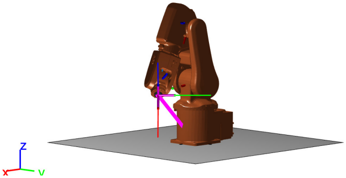
Last time: Inverse Kinematics

Analytic:

```
% generate a function called 'robotIK'
generateIKFunction(aik,'robotIK');
% find the pose for the target end-effector position
q = robotIK(Td);
```

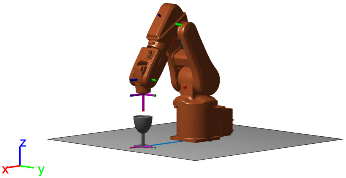


Numeric:

```
% BFGSGradientProjection IK object
ik = inverseKinematics('RigidBodyTree',r);

% find pose with numerical IK
[q,solnInfo] = ik('link_6',Td,ones(6,1),zeros(6,1));
```

Generalized Numeric:

```
% Generalized IK object
gik = generalizedInverseKinematics('RigidBodyTree',r);
gik.ConstraintInputs = {'cartesian','position','aiming'};

...

% Get IK
[q,solnInfo] = gik(q0,heightAboveFloor, ...
                      distanceFromCup, ...
                      alignWithCup);
```



Today's Agenda

- Time Derivatives
- Angular Velocity Vector / Matrix
- Rigid Body Velocity
- Velocity Transformation Matrix
- Homogeneous Transformation Derivatives

Time Derivatives and Coordinate Frames

General case: The time derivative of a vector depends on *which* coordinate frame we want to take the derivative in:

G-derivative:
(Derivative in the Global Frame)

$$\frac{{}^{G}d}{dt}\,\boldsymbol{r}_p$$

B-derivative:
Derivative in the Body Frame

$$\frac{{}^{B}d}{dt}\,\boldsymbol{r}_p$$

Time Derivatives and Coordinate Frames

If vector $\boldsymbol{r}_p$ is represented in the same coordinate frame as the derivative, the frame's *unit vectors* are constant:

G-derivative:
(Derivative in the Global Frame)

$$\frac{{}^G d}{dt}\,{}^G\boldsymbol{r}_p = {}^G\dot{\boldsymbol{r}}_p = {}^G\boldsymbol{v}_p$$
$$= \dot{X}\hat{I} + \dot{Y}\hat{J} + \dot{Z}\hat{K}$$

B-derivative:
Derivative in the Body Frame

$$\frac{{}^B d}{dt}\,{}^B\boldsymbol{r}_p = {}^B\dot{\boldsymbol{r}}_p = {}^B\boldsymbol{v}_p$$
$$= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$$

Time Derivatives and Coordinate Frames

We can also find the **G-derivative** of ${}^{B}\boldsymbol{r}_p$:

G-derivative:

(Derivative in the Global Frame)

$$ {}^{B}_{G}\boldsymbol{v}_p = \frac{{}^{G}d}{dt}\,{}^{B}\boldsymbol{r}_p = {}^{B}\dot{\boldsymbol{r}}_p + {}^{B}_{G}\boldsymbol{\omega}_B \times {}^{B}\boldsymbol{r}_p $$

where ${}^{B}_{G}\boldsymbol{\omega}_B$ is the angular velocity of B relative to G.

How to derive?

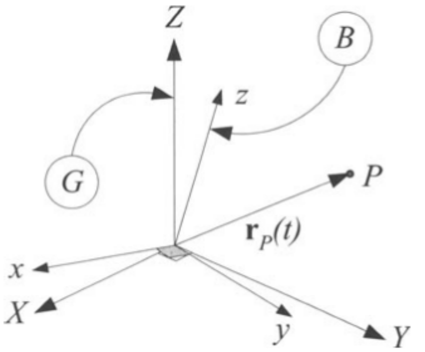


Figure: 7.3 - Moving point P in rotating frame B

Position in the body frame:

$$^{B}\boldsymbol{r}_p = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

Take time derivative:

$$\frac{^{G}d}{dt}\,^{B}\boldsymbol{r}_p = \frac{^{G}d}{dt}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$
$$= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} + x\frac{^{G}d\hat{\imath}}{dt} + y\frac{^{G}d\hat{\jmath}}{dt} + z\frac{^{G}d\hat{k}}{dt}$$

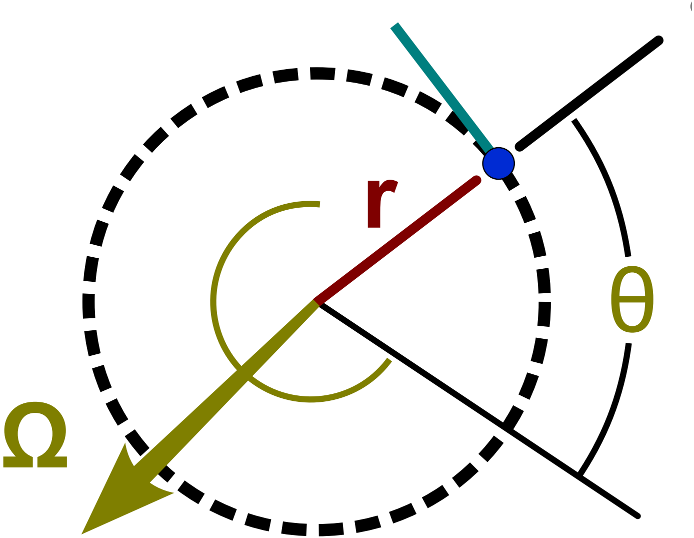
Time derivative of <u>unit vectors</u>:

$$\frac{^{G}d\hat{\imath}}{dt} = {}^{B}_{G}\boldsymbol{\omega}_B \times \hat{\imath}, \quad \frac{^{G}d\hat{\jmath}}{dt} = {}^{B}_{G}\boldsymbol{\omega}_B \times \hat{\jmath}, \quad \frac{^{G}d\hat{k}}{dt} = {}^{B}_{G}\boldsymbol{\omega}_B \times \hat{k}$$

Therefore:

$${}^{B}_{G}\boldsymbol{v}_p = {}^{B}\dot{\boldsymbol{r}}_p + {}^{B}_{G}\boldsymbol{\omega}_B \times {}^{B}\boldsymbol{r}_p$$

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$$



Time Derivatives and Coordinate Frames

G-derivative of ${}^{B}\boldsymbol{r}_p$ and the **B-derivative** of ${}^{G}\boldsymbol{r}_p$:

G-derivative:
(Derivative in the Global Frame)

$$
\begin{aligned}
{}^{B}_{G}\boldsymbol{v}_p &= \frac{{}^{G}d}{dt}\ {}^{B}\boldsymbol{r}_p \\
&= {}^{B}\dot{\boldsymbol{r}}_p + {}^{B}_{G}\boldsymbol{\omega}_B \times {}^{B}\boldsymbol{r}_p
\end{aligned}
$$

where ${}^{B}_{G}\boldsymbol{\omega}_B$ is the angular velocity of B relative to G.

B-derivative:
(Derivative in the Body Frame)

$$
\begin{aligned}
{}^{G}_{B}\boldsymbol{v}_p &= \frac{{}^{B}d}{dt}\ {}^{G}\boldsymbol{r}_p \\
&= {}^{G}\dot{\boldsymbol{r}}_p + {}^{G}_{B}\boldsymbol{\omega}_G \times {}^{G}\boldsymbol{r}_p
\end{aligned}
$$

where ${}^{G}_{B}\boldsymbol{\omega}_G$ is the angular velocity of G relative to B.

Angular Velocity Vector / Matrix

Specific case: Consider a rigid body rotating about a fixed point. In this case ${}^B\boldsymbol{r}_p$ represents a point on the rigid body (e.g., center of mass), thus is constant.

$$ {}^G\boldsymbol{r}_p(t) = {}^G R_B(t)\ {}^B\boldsymbol{r}_p $$

What is the velocity of point p?

$$ \begin{aligned} {}^G\boldsymbol{v}_p(t) = {}^G\dot{\boldsymbol{r}}_p(t) &= {}^G\dot{R}_B(t)\ {}^B\boldsymbol{r}_p \\ &= {}_G\boldsymbol{\omega}_B \times {}^G\boldsymbol{r}_p(t) \\ &= {}_G\tilde{\omega}_B\,{}^G\boldsymbol{r}_p(t) \end{aligned} $$

where the skew symmetric matrix:

$$ {}_G\tilde{\omega}_B = {}^G\dot{R}_B(t)\,{}^G R_B^T(t) = \dot{\phi}\tilde{u} $$

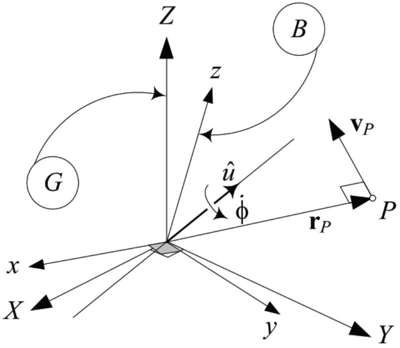


Figure: 7.1 - Rotating body frame.

$$ {}^{G}\boldsymbol{v}_p(t) = {}^{G}\dot{\boldsymbol{r}}_p(t) = {}^{G}\dot{R}_B(t)\ {}^{B}\boldsymbol{r}_p $$

But we can write:

$$ {}^{B}\boldsymbol{r}_p(t) = {}^{G}R_B^T(t)\,{}^{G}\boldsymbol{r}_p $$

so

$$ {}^{G}\boldsymbol{v}_p(t) = {}^{G}\dot{\boldsymbol{r}}_p(t) = \underbrace{{}^{G}\dot{R}_B(t)\,{}^{G}R_B^T(t)}_{{}_{G}\tilde{\omega}_B}\,{}^{G}\boldsymbol{r}_p(t) $$

Why should ${}_{G}\tilde{\omega}_B$ be skew symmetric?

Start from orthogonality conditions:

$$
\begin{aligned}
{}^G R_B \, {}^G R_B^T &= I & {}^G R_B^T \, {}^G R_B &= I \\
\frac{d}{dt}\left({}^G R_B \, {}^G R_B^T = I\right) && \frac{d}{dt}\left({}^G R_B^T \, {}^G R_B = I\right) \\
{}^G \dot{R}_B \, {}^G R_B^T + {}^G R_B \, {}^G \dot{R}_B^T &= 0 & {}^G \dot{R}_B^T \, {}^G R_B + {}^G R_B^T \, {}^G \dot{R}_B &= 0 \\
{}^G \dot{R}_B \, {}^G R_B^T &= -\, {}^G R_B \, {}^G \dot{R}_B^T & {}^G \dot{R}_B^T \, {}^G R_B &= -\, {}^G R_B^T \, {}^G \dot{R}_B \\
\left({}^G R_B \, {}^G \dot{R}_B^T\right)^T &= -\, {}^G R_B \, {}^G \dot{R}_B^T & \left({}^G R_B^T \, {}^G \dot{R}_B\right)^T &= -\, {}^G R_B^T \, {}^G \dot{R}_B
\end{aligned}
$$

Note that:

$B^T A = (A^T B)^T$ and $A^T = -A$ for skew symmetric matrix

The group of <u>all</u> skew-symmetric 3×3 matrices forms the Lie algebra $\mathfrak{so}(3)$ of the Lie group $SO(3)$ (special rotation group). They represent *infinitesimal generators* of rotations, e.g., **angular velocity!**

$$ {}^{G}\dot{\boldsymbol{r}}_p(t) = {}^{G}\dot{R}_B(t)\ {}^{B}\boldsymbol{r}_p $$

Left multiply rotation matrix:

$$ \underbrace{{}^{G}R_B^T(t)\,{}^{G}\dot{\boldsymbol{r}}_p(t)}_{{}^{B}\dot{\boldsymbol{r}}_p} = \underbrace{{}^{G}R_B^T(t)\,{}^{G}\dot{R}_B(t)}_{{}^{B}_{G}\tilde{\omega}_B}\ {}^{B}\boldsymbol{r}_p $$

Substitute ${}^{G}\boldsymbol{r}_p$ on RHS:

$$ {}^{G}\dot{\boldsymbol{r}}_p(t) = \underbrace{{}^{G}\dot{R}_B(t)\,{}^{G}R_B^T(t)}_{{}^{G}_{G}\tilde{\omega}_B}\ {}^{G}\boldsymbol{r}_p $$

${}^{B}_{G}\tilde{\omega}_B$: instantaneous angular velocity of B relative to G as seen from B.

${}^{G}_{G}\tilde{\omega}_B$: instantaneous angular velocity of B relative to G as seen from G.

Rigid Body Velocity

Now consider a rigid body rotating and translating (freely moving) relative to the Global frame.

$$^{G}\boldsymbol{r}_p(t) = {}^{G}R_B(t)\ {}^{B}\boldsymbol{r}_p + {}^{G}\boldsymbol{d}_B$$

What is the velocity of point p in the Global frame?

$$^{G}\boldsymbol{v}_p(t) = {}^{G}\dot{\boldsymbol{r}}_p(t) = {}^{G}\dot{R}_B(t)\ {}^{B}\boldsymbol{r}_p + {}^{G}\dot{\boldsymbol{d}}_B$$
$$= {}_{G}\tilde{\omega}_B\left({}^{G}\boldsymbol{r}_p - {}^{G}\boldsymbol{d}_B\right) + {}^{G}\dot{\boldsymbol{d}}_B$$

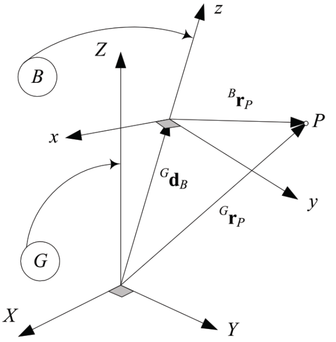


Figure: 7.5 - Rotating and translating body frame.

Geometric Interpretation of Rigid Body Velocity

$$^{G}\boldsymbol{v}_p = {}_G\tilde{\omega}_B \left({}^{G}\boldsymbol{r}_p - {}^{G}\boldsymbol{d}_B \right) + {}^{G}\dot{\boldsymbol{d}}_B$$

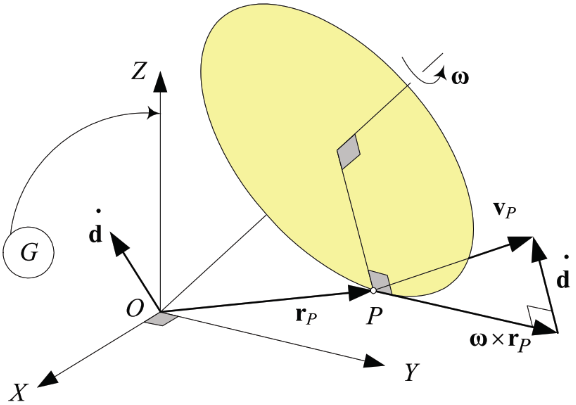


Figure: 7.6 - Rigid body velocity.

Velocity Transformation Matrix

The Velocity Transformation Matrix is a more convenient representation of rigid body velocity by utilizing a matrix operator

$$ {}^{G}\boldsymbol{v}_p(t) = {}^{G}V_B \; {}^{G}\boldsymbol{r}_p $$

Note that we need to represent the point in the Global frame

$$ \begin{aligned} {}^{G}V_B &= {}^{G}\dot{T}_B \, {}^{G}T_B^{-1} \\ &= \begin{bmatrix} {}^{G}\dot{R}_B \, {}^{G}R_B^T & {}^{G}\dot{\boldsymbol{d}}_B - {}^{G}\dot{R}_B \, {}^{G}R_B^T \, {}^{G}\boldsymbol{d}_B \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}^{G}\tilde{\omega}_B & {}^{G}\dot{\boldsymbol{d}}_B - {}^{G}\tilde{\omega}_B \, {}^{G}\boldsymbol{d}_B \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}^{G}\tilde{\omega}_B & {}^{G}\boldsymbol{v}_B \\ 0 & 0 \end{bmatrix} \end{aligned} $$

(same as $\tilde{\mathcal{S}}$!)

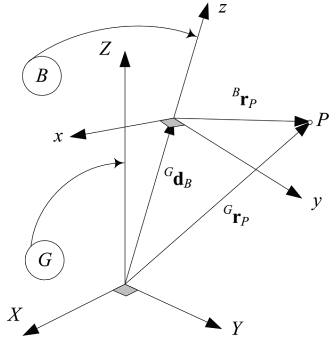


Figure: 7.5 - Rotating and translating body frame.

Velocity Transformation Matrix

The Velocity Transformation Matrix is a matrix operator that provides the global velocity of any point attached to the B frame.

$$^{G}\boldsymbol{v}_p(t) = {}^{G}V_B \; {}^{G}\boldsymbol{r}_p$$

We can also represent as a velocity transformation vector, also called a *twist*:

$$^{G}V_B = \begin{bmatrix} {}^{G}\tilde{\omega}_B & {}^{G}\boldsymbol{v}_B \\ 0 & 0 \end{bmatrix}$$

$$^{G}\boldsymbol{t}_B = \begin{bmatrix} {}^{G}\boldsymbol{v}_B \\ {}^{G}\boldsymbol{\omega}_B \end{bmatrix}$$

The group of all twist matrices form the Lie algebra $\mathfrak{se}(3)$ of the Lie group $SE(3)$ (special Euclidean group). They represent *infinitesimal generators* of screw motions, e.g., **angular and translational velocity!**

Derivative of a Homogeneous Transformation Matrix

The velocity transformation matrix is related to the derivative of a homogeneous transformation matrix (e.g., can be used for successive link transformation for velocity forward kinematics)

$$ {}^{G}\dot{T}_{B} = {}^{G}V_{B} \; {}^{G}T_{B} $$

$$ {}^{i-1}\dot{T}_{i} = {}^{i-1}V_{i} \; {}^{i-1}T_{i} $$

$$ {}^{i-1}V_{i} = \underbrace{{}^{i-1}\dot{T}_{i} \; {}^{i-1}T_{i}^{-1}}_{\text{known for each joint!}} $$

Next time we will apply this to robot manipulators for velocity (forward/inverse) kinematics.

Informal Instructor Feedback Exercise

- Do you feel that you are learning?
- What do you like about the course? What is helping you learn?
- What do you not like about the course? What is inhibiting your learning?
- What changes would you implement if you were the instructor?
  - The homeworks are too hard (or too easy).
  - Lectures are boring. I would make them more exciting.
  - I would implement slides only instead of handwritten notes.
  - etc.