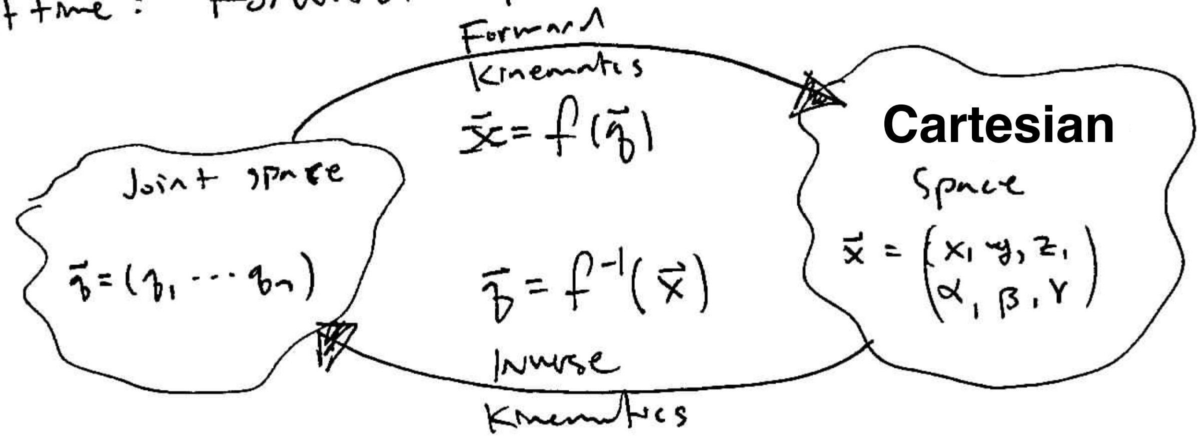
Lecture 5 10/11/2021 ME221

Last time: Forward Kinematics

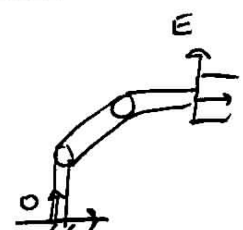


- use DH-parameters as a systematic approach to assign coordinates frame to joints

$$\left.\begin{array}{l} a_i : \text{ link length} \\ \alpha_i : \text{ link twist} \\ d_i : \text{ joint distance} \\ \theta_i : \text{ joint angle} \end{array}\right] \text{ completely define coordinate transform } i-1 \rightarrow i$$

$$q_i : \left[\begin{array}{l} d_i : \text{ joint distance} \\ \theta_i : \text{ joint angle} \end{array}\right.$$

$$^{i-1}T_i = f(a_i, \alpha_i, d_i, \theta_i)$$



$$\boxed{^0T_E = {^0T_1}\, {^1T_2} \cdots {^{n-1}T_e}} \longleftarrow \text{Forward Kinematics } \cancel{\text{Kinemat}}$$

$$^0T_E = \begin{bmatrix} {^0R_E} & \begin{matrix} {^0x_e} \\ {^0y_e} \\ {^0z_e} \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$\longrightarrow x, y, z$ of end-effector

$\longrightarrow \alpha, \beta, \gamma$

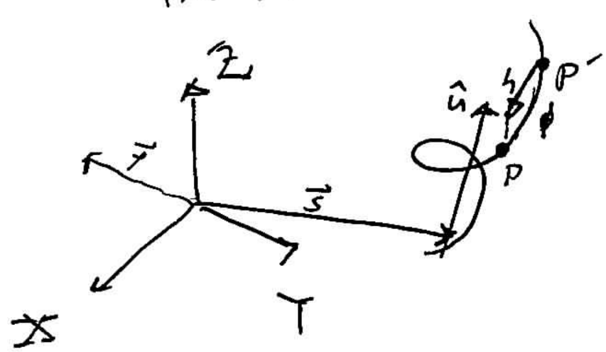
Today:

- screw coordinates
- Product of Exponentials (PoE)
- Intro. Inverse Kinematics

Screw Coordinates: ~~generalization~~ generalized approach to multi-body mechanics

Basic Idea: (Chasles Theorem)

All displacements can be produced by a single rotation combined with a translation along some axis:



$$p = \frac{h}{\phi}$$

$$ {}^{G}\breve{S}_{B}(h, \phi, \hat{u}, \vec{s}) = \begin{bmatrix} {}^{G}R_{B} & {}^{G}\vec{s} - {}^{G}R_{B}\,{}^{G}\vec{s} + h\hat{u} \\ 0 & 1 \end{bmatrix} $$

h: translation
ϕ: rotation
$\hat{u}$: axis
$\vec{s}$: location vector

Inverse Screw :

$${}^{G}\breve{s}^{-1}_{B}(h, \phi, \hat{u}, \vec{s}) = {}^{B}\breve{s}_{G}(h, \phi, \hat{u}, \vec{s})$$

$$= \begin{bmatrix} {}^{G}R^{T}_{B} & {}^{G}\vec{s} - {}^{G}R^{T}_{B}\,{}^{G}\vec{s} - h\hat{u} \\ 0 & 1 \end{bmatrix}$$

$$T = \breve{s}(h, \phi, \hat{u}, \vec{s}) = \begin{bmatrix} e^{\phi\tilde{u}} & (I - e^{\phi\tilde{u}})\vec{s} + h\hat{u} \\ 0 & 1 \end{bmatrix}$$

$\tilde{u}$: skew symmetric version of $\hat{u}$

Q: Can we write entire transformation as a matrix exponential ? Yes!

Screw Axis

Plücker Coordinate

$$\mathcal{S} = \underbrace{\begin{bmatrix} \hat{u} \\ \vec{\rho} + p\hat{u} \end{bmatrix}}_{\text{screw axis}} \underbrace{\phi}_{\text{displacement}}$$

$$\begin{bmatrix} \vec{\rho} = -\hat{u} \times \vec{s} \\ p = \dfrac{h}{\phi} \end{bmatrix}$$

$\underline{\underline{6 \times 1}}$

$$\tilde{\mathcal{S}} = \begin{bmatrix} \tilde{u} & \vec{\rho} + p\hat{u} \\ 0 & 0 \end{bmatrix} \phi$$

$$\boxed{T = e^{\tilde{\mathcal{S}}\phi}}$$

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \vec{r} + p\hat{u} \end{bmatrix}$$ Plücker Coordinate (Screw Axis)

$$r_p = e^{\tilde{\mathcal{S}}\phi} \, r_p(0)$$

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix}$$

$$\tilde{\mathcal{S}} = \begin{bmatrix} \tilde{u} & \vec{r} + p\hat{u} \\ 0 & 0 \end{bmatrix}$$

4x4

$P = \infty$ pure translation

$$\mathcal{S} = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix}$$ Prismatic joint

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$P = 0$ pure rotation

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \vec{r} \end{bmatrix} \quad \vec{r} = -\hat{u} \times \vec{s}$$ revolute joint

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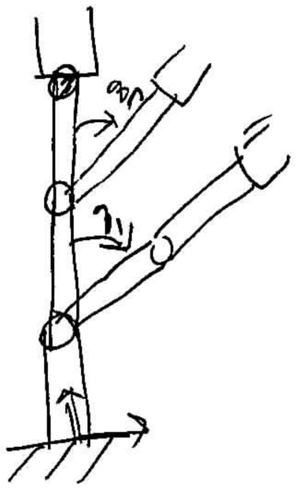
Product of Exponentials For Forward Kinematics

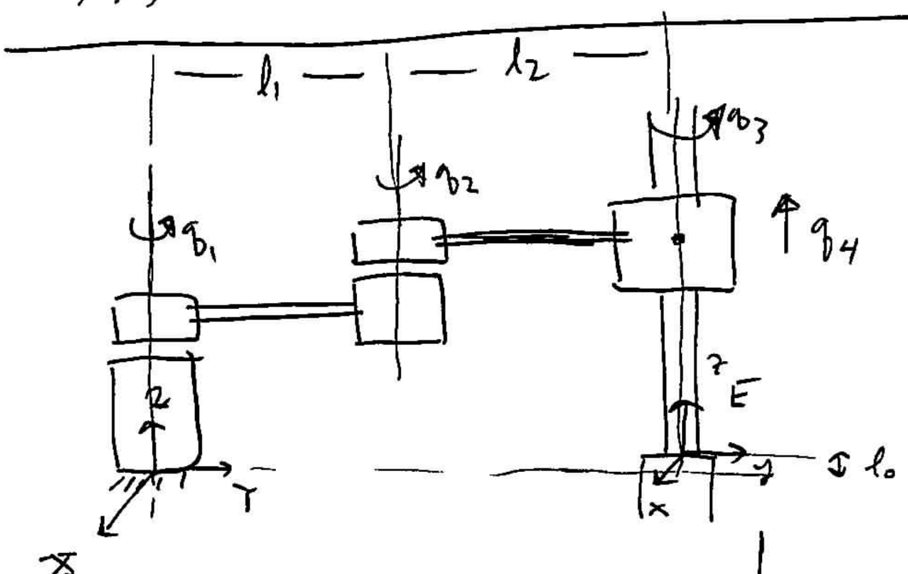
Murray Ch. 3.2
Lynch Ch. 4

- base frame & End effector
  - Homogenous Transformation M $(q=0)$
- define axis of motion
- Find screw axis $\mathcal{S}$ for each joint
- ~~PoE ^0T_E~~ - PoE $^0T_E = e^{\tilde{\mathcal{S}}_{n-1}\phi_{n-1}} e^{\tilde{\mathcal{S}}_n \phi_n} M$

• POE:

$$ {}^{0}T_E = e^{\tilde{S}_1 q_1} \cdots e^{\tilde{S}_{n-1} q_{n-1}} \underbrace{e^{\tilde{S}_n q_n} M} $$





$$ \mathcal{S} = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix} \quad \text{prismatic joint} $$

$$ \mathcal{S} = \begin{bmatrix} \hat{u} \\ \vec{\rho} \end{bmatrix} \quad \text{revolute} \quad \vec{\rho} = -\hat{u} \times \vec{s} $$

$$ \hat{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathcal{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ \vec{s}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ M = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} $$

$$ \hat{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} $$

$$ \vec{s}_1 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} \qquad \mathcal{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ l_1 \\ 0 \\ 0 \end{bmatrix} $$

$\mathcal{S}_3 = ?$

$\hat{u}_3 = ?$

$\vec{s} = ?$

$$\Rightarrow \begin{bmatrix} \hat{u}_3 \\ -\hat{u}_3 \times \vec{s}_3 \end{bmatrix}$$

$$\hat{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{s} = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix} \qquad \mathcal{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

$$-\hat{u}_3 \times \vec{s} = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathcal{S}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{s} = 0$

$$\boxed{{}^{0}T_4 = e^{\tilde{\mathcal{S}}_1 q_1} e^{\tilde{\mathcal{S}}_2 q_2} e^{\tilde{\mathcal{S}}_3 q_3} e^{\tilde{\mathcal{S}}_4 q_4} M}$$

<u>Some advantages</u>

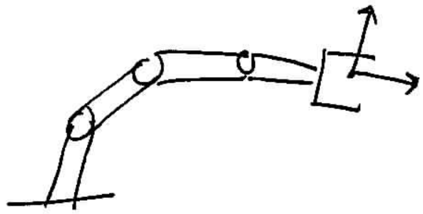
- ignore the in-between transformation
- thrs velocity -

<u>Some disadvantages</u>

- ${}^{i-1}T_i$ not given explicitly

<u>POE</u>

Inverse Kinematics



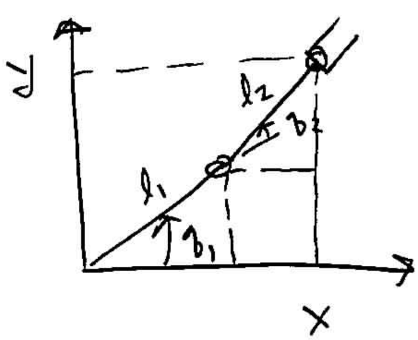
$\begin{pmatrix} x, y, z \\ \alpha, \beta, \gamma \end{pmatrix}$ desired

what q's get us there?

$${}^0T_{e,\,desired} = \begin{bmatrix} {}^0R_e & {}^0x_e \\ & {}^0y_e \\ 0 & {}^0z_e \\ & 1 \end{bmatrix}$$

$${}^0T_e(\vec{q}) = {}^0T_{e,\,desired}$$

Planar RR robot IK



$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \end{bmatrix}$$

$x^2 + y^2$

$$x^2 = (l_1 c_1 + l_2 c_{12})^2$$
$$= l_1^2 c_1^2 + 2 l_1 l_2 c_1 c_{12} + l_2^2 c_{12}^2$$
$$y^2 = l_1^2 s_1^2 + 2 l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2$$

$$\cancel{x^2 + y^2 = l_1^2 + l_2^2}$$

$$x^2 + y^2 = l_1^2 \underbrace{(c_1^2 + s_1^2)}_{1} + l_2^2 \underbrace{(c_{12}^2 + s_{12}^2)}_{1}$$
$$+ 2 l_1 l_2 [c_1 c_{12} + s_1 s_{12}]$$
$$= l_1^2 + l_2^2 + 2 l_1 l_2 [c_1 c_{12} + s_1 s_{12}]$$

$$c_{12} = \cos(q_1 + q_2) = \cos q_1 \cos q_2 - \sin q_1 \sin q_2$$
$$s_{12} = \sin(q_1 + q_2) = \sin q_1 \cos q_2 + \cos q_1 \sin q_2$$

$$2 l_1 l_2 [c_1 c_{12} + s_1 s_{12}] = 2 l_1 l_2 [c_1 (c_1 c_2 - s_1 s_2) + s_1 (s_1 c_2 + c_1 s_2)]$$

$$2 l_1 l_2 [c_1^2 c_2 - \cancel{s_1 s_2 c_1} + s_1^2 c_2 + \cancel{s_1 s_2 c_2}]$$

$$2 l_1 l_2 [c_2] = 2 l_1 l_2 \cos q_2$$

$$\cancel{x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos q_2}$$

$$\boxed{\cos q_2 = \frac{(x^2 + y^2) - (l_1^2 + l_2^2)}{2 l_1 l_2}}$$