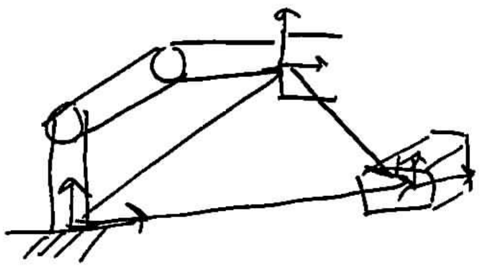
Lecture 10 11|1|2021 ME221

What do we know so far?



1. Describe objects in 3D ; we relate them to each ~~the~~ other

$$T = \begin{bmatrix} R & \vec{d} \\ 0 & 1 \end{bmatrix}$$ Homogeneous Transformation

2. Provide a mathematical description of serial manipulators

$${}^{0}T_{E}(\vec{q}) = {}^{0}T_{1}\,{}^{1}T_{2} \cdots {}^{n-1}T_{E} \quad \text{(Forward Kinematics)}$$

$$\begin{cases} {}^{i-1}T_{i} = f(\text{DH-parameters}) \\ {}^{0}T_{E} = e^{\tilde{S}_1 q_1} \cdots e^{\tilde{S}_n q_n} M \end{cases}$$

3. We know several ways to attack the Inverse Kinematics: $\vec{x} = f(\vec{q})$

$$\vec{q} = f^{-1}(\vec{x})$$

1. analytic solutions
2. ~~m~~ numerical approaches

4. "Differential Kinematics" allows us to understand the velocity relationship between joint variables ; end-effector pose: $\dot{\vec{x}} = J(\vec{q})\,\dot{\vec{q}}$

Jacobians are very rich information. let dive a little deeper:

Today:

- Jacobian generating vector 8.3 ; 9.4
- Singularities
- Manipulability

] Lynch ; Park 5.3 - 5.4

Direct differentiation is tedious! there ~~is better~~ is a better way! Do in <u>tandem</u> with forward Kinematics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \dot{X} = J(\vec{q})\,\dot{\vec{q}}$$

$$J = \begin{bmatrix} J_P \\ J_R \end{bmatrix} = \begin{bmatrix} {}^0\hat{k}_0 \times {}^0_0\vec{d}_n & {}^0\hat{k}_1 \times {}^0_1\vec{d}_n & \cdots & {}^0\hat{k}_{n-1} \times {}^0_{n-1}\vec{d}_n \\ {}^0\hat{k}_0 & {}^0\hat{k}_1 & & {}^0\hat{k}_{n-1} \end{bmatrix}$$

$$c_i(\vec{q}) = \begin{bmatrix} {}^0\hat{k}_{i-1} \times {}^0_{i-1}\vec{d}_n \\ {}^0\hat{k}_{i-1} \end{bmatrix}$$ "Jacobian generating vector"

${}^0_{i-1}\vec{d}_n$: origin of the end-effector coordinate frame with respect to the coordinate $i-1$, expressed in the global frame

${}^0\hat{k}_{i-1}$: unit vector in the direction of the joint axes i expressed in the global frame

Revolute Joint

$$c_i = \begin{bmatrix} {}^0\hat{k}_{i-1} \times {}^0_{i-1}\vec{d}_n \\ {}^0\hat{k}_{i-1} \end{bmatrix}$$

Prismatic Joint

$$c_i = \begin{bmatrix} {}^0\hat{k}_{i-1} \\ 0 \end{bmatrix}$$

Easier to compute:

$${}^0\hat{k}_{i-1} = {}^0R_{i-1} \, {}^{i-1}\hat{k}_{i-1}$$

$${}^0\hat{k}_{i-1} \times {}^0_{i-1}\vec{d}_n = {}^0R_{i-1}\left({}^{i-1}\hat{k}_{i-1} \times {}^{i-1}\vec{d}_n \right)$$

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$${}^0T_6 = {}^0T_1 \, {}^1T_2 \, {}^2T_3 \cdots {}^5T_6$$

$$= \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ - & - & \cdots & t_{24} \\ & & & t_{34} \\ & 0 & & 1 \end{bmatrix}$$

Column 1:

$${}^0\hat{k}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0_0\vec{d}_n = {}^0\vec{d}_n = \begin{bmatrix} t_{14} \\ t_{24} \\ t_{34} \end{bmatrix}$$

$${}^0\hat{k}_0 \times {}^0\vec{d}_6 = {}^0\tilde{k}_0 \, {}^0\vec{d}_6$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_{14} \\ t_{24} \\ t_{34} \end{bmatrix} = \begin{bmatrix} -t_{24} \\ t_{14} \\ 0 \end{bmatrix}$$

$$c_1 = \begin{bmatrix} -t_{24} \\ t_{14} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Column 2:

$$ {}^0\hat{k}_1 = {}^0R_1 \, {}^1\hat{k}_1 = {}^0R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ {}^0T_6 = {}^0T_1 \, {}^1T_2 \cdots {}^5T_6 $$

$$ {}^0\hat{k}_1 \times {}^1_0\vec{d}_6 = {}^0R_1 \left({}^1\hat{k}_1 \times {}^1\vec{d}_6 \right) $$

$$ \underbrace{{}^1T_0 \, {}^0T_6}_{} = {}^1T_2 \cdots {}^5T_6 $$

· Singularities

$$ \dot{x} = J(q)\dot{q} \Rightarrow \dot{q} = J^{-1}(q)\,\dot{x} $$

- Jacobian describe the instantaneous velocity relationship between end-effector & joint variables

- singular configuration occurs when the Jacobian drops rank

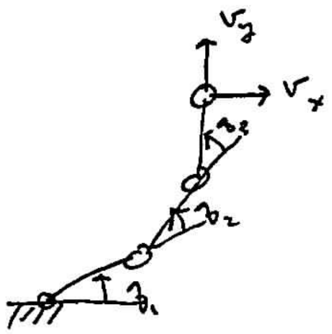
Matrix rank: max # of independent columns

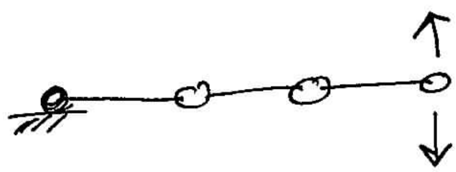
→ if we drop rank two or more columns become linearly dependent if rank < # columns

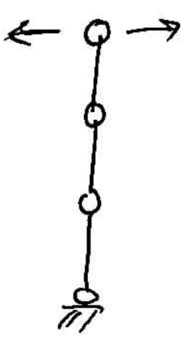
matrix become non-invertible

$$\dot{q} = J^{-1}(q)\,\dot{x}$$

- near ~~sig~~ singular configuration may require extremely large joint velocities to maintain desired end-effector velocity.







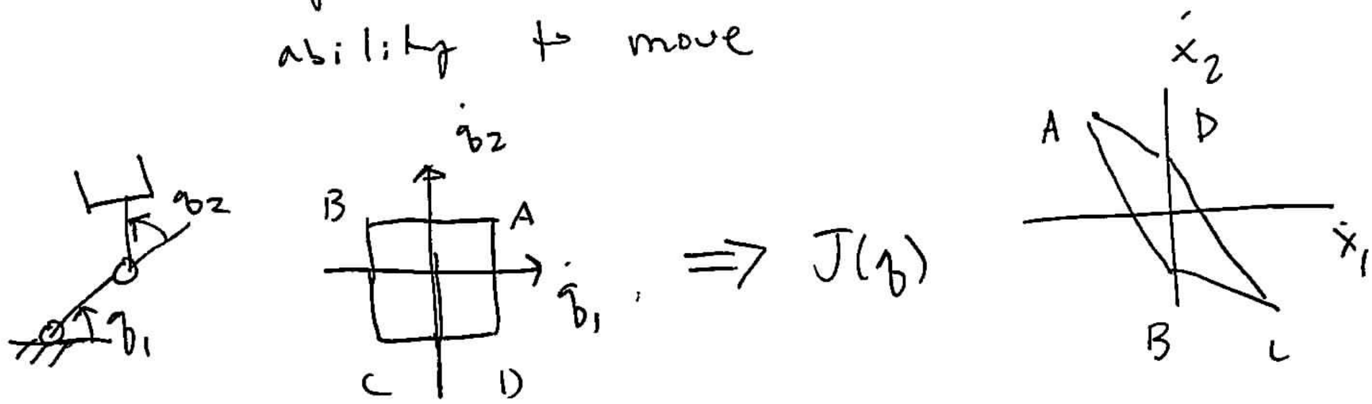
$$\min_{q} \|\vec{x}_d - f(q)\|$$

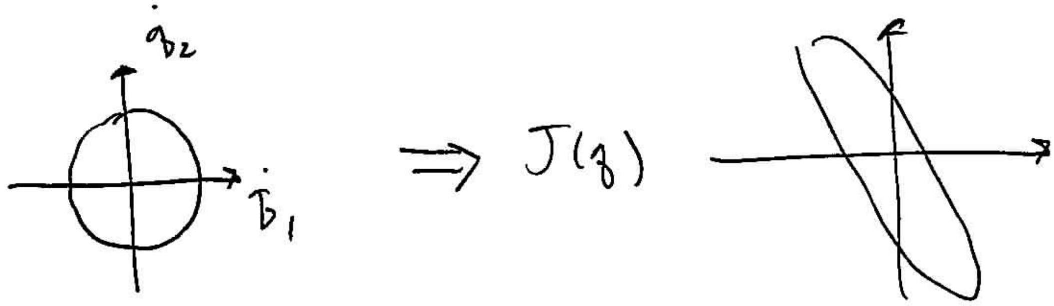
$$\text{s.t.} \quad \mathrm{rank}(J)\ \cancel{\geq}\ \cancel{k}\ n$$

- Can we determine whether a certain configuration is "close" to becoming singular?

- Manipulability

  → geometric representation of the end-effector's ability to move





$$1 = \dot{q}^T \dot{q}$$

$$= \left(J(q)^{-1} \dot{x} \right)^T \left(J(q)^{-1} \dot{x} \right)$$

$$= \dot{x}^T \left(J(q)^{-1} \right)^T \left(J(q)^{-1} \right) \dot{x}$$

$$= \dot{x}^T \underbrace{\left(J J^T \right)}_{A}{}^{-1} \dot{x} \quad \Rightarrow \quad \boxed{\dot{x}^T A^{-1} \dot{x} = 1}$$

$A = J J^T$: square (6×6), symmetric $(A = A^T)$, positive definite $(x^T A x > 0 \ \forall x)$

$$A = V D V^T$$

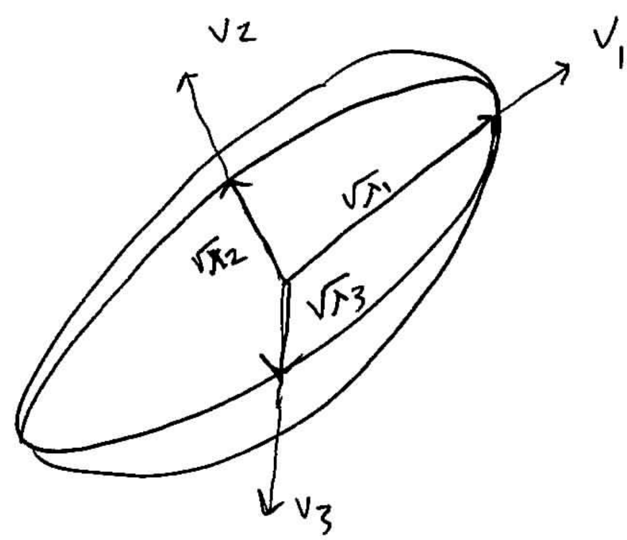
$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_n & — \end{bmatrix}$$

$$\dot{x}^T A^{-1} \dot{x} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow [x\ y\ z] \begin{bmatrix} 1/a^2 & & 0 \\ & 1/b^2 & \\ 0 & & 1/c^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\sqrt{\lambda_n}$: axis

V_n : unit vectors



$$\mu_1(JJ^T) = \frac{\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}} = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \geq 1$$

$\mu_1(JJ^T) = 1$? isotropic

$\mu_1(JJ^T) \rightarrow \infty$

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condition

$$\mu_2(JJ^T) = \frac{\lambda_{max}}{\lambda_{min}} \geq 1$$

$$\mu_3(\cancel{A}JJ^T) = \sqrt{\lambda_1 \cdots \lambda_n} = \sqrt{\det(JJ^T)} \sim \text{volume}$$

$$\max \quad \mu_3(JJ^T)$$
$$\text{s.t.} \quad x - f(\theta) = 0$$

vs

$$\min \quad x - f(\theta)$$