ME 221 Lecture 2 9/29/2021

Today : Rotation & Orientations

Goal :

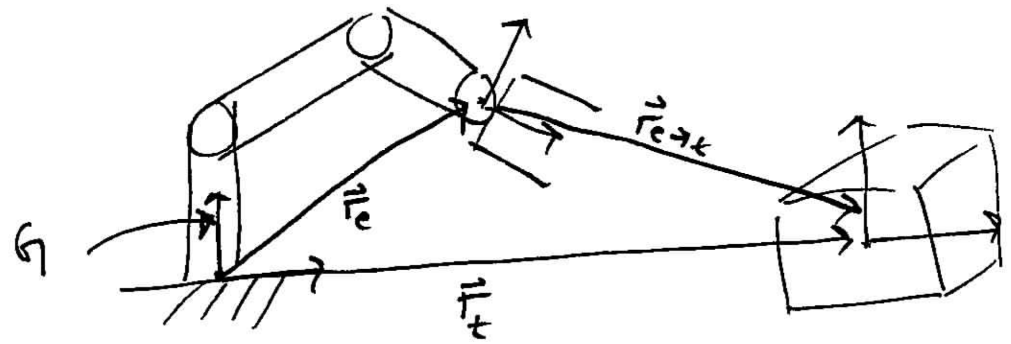
① Review coordinates frame & vectors

② Introduce rotation matrices (R)

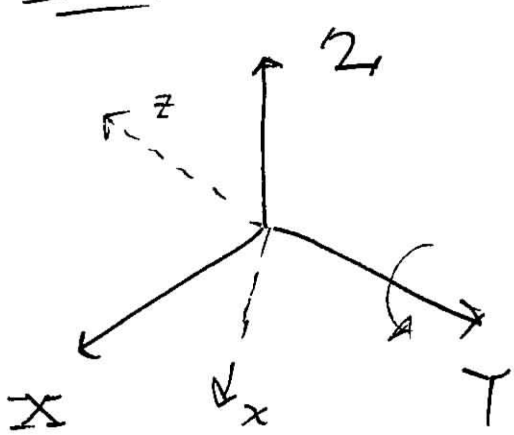
③ Show a couple to parameterize R

Motivation :

$\vec{r}$: vector



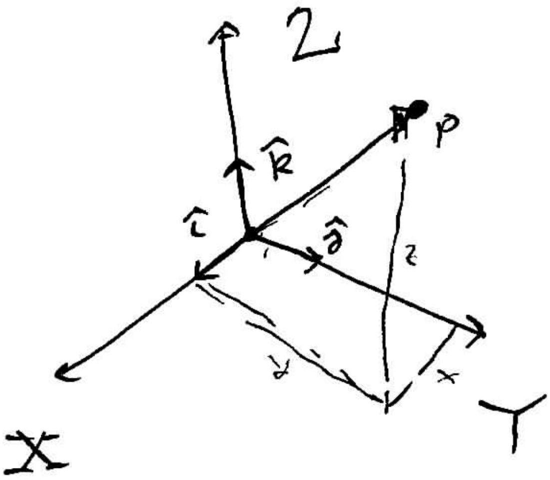
How to mathematically represent robot / objects in global frame?



Reference frame

$\mathbb{R}^3 \rightarrow x, y, z$

- a coordinate system identifies geometric points in space
  - → cartesian
  - spherical
  - etc.



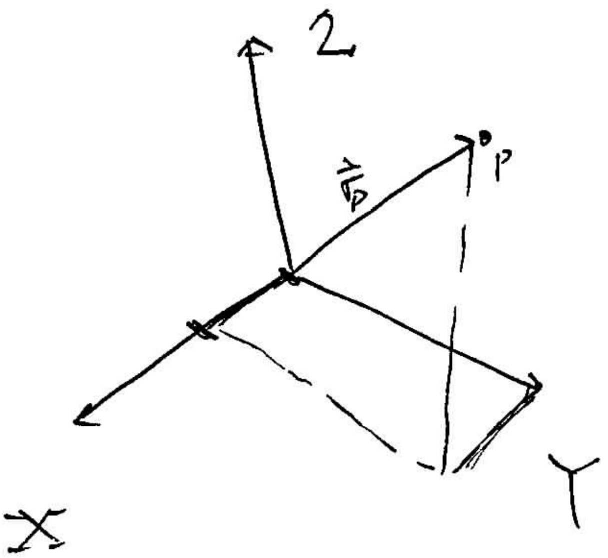
$$\vec{r}_P = \begin{bmatrix} r_{P_x} \\ r_{P_y} \\ r_{P_z} \end{bmatrix}$$

unit vectors

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{i} \cdot \hat{j} = 0 \quad \rightarrow \quad \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

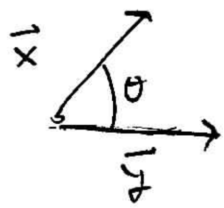
$$\vec{r}_p = \underbrace{r_{p_x}\,\hat{i} + r_{p_y}\,\hat{j} + r_{p_z}\,\hat{k}}_{\text{linear combination of the unit vectors!}}$$

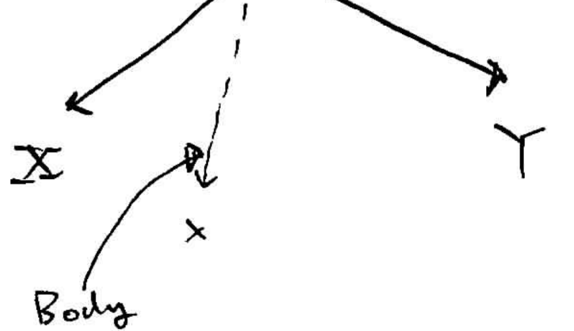


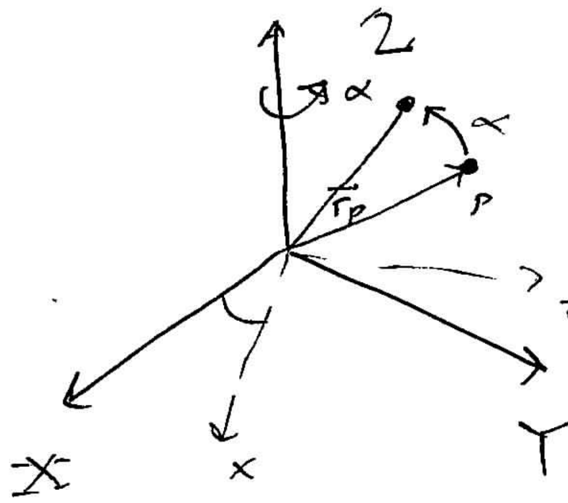
$$\vec{r}_p = (\vec{r}_p \cdot \hat{i})\,\hat{i} + (\vec{r}_p \cdot \hat{j})\,\hat{j} + (\vec{r}_p \cdot \hat{k})\,\hat{k}$$

dot product: $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

$$= |\vec{x}|\,|\vec{y}|\cos\theta$$







How find coordinates in Global frame?

$${}^{B}\vec{r}_p = x_p\,\hat{i} + y_p\,\hat{j} + z_p\,\hat{k}$$

$${}^{G}\vec{r}_p = ?$$

$$X_p = \left(\vec{r}_p \cdot \hat{I}\right) = x_p\,\hat{i}\cdot\hat{I} + y_p\,\hat{j}\cdot\hat{I} + z_p\,\hat{k}\cdot\hat{I}$$

$$Y_p = \left(\vec{r}_p \cdot \hat{J}\right) = x_p\,\hat{i}\cdot\hat{J} + y_p\,\hat{j}\cdot\hat{J} + z_p\,\hat{k}\cdot\hat{J}$$

$$Z_p = \left(\vec{r}_p \cdot \hat{K}\right) = x_p\,\hat{i}\cdot\hat{K} + y_p\,\hat{j}\cdot\hat{K} + z_p\,\hat{k}\cdot\hat{K}$$

$$\begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} = \underbrace{\begin{bmatrix} \overset{\cos\theta}{\hat{i}\cdot\hat{I}} & \hat{j}\cdot\hat{I} & \hat{k}\cdot\hat{I} \\ \hat{i}\cdot\hat{J} & \hat{j}\cdot\hat{J} & \hat{k}\cdot\hat{J} \\ \hat{i}\cdot\hat{K} & \hat{j}\cdot\hat{K} & \hat{k}\cdot\hat{K} \end{bmatrix}}_{\text{directional cosine}} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$\underbrace{R_z(\alpha) \quad R_y(\beta) \quad R_x(\gamma)}$$

Basic rotation matrices

<u>Any</u> rotation about fixed point can be realized by <u>at most</u> three consecutive rotations

Properties

① Orthogonality : $R^T = R^{-1}$

$$\Rightarrow RR^T = I$$

R has $\underline{\underline{9}}$ elements

R has $\underline{\underline{6}}$ constraints

$\therefore$ R has $9 - 6 = \underline{\underline{3}}$ degrees of freedom.

② $\det(R) = +1$

$$RR^T = I$$

$$\underbrace{\det(RR^T)}_{\det(R)\det(R^T)} = \cancel{I}\,\det(I)$$

$$= 1$$

$$\det(R)\det(R^T) = 1$$

$$\det(R)^2 = 1$$

$$\boxed{\det(R) = \pm 1}$$

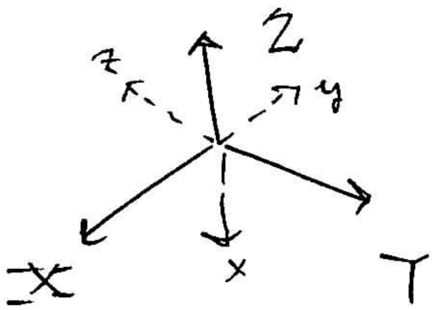
We call group of $\underline{\text{all}}$ R's the "Special Orthogonal" group SO(3)

$$SO(3) = \{ R \in \mathbb{R}^3 \times \mathbb{R}^3 : RR^T = I;\ \det(R) = +1 \}$$

Lie group ('Lee') → represent all rotations of rigid bodies

Interpretations of R

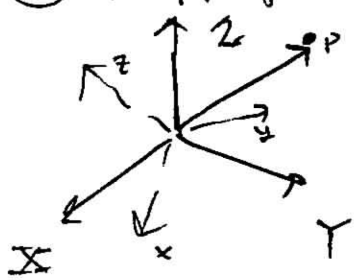
① Represents orientation



→ ${}^{G}R_{B}$: represents frame 'B' interms of frame 'G'

→ ${}^{G}R_{B}$: takes frame 'G' to a new frame 'B' (given interms of global)

② Mapping between frames



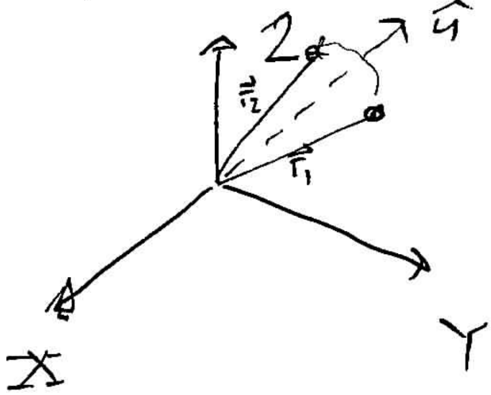
→ ${}^{G}\vec{r}_{p} = {}^{G}R_{B}\,{}^{B}\vec{r}_{p}$

→ ${}^{B}\vec{r}_{p} = {}^{B}R_{G}\,{}^{G}\vec{r}_{p}$

$${}^{B}R_{G} = \left({}^{G}R_{B}\right)^{T}$$

(3.) Rotation operator

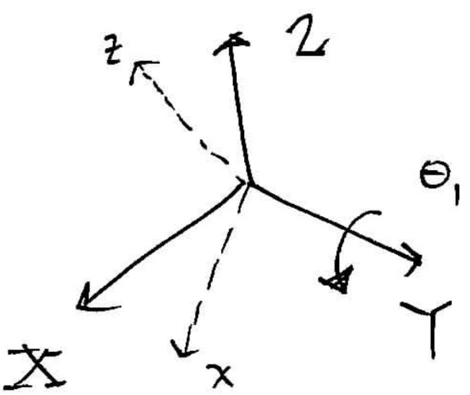
→ use `R' to rotate vectors

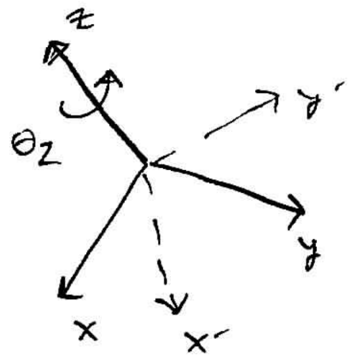


$$ {}^G\vec{r}_2 = R(\theta, \hat{u}) \cdot {}^G\vec{r}_1 $$

<u>Compositions</u>

1. Rotations with respect to "moving" frame

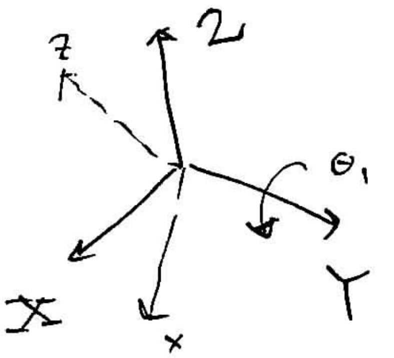




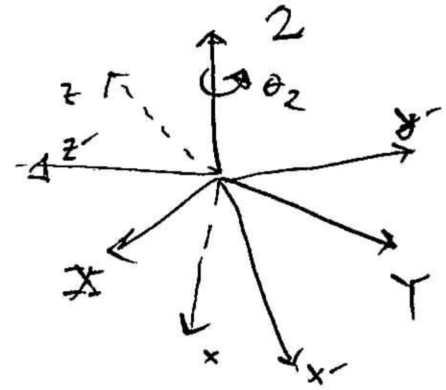
$R_y(\theta_1)$ $R_z(\theta_2)$

<u>Post-multiplication</u> $R = R_y(\theta_1)\, R_z(\theta_2)$

2. Rotations with respect to the fixed frame



$R_y(\theta_1)$



$R_z(\theta_2)$

Pre-multiplication

$$\boxed{R = R_z(\theta_2) R_y(\theta_1)}$$

Example: Find overall rotation matrix

1. $R_x(\theta)$ moving frame
2. $R_z(\phi)$ moving frame
3. $R_z(\alpha)$ Fixed frame
4. $R_y(\beta)$ moving frame
5. $R_x(\gamma)$ Fixed frame

$R = ?$

$$R = R_x(\gamma) R_z(\alpha) R_x(\theta) R_z(\phi) R_y(\beta)$$

~~Paramett~~

Parameterize R

- orientation has 3 degrees of freedom
- R has 9 elements
- R is redundant

Typically:

1. Euler Angles
2. Roll, pitch, yaw
3. Axis / angle representation
4. Quaterions

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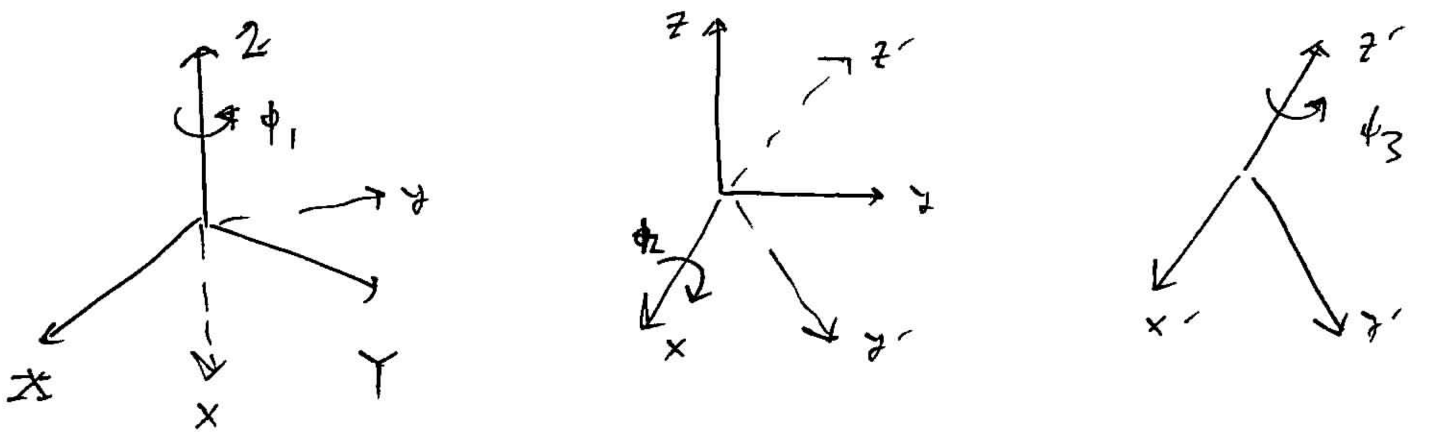
① Euler Angles

- 3 successive rotations about moving frame
- 12 possible combos: XYZ
  XYX
  ⋮

$3 \cdot 2 \cdot 2 = 12$ total Euler Angles

Z - Y - Z



$R_z(\phi_1)$ $\qquad$ $R_{x'}(\phi_2)$ $\qquad$ $R_{z'}(\phi_3)$

$$R(\phi_1, \phi_2, \phi_3) = R_z(\phi_1) \cdot R_x(\phi_2) \cdot R_{z'}(\phi_3)$$

$$ZYZ = R_z(\phi_1)\, R_y(\phi_2)\, R_z(\phi_3)$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} - & - & c\phi_1 s\phi_2 \\ - & - & s\phi_1 s\phi_2 \\ -s\phi_2 c\phi_3 & s\phi_2 s\phi_3 & c\phi_2 \end{bmatrix}$$

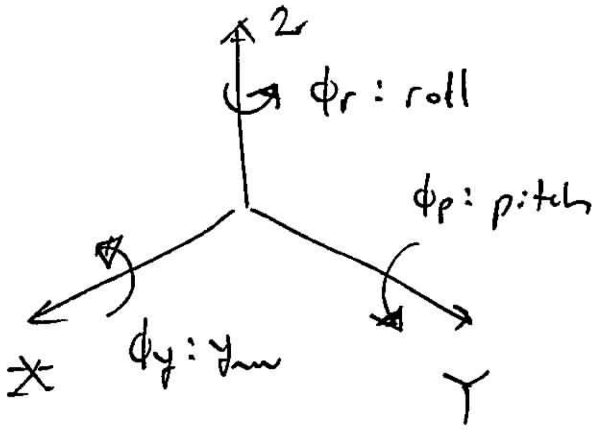
$$\phi_2 = \operatorname{atan2}\left(\pm\sqrt{r_{13}^2 + r_{23}^2},\ r_{33} \right)$$

$$\phi_1 = \operatorname{atan2}\left(\frac{r_{23}}{\sin\phi_2},\ \frac{r_{13}}{\sin\phi_2} \right)$$

$$\phi_3 = \operatorname{atan2}\left(\frac{r_{32}}{\sin\phi_2},\ \frac{r_{31}}{\sin\phi_2} \right)$$

$\sin\phi_2 = 0 \qquad \phi_2 = 0, \pi, 2\pi$

(2.) Roll, pitch, yaw (Fixed frame)



Sequence (1) yaw
(2) pitch
(3.) roll

$$\boxed{R = R_z R_y R_x}$$

$\longrightarrow$ $\underline{\cos \phi_p = 0}$ singularity