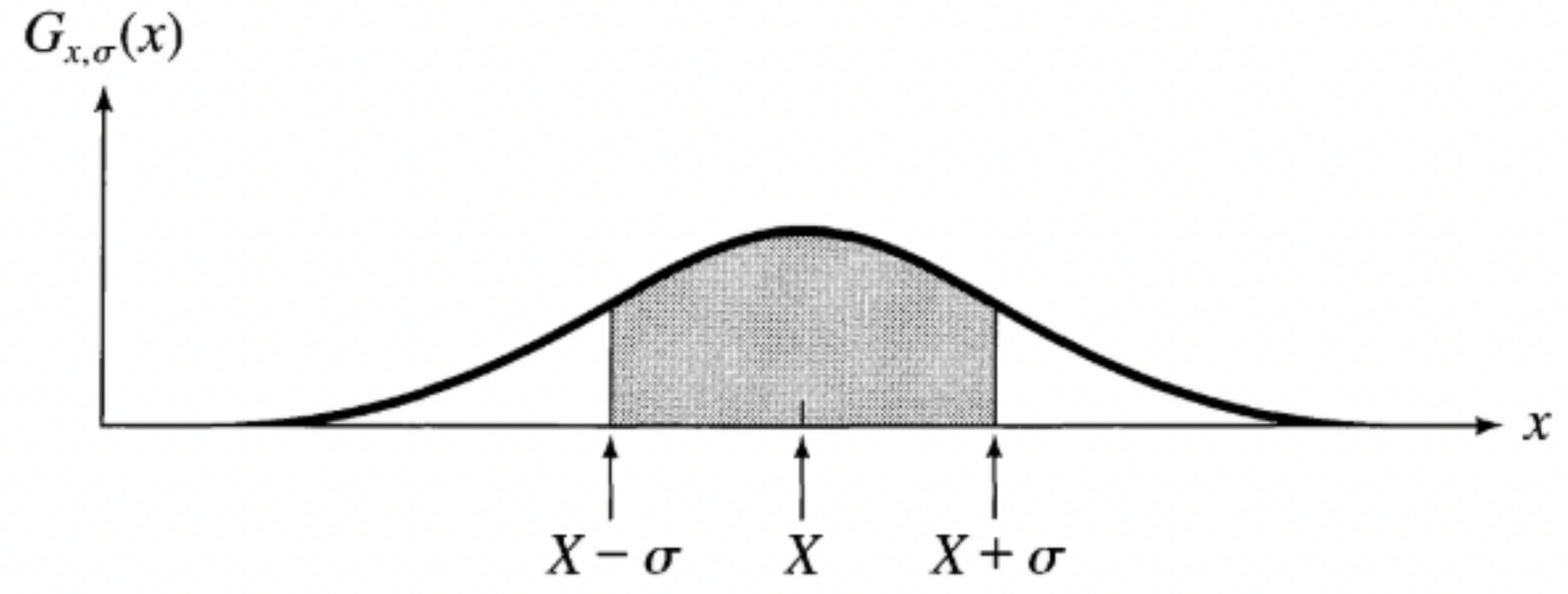
substituting $(x - X)/\sigma = z$.

$$Prob(\text{within } \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2}\,dz.$$



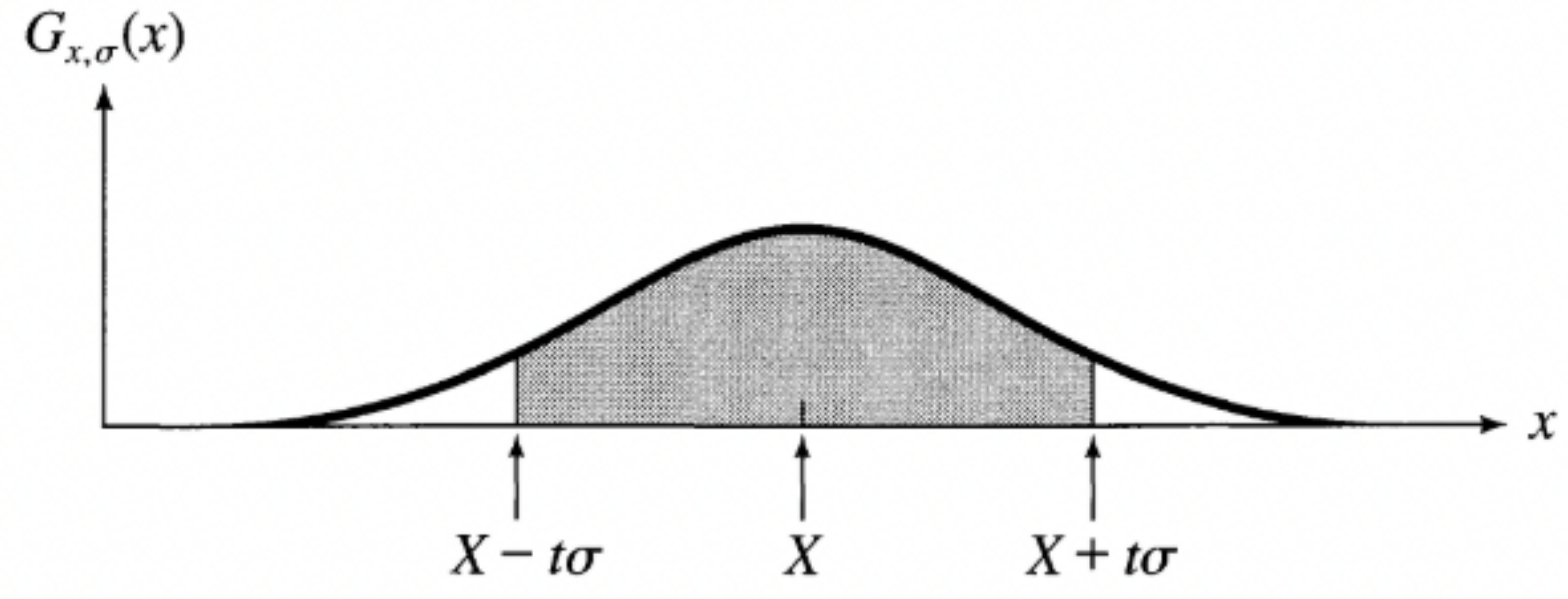
The standard deviation as 68% confidence limit

$$Prob(\text{within } \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2}\, dz.$$



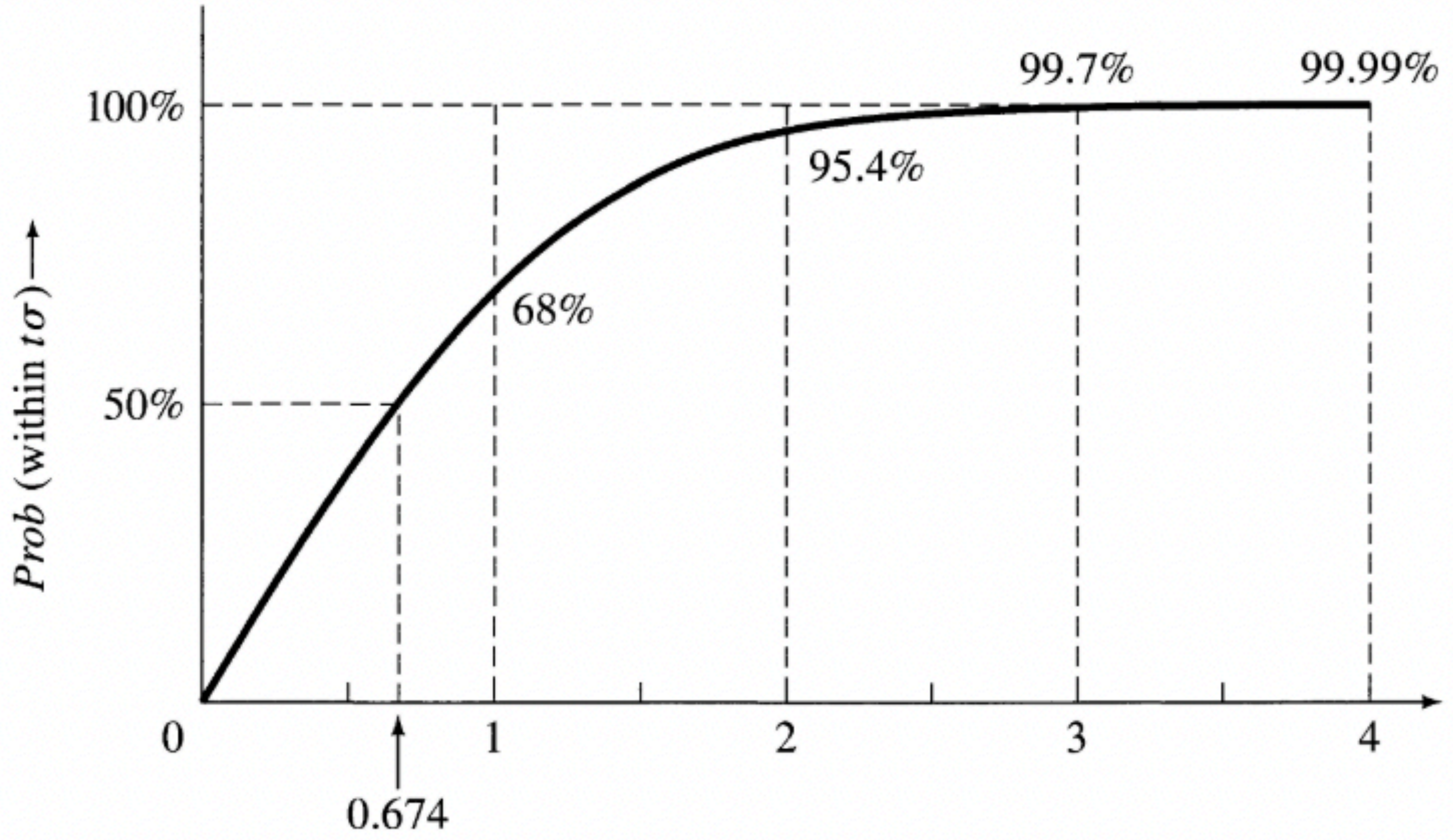
More generally, what is the probability a measurement falls within t*sigma?

$$\boxed{Prob(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz}$$



The error function

$$Prob(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz$$



t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
Prob (%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Quick Check 5.4. The measurements of a certain distance x are distributed normally with $X = 10$ and $\sigma = 2$. What is the probability that a single measurement will lie between $x = 7$ and $x = 13$? What is the probability that it will lie *outside* the range from $x = 7$ to 13?

t	0	0.25	0.5	0.75	1.0	1.25	$\boxed{1.5}$	1.75	2.0	2.5	3.0	3.5	4.0
Prob (%)	0	20	38	55	68	79	$\boxed{87}$	92	95.4	98.8	99.7	99.95	99.99

$$\text{Prob(with } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2} \, dz$$

$$X - x = 10 - 7 = 3 \Rightarrow 3 = t\sigma$$

$$3 = t \cdot 2$$

$$t = 1.5$$

Summary of what we've discussed so far:

> 'limiting distribution' is the distribution is infinite measurements were taken

> we call this 'limiting distribution' f(x) : PDF: probability Density Function

> if f(x) is known (or approximated) we can directly calculate mean and standard deviation from f(x) alone

> if the distribution is normal, than the mean x corresponds to the 'true value' (center) of the distribution

Main problem: we never actual know f(x), and in practice only have a finite number of measurements and our problem is to find the best estimate based on these!

Maximum likelihood estimator

$x_1, x_2, \ldots, x_N,$ data points

Suppose we know the 'center' and 'width' parameters of a Gaussian that describes our finite set of data points

We can estimate the probability of observing x_1 given our Gaussian parameters :

$$\text{Prob}(x \text{ between } x_1 \;\&\; x_1 + dx_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_1 - \bar{X})^2/2\sigma^2} dx_1$$

$$\text{Prob}(x_1) \propto \frac{1}{\sigma} e^{-(x_1 - \bar{X})^2/2\sigma^2}$$

We can do this for all measurements

$$\text{Prob}(x_2) \propto \frac{1}{\sigma} e^{-(x_2 - \bar{X})^2/2\sigma^2} \quad \ldots\ldots \quad \text{Prob}(x_N) \ldots$$

The Principle of Maximum Likelihood

We can estimate the probability of obtaining each of the readings, x_1, x_2 … x_n:

$$\mathrm{Prob}_{X,\sigma}(x_1, \ldots x_n) = \mathrm{Prob}(x_1) \times \mathrm{Prob}(x_2) \cdots \mathrm{Prob}(x_n)$$

or

$$\mathrm{Prob}_{X,\sigma}(x_1 \cdots x_N) \propto \frac{1}{\sigma^N} e^{-\sum (x_i - X)^2 / 2\sigma^2}$$

In reality, the Gaussian parameters X and sigma can not be known!

By iteratively adjusting X and sigma to maximize the probability of observing the data we can get a good estimate of X and sigma from our data points!

Maximum likelihood estimator: summary

Given: N observations, x_1, x_2 … x_n
Find: X and Sigma, expected value (mean) and standard deviation of the limiting distributions

The best estimate, maximizes the following probability:

$$Prob_{X,\sigma}(x_1, \ldots, x_N) \propto \frac{1}{\sigma^N} e^{-\Sigma(x_i - X)^2/2\sigma^2}.$$

mle
Maximum likelihood estimates

R2022b
collapse all in page

MATLAB MLE function

Syntax

```
phat = mle(data)
phat = mle(data,Name,Value)
[phat,pci] = mle( __ )
```

Justification of mean as the best estimate

$$Prob_{X,\sigma}(x_1, \ldots, x_N) \propto \frac{1}{\sigma^N} e^{-\Sigma(x_i - X)^2/2\sigma^2}.$$

When is this maximum?

when sum term is minimum!

$$\sum_{i=1}^{N} (x_i - X)^2/\sigma^2$$

When is this minimum?

differentiate with respect to x, set to zero:

$$\sum_{i=1}^{N} (x_i - X) = 0 \rightarrow (\text{best estimate for } X) = \frac{\Sigma x_i}{N}.$$

This proves that the mean is the best estimate if the limiting distribution is Gaussian!

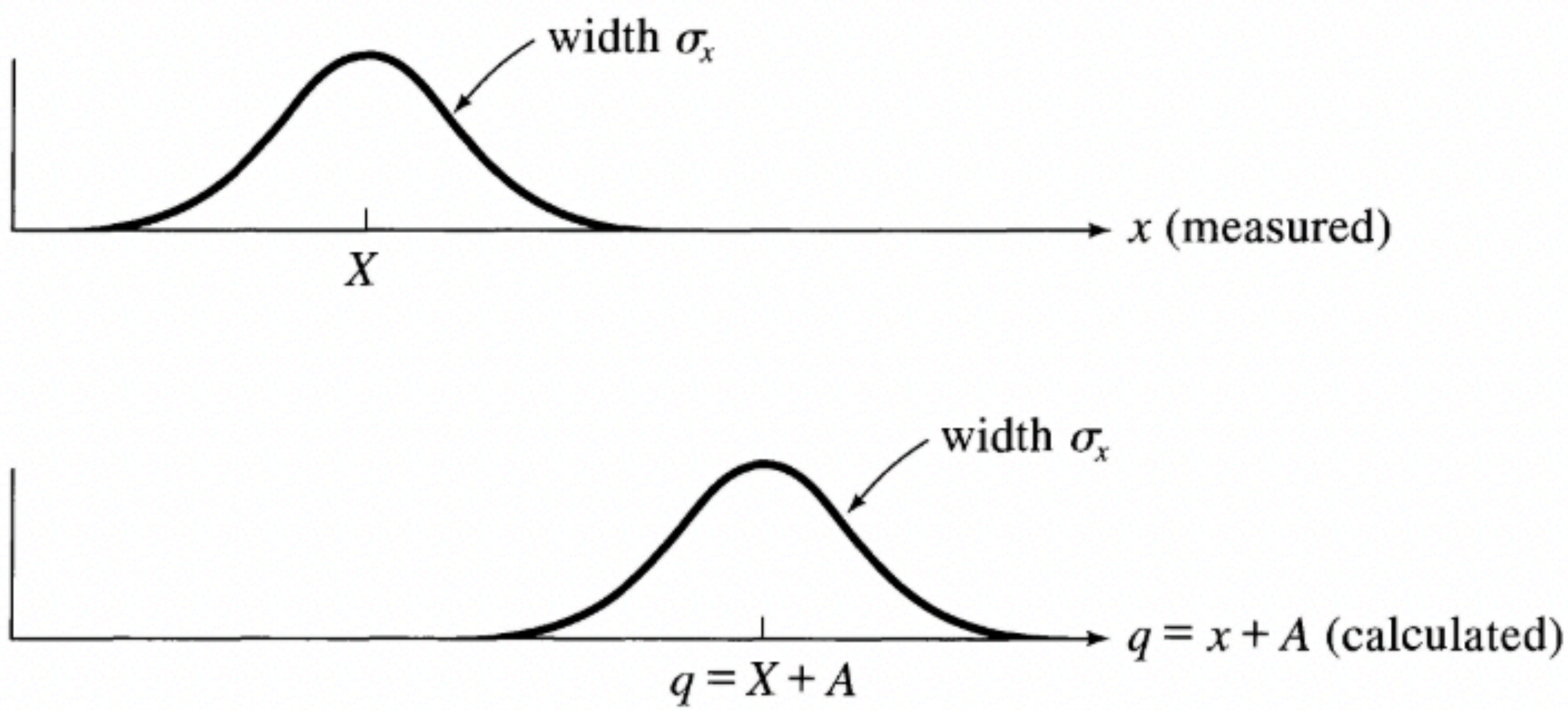
Justification of mean as the best estimate

We can use same arguments for sigma:

$$\text{(best estimate for } \sigma) = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}.$$

Let's revisit our previous uncertainty estimate with our new framework

$$q = x + A \qquad A \text{ is a fixed number with no uncertainty}$$



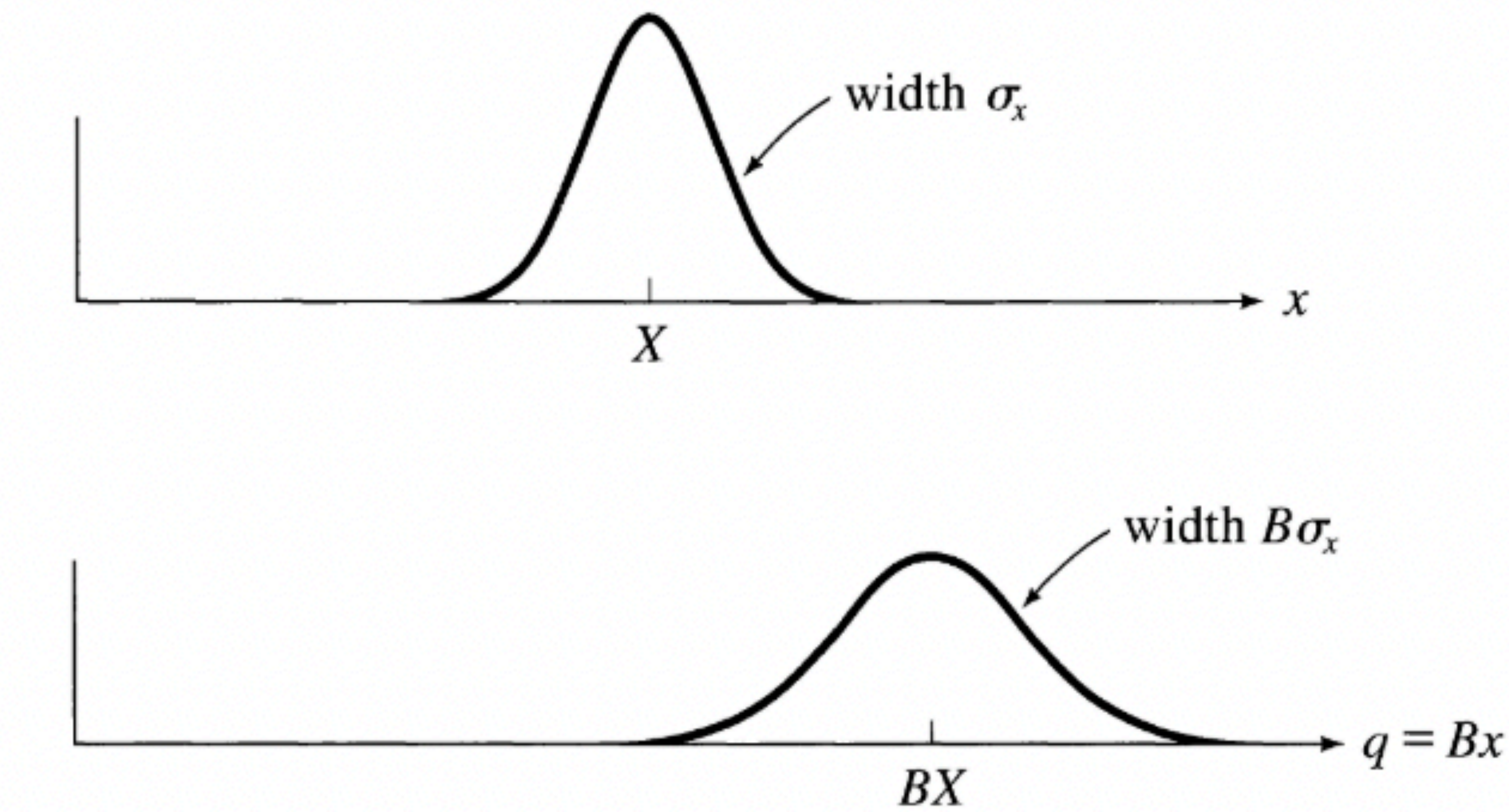
x is our measurement, but q is our experimental outcome, e.g., we need an uncertainty measure of q from x

width (sigma) doesn't change!

$\delta q \sim \delta x$

Let's revisit our previous uncertainty estimate with our new framework

$$q = Bx \qquad \text{where } B \text{ is a fixed number}$$



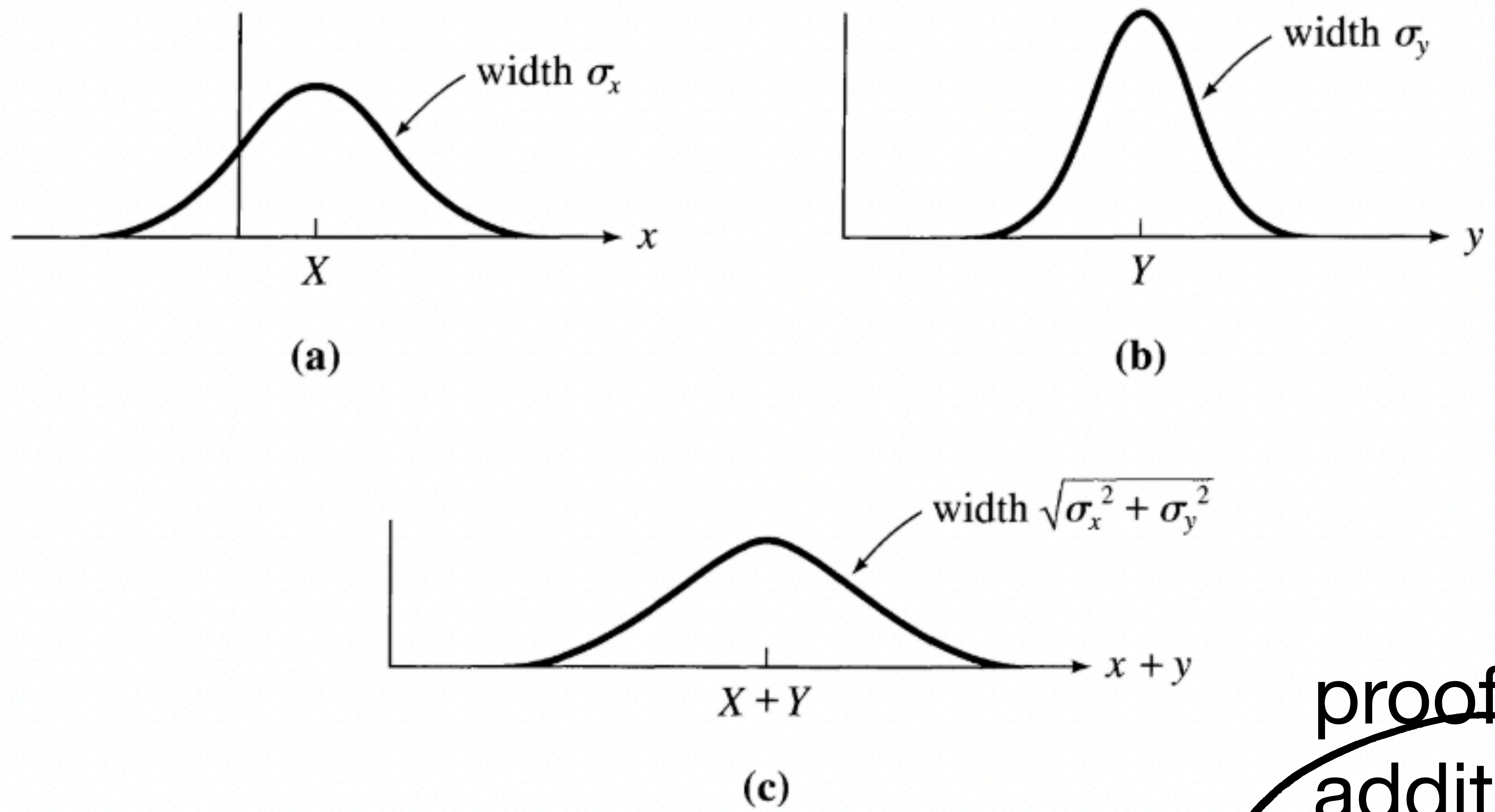
new sigma after B is B*sigma!

$$\sigma_q = B \cdot \sigma_x$$

Let's revisit our previous uncertainty estimate with our new framework

$$q = x + y.$$

both x and y have their own sigmas



new width = $\sqrt{\sigma_x^2 + \sigma_y^2}$

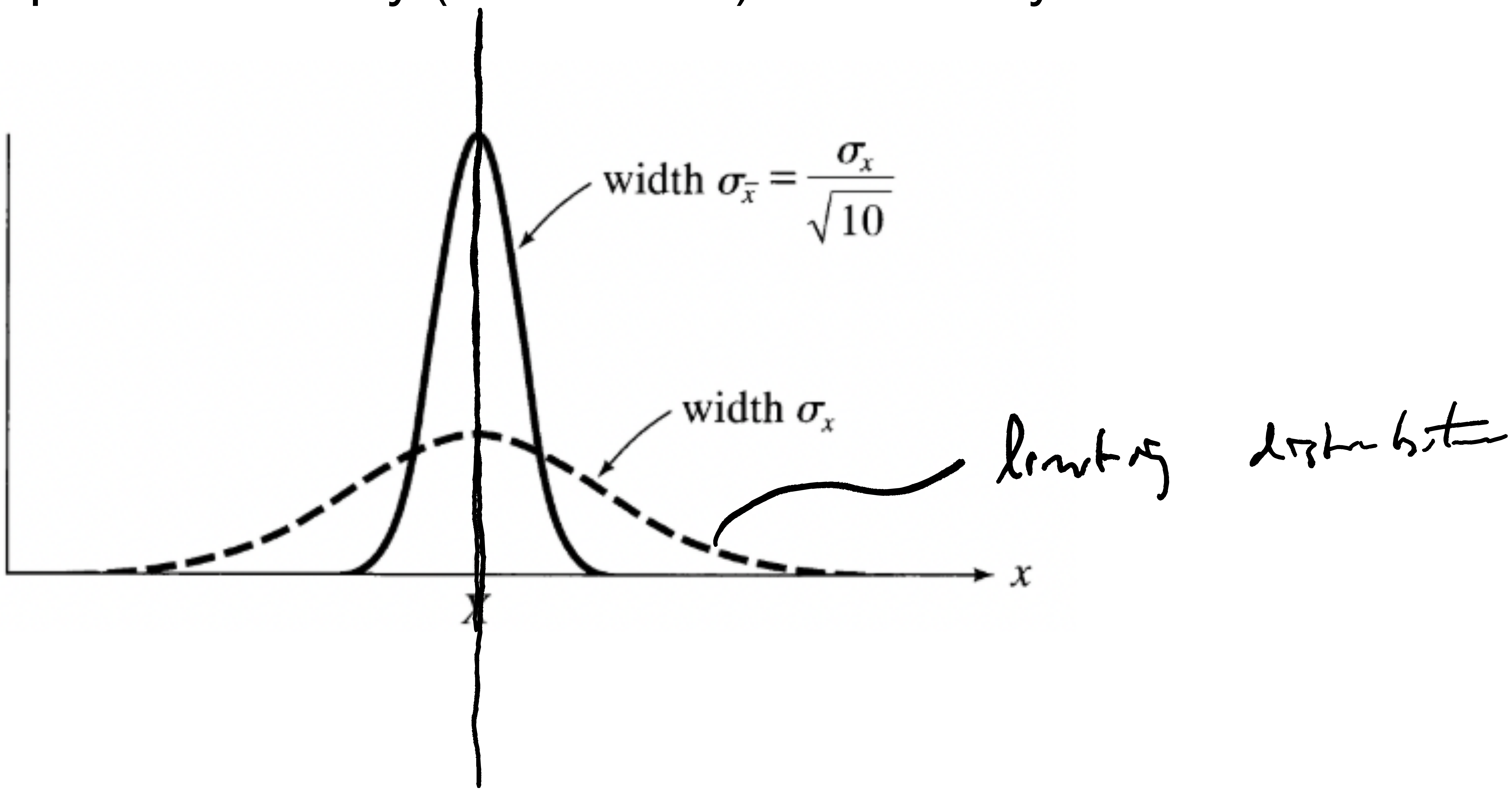
proof in book — this proves addition quadrature is correct ✓

Standard Deviation of the Mean

$$\sigma_{\bar{x}} = \sigma_x/\sqrt{N}$$

recall that the SDM is best estimate
of uncertainty from N measurements

This can be proved directly (in the book). Take away:



Summary

If we measure a quantity x many times, the mean of the measurements corresponds to our best estimate, and the standard deviation of the mean a measure of our uncertainty

$$\boxed{(\text{value of } x) = \bar{x} \pm \sigma_{\bar{x}},}$$

This statement means: we expect 68% of measurements, take in the same way, to fall within our estimated value

Using the Gaussian framework, we can now calculate probabilities directly.
You can use this to determine if a 'discrepancy' is significant or not.
Roughly, this is how 'p-values' or significance is calculated is practice.