11/8/2021 Lecture 12

Last time:

- static force relationship
- intro dynamics

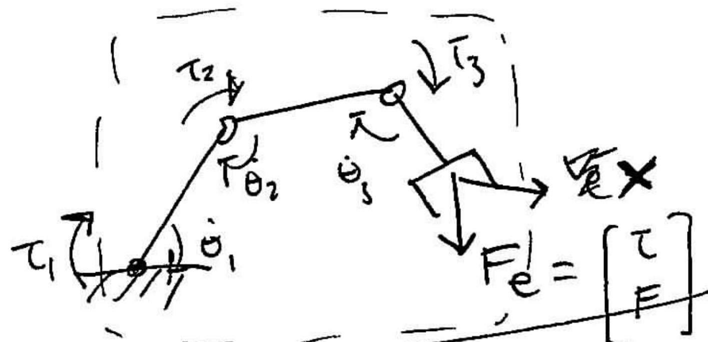
Today:

- Review static force
- Overview of dynamics methods
- Lagrangian Mechanics

Velocity Force Duality

Conservation of energy.

$$\boxed{P = F \cdot V}$$



$$\dot{q}^T \tau = \dot{x}^T F_e$$

Power In = Power Out

$$\dot{q}^T \tau = (J(q)\dot{q})^T F_e$$

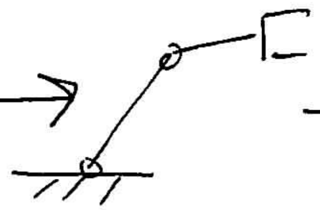
$$\dot{q}^T \tau = \dot{q}^T J(q)^T F_e$$

$$\boxed{\tau = J(q)^T F_e}$$

Forward Dynamics

given a set of joint torques, what is the motion that results?

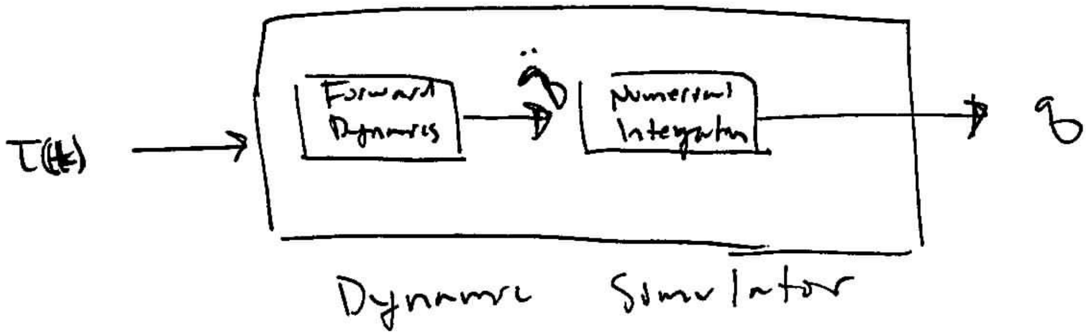
$$\tau(t) = \begin{bmatrix} \tau_1(t) \\ \vdots \\ \tau_n(t) \end{bmatrix} \longrightarrow$$



$$\longrightarrow \quad \ddot{q}(t) = \begin{bmatrix} \ddot{q}_1(t) \\ \vdots \\ \ddot{q}_n(t) \end{bmatrix}$$

$$\underbrace{\ddot{q} = f(\tau)}_{\text{Equations of Motion}} = f(q, \dot{q}, \tau)$$

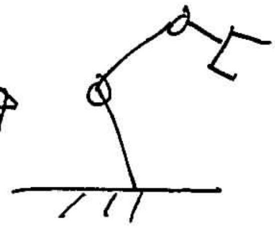
Main use: Simulation



Inverse Dynamics

What are the forces to achieve a certain motion ?

$q_d, \dot{q}_d, \ddot{q}_d \longrightarrow$



$\longrightarrow \tau_d = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$

Main use: control

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \quad *$$

q : generalized joint coordinates

$M(q)$: Mass matrix

$C(q,\dot{q})$: centrifugal & Coriolis forces

$g(q)$: gravity

τ : generalized torque vector

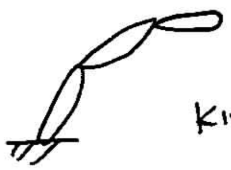
$* + J(q)^T F_e$

Building a "dynamic model"
we mean we finding all the matrices
$M(q)$, $C(q,\dot{q})$, vector $g(q)$

Methods for computing Robot Dynamics

Euler - Lagrange

- energy based



Kinetic energy

$T(q,\dot{q})$

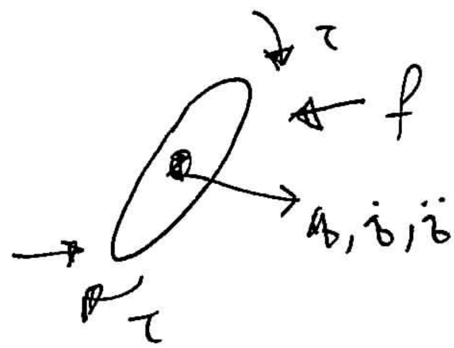
Potential energy

$V(q)$

- derive the EOM
- study dynamic properties
- Analyze control schemes

Newton - Euler

- Force balance



- <u>numeric</u> approach
- recursive algorithm
- Inverse Dynamics
- ~~cate~~ control implementations

Basic Idea:

"Principle of Least Action"

↳ solutions to EOM are stationary points of the system action functional

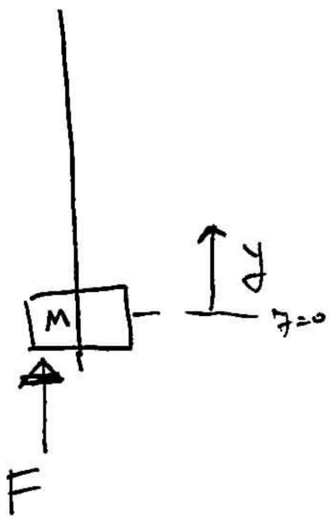
- thermodynamics
- fluid mechanics
- relativity
- Quantum mechanics
- etc

The Lagrangian

$$\mathcal{L} = T(q, \dot q) - V(q) \qquad T = \frac{1}{2} m v^2$$

$$\boxed{f = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot q} - \frac{\partial \mathcal{L}}{\partial q}}$$

Example: Vertical Mass



$$L = T - V$$

$$f = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot q} - \frac{\partial \mathcal{L}}{\partial q}$$

$$T = \frac{1}{2} m \dot y^2$$

$$V = mgy$$

$$\mathcal{L} = \frac{1}{2} m \dot y^2 - mgy$$

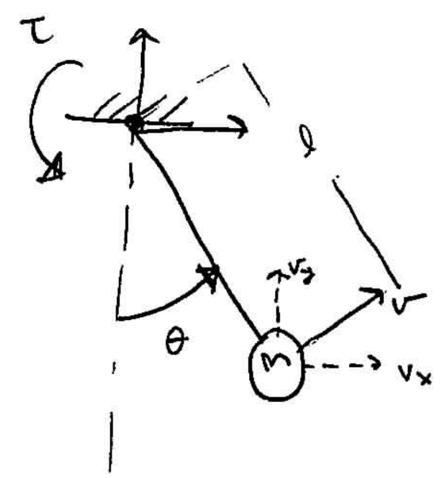
$$\frac{\partial \mathcal{L}}{\partial \dot q} = m\dot y \rightarrow \frac{d}{dt}(m\dot y) = m\ddot y$$

$$\frac{\partial \mathcal{L}}{\partial q} = -mg$$

$$F = m\ddot y + mg$$

$$\boxed{m\ddot y = F - mg}$$

Example: Pendulum



$$\mathcal{L} = T - V$$

$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}$$

$V = mgy$

$= mg(l - l\cos\theta)$

$T =$

$x = l\sin\theta \Rightarrow v_x = l\cos\theta\,\dot{\theta}$

$y = l - l\cos\theta \Rightarrow v_y = l\sin\theta\,\dot{\theta}$

$T = \frac{1}{2}m\left[v_x^2 + v_y^2\right]$

$= \frac{1}{2}m\left[l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2\right]$

$= \cancel{\frac{1}{2}ml^2\dot{\theta}^2} = \frac{1}{2}ml^2\dot{\theta}^2$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl + mgl\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = ml^2\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial q} = -mgl\sin\theta$$

$$\tau = ml^2\ddot{\theta} + mgl\sin\theta$$

$$\ddot{\theta} = \left(\frac{1}{ml^2}\right)\tau - \frac{g}{l}\sin\theta$$

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ -\frac{g}{l}\sin\theta + \left[\frac{1}{ml^2}\right]\tau\end{bmatrix}$$

$$\uparrow$$

$$\vec{q} = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

$$\boxed{\cancel{\dot{q} = f(q, \dot{q}, \tau)}}$$

$$\boxed{\dot{q} = f(q, \tau(q))}$$