Lecture 3 ME 221 10/4/2021

Recap:

Basic Rotation matrices $R_z(\phi)$ $R_y(\phi)$ $R_x(\phi)$

- orthogonal

$$R^T = R^{-1} \Rightarrow RR^T = R^TR = I$$

$\hookrightarrow$ 6 constraints

- $\det(R) = 1$

- Composition
  - $\rightarrow$ Fixed frame (pre-multiplication)

$$R = R_2 R_1$$

  - $\rightarrow$ Moving frame (post-multiplication)

$$R = R_1 R_2$$

- Parameterize

In general use <u>three</u> successive rotation to ~~to get~~ get to general orientation.

$\rightarrow$ Euler Angles (12 total)

$\rightarrow$ successive rotation moving frame

$\rightarrow$ (Global) Roll pitch yaw

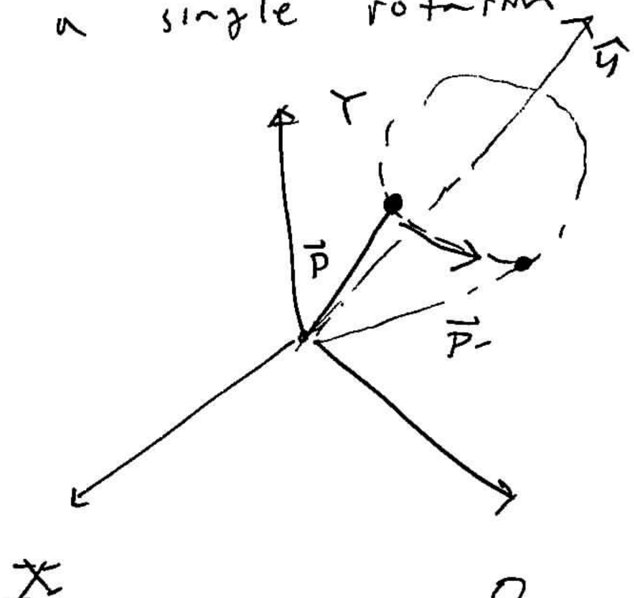
$\rightarrow$ successive rotation fixed frame

Main Problem: they singularities (gimbal lock)

Today:

- Axis / Angle representation
  - Exponential Coordinates
  - Quaternion
- Motion Kinematics
  - Homogeneous Transformation
  - Inverse
  - Composition

Euler's Theorem: Any rigid body ~~rotations~~ displacement ~~with~~ about fixed point is equivalent to a single rotation about some fixed axis.

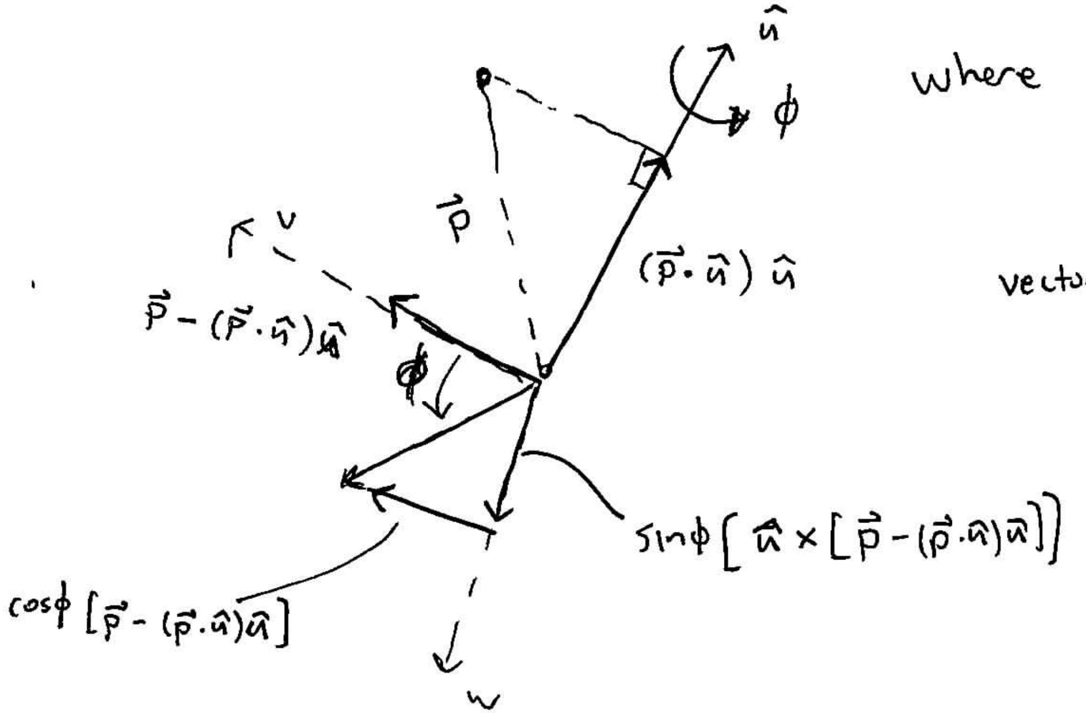


$$\vec{r}_{p'} = R\,\vec{r}_p$$

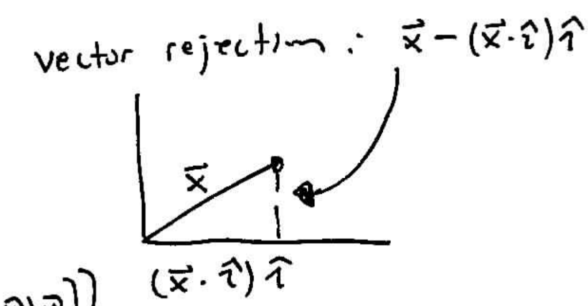
Is there an <u>axis</u> that remains fixed?

$$\vec{u} = R\vec{u} \Rightarrow R\vec{u} = \lambda\vec{u} \qquad \lambda = 1$$

How can we create R from a given axis angle?



where does it end up?



$$\underbrace{(\vec{p}\cdot\hat{u})\,\hat{u}}_{\hat{u}\hat{u}^T\vec{p}} + \cos\phi\,[\vec{p}-(\vec{p}\cdot\hat{u})\hat{u}] + \sin\phi\left[\hat{u}\times[\vec{p}-(\vec{p}\cdot\hat{u})\hat{u}]\right]$$

$$\hat{u}\hat{u}^T\vec{p} + \cos\phi\,\vec{p} - \cos\phi\,\hat{u}\hat{u}^T + \sin\phi\left[\underbrace{\hat{u}\times\vec{p}}_{\tilde{u}\vec{p}} - \cancelto{0}{\hat{u}\times\hat{u}}\,(\vec{p}\cdot\vec{u})\right]$$

$$\underbrace{\left[(1-\cos\phi)\hat{u}\hat{u}^T + \cos\phi\,I + \tilde{u}\sin\phi\right]}_{R(\hat{u},\phi)}\vec{p}$$

Rodrigues Formula

Eq. 3.4

$$\tilde{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

$A^T = -A$

$$\tilde{u} = \frac{1}{2\sin\phi}\left[R(\hat{u},\phi) - R(\hat{u},\phi)^T \right]$$

$$\cos\phi = \frac{1}{2}\left[\underbrace{tr\left(R(\hat{u},\phi)\right)}_{\text{trace}} - 1 \right]$$

$$R(\hat{u}, \phi + 2\pi k) = R(\hat{u},\phi)$$

<u>Exponential Coordinates</u> Murray Ch. 2

$\hat{u}\phi \Rightarrow$ what is $\underbrace{e^{\tilde{u}\phi}}_{\text{matrix exponential}}$

Taylor Exp.

$$e^{\tilde{u}\phi} = I + \tilde{u}\phi + \frac{\phi^2}{2!}\tilde{u}^2 + \frac{\phi^3}{3!}\tilde{u}^3 \cdots$$

some useful relationships

$$\tilde{u}^2 = \hat{u}\hat{u}^T - I$$
$$\tilde{u}^3 = -\tilde{u}$$

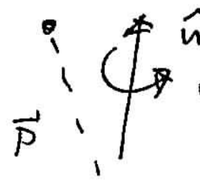
$$e^{\tilde{u}\phi} = I + \underbrace{\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \cdots\right)}_{\sin\phi}\tilde{u} + \underbrace{\left(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \frac{\phi^6}{6!}\right)}_{1-\cos\phi}\tilde{u}^2$$

$$e^{\tilde{u}\phi} = I + \tilde{u}\sin\phi + \tilde{u}^2(1-\cos\phi)$$

$$e^{\tilde{u}\phi} = \cos\phi\, I + \tilde{u}\sin\phi + \hat{u}\hat{u}^T(1-\cos\phi)$$

Rodigues Formula!

$$R(\hat{u},\phi) = e^{\tilde{u}\phi} \qquad \tilde{u} \Rightarrow \hat{u}$$



$$\vec{\omega} = \frac{d\phi}{dt}\hat{u}$$

$$\dot{p} = \vec{\omega} \times p$$

$$\dot{p} = \tilde{\omega} p$$

$$\frac{dp}{dt} = \tilde{\omega} p \qquad \text{linear time invariant ODE}$$

$$\int \frac{dp}{p} = \int \tilde{\omega}\, dt$$

$$\ln|p| = \tilde{\omega} t \Rightarrow p = e^{\tilde{\omega} t} \Rightarrow \boxed{p = e^{\hat{u}\phi}}$$

We revisit when we get velocities.

Quaternions (Section 3.4)

* William Hamilton 1843
* extension of complex numbers

Def: $q = q_0 + \vec{q} = q_0 + q_1 \hat{I} + q_2 \hat{J} + q_3 \hat{K}$

q_0 ↑ pure scalar; $\vec{q}$ ↑ vector

Properties:

① Addition: $q + p = (q_0 + \vec{q}) + (p_0 + \vec{p})$

$$= (q_0 + p_0) + (q_1 + p_1)\hat{I} + (q_2 + p_2)\hat{J} + (q_3 + p_3)\hat{K}$$

② Multiplication:

$$qp = q \circ p = (q_0 + \vec{q})(p_0 + \vec{p})$$

$$= q_0 p_1 + q_0 \vec{p} + p_0 \vec{q} + \underbrace{\vec{q}\vec{p}}$$

$\rhd\ \vec{q} \times \vec{p} - \vec{q} \cdot \vec{p}$

③ Conjugate:

$$q^* = q_0 - \vec{q}$$

$$\therefore\ qq^* = (q_0 + \vec{q})(q_0 - \vec{q})$$

$$= |q|$$

Inverse: $q^{-1} = \frac{1}{q} = \frac{q^*}{|q|^2}$

let's consider a unit quaternion: $|q| = 1$

$\Rightarrow q^{-1} = q^*$

$$|e(\hat{u}, \phi)| = 1$$

$$e(\hat{u}, \phi) = e_0 + \vec{e}$$
$$= e_0 + e_1 \hat{I} + e_2 \hat{J} + e_3 \hat{K}$$

$$\left.\begin{aligned} e_0 &= \cos\frac{\phi}{2} \\ \vec{e} &= \sin\frac{\phi}{2}\,\hat{u} \end{aligned}\right] \quad \text{Euler Parameters}$$

$\cancel{{}^G\vec{r} \quad e(\phi, \hat{u})}$

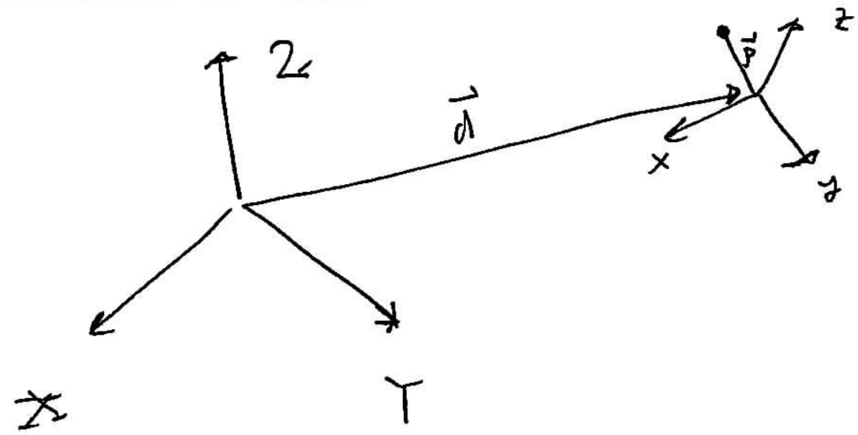
$$\boxed{{}^G\vec{r} = e(\hat{u}, \phi)\, {}^B\vec{r}\, e^*(\hat{u}, \phi)} \; *$$

$$R(\hat{u}, \phi) = (e_0^2 - \vec{e}^{\,2}) I + 2\vec{e}\vec{e}^{\,T} + 2e_0\tilde{e}$$

Rodigues Formula

* **Must convert Br_p to quaternion, with 0 scalar part, first. Then do quaternion multiplication. Then convert back to vector (drop the scalar part).**

Motion Kinematics Ch. 4



$$ {}^G\vec{r}_p = \underbrace{{}^G R_B \; {}^B\vec{r}_p + {}^G\vec{d}}_{\text{Clunky}} $$

Homogeneous Transformation Matrix

$$ {}^G\vec{r}_p = {}^G T_B \; {}^B\vec{r}_p \quad \leftarrow \text{convenient} $$

$$ {}^G T_B = \left[\begin{array}{ccc|c} & {}^G R_B & & {}^G\vec{d} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} {}^G R_B & {}^G\vec{d} \\ 0 & 1 \end{bmatrix} \quad 4\times 4 $$

$$ {}^G\vec{r}_p = \left[\begin{array}{c} X_p \\ Y_p \\ Z_p \\ \hline 1 \end{array}\right] \qquad {}^B\vec{r}_p = \left[\begin{array}{c} x_p \\ y_p \\ z_p \\ \hline 1 \end{array}\right] $$

Scale

· Homogeneous Transformation Matrix

- Rotate
- Translate

] robotics $\Rightarrow$ Special Euclidean group 'Lie group' SE(3)

- Reflect
- Shear
- Scale

] graphics

Inverse :

$$ {}^{G}T_{B} = \underbrace{\begin{bmatrix} I & {}^{G}\vec{d} \\ 0 & 1 \end{bmatrix}}_{\text{translation}} \underbrace{\begin{bmatrix} {}^{G}R_{B} & 0 \\ 0 & 1 \end{bmatrix}}_{\text{rotation}} = \begin{bmatrix} {}^{G}R_{B} & {}^{G}\vec{d} \\ 0 & 1 \end{bmatrix} $$

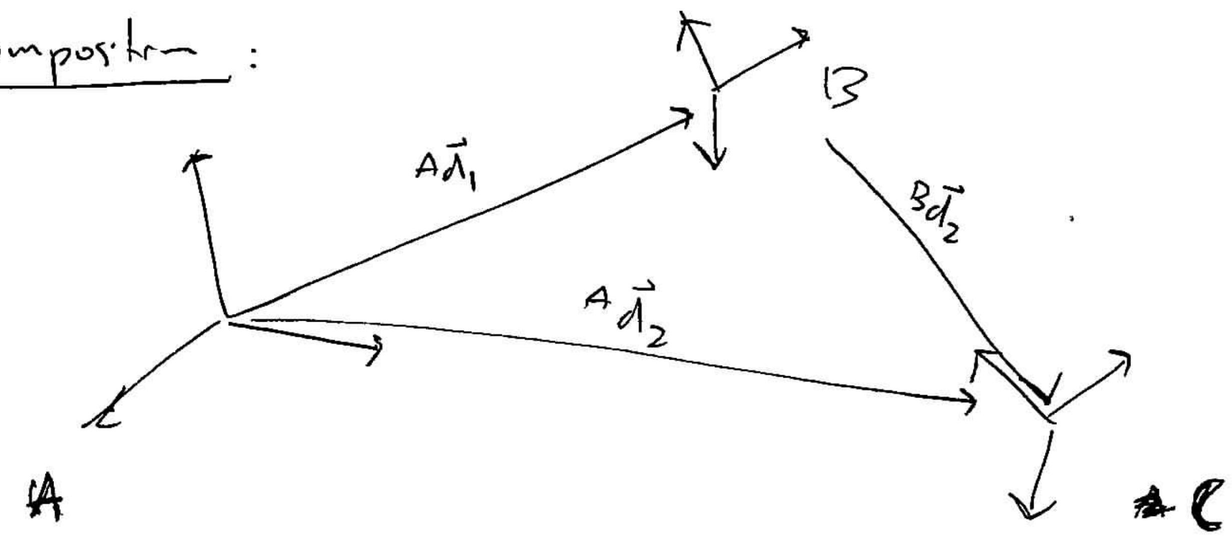
$$ {}^{B}T_{G} = {}^{G}T_{B}^{-1} = \begin{bmatrix} {}^{G}R_{B} & {}^{G}\vec{d} \\ 0 & 1 \end{bmatrix}^{-1} $$

$$ = \begin{bmatrix} {}^{G}R_{B}^{T} & -{}^{G}R_{B}^{T}\,{}^{G}\vec{d} \\ 0 & 1 \end{bmatrix} $$

$$ \therefore \; {}^{G}T_{B}\,{}^{B}T_{G} = I $$

(Not orthogonal $\;{}^{G}T_{B}^{-1} \neq {}^{G}T_{B}^{T}$)

Composition :



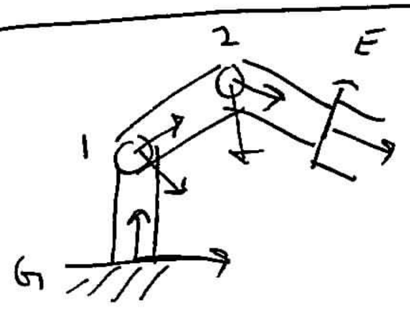
$$^A T_B = \begin{bmatrix} ^A R_B & ^A\vec{d}_1 \\ 0 & 1 \end{bmatrix}$$

$$^B T_C = \begin{bmatrix} ^B R_C & ^B\vec{d}_2 \\ 0 & 1 \end{bmatrix}$$

$$^A T_C = {}^A T_{\cancel{B}}\, {}^{\cancel{B}} T_C = \begin{bmatrix} ^A R_B & ^A\vec{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ^B R_C & ^B\vec{d}_2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ^A R_B\, ^B R_C & ^A R_B\, ^B\vec{d}_2 + {}^A\vec{d}_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ^A R_C & ^A\vec{d}_2 \\ 0 & 1 \end{bmatrix}$$



$$\boxed{^G T_e = {}^G T_1\, {}^1 T_2\, {}^2 T_e}$$

Forward Kinematics.