

Last time: Angular Velocity

- angular velocity is always skew symmetric because of the orthogonality condition:

$${}^G \dot{\mathbf{r}}_p(t) = {}^G \dot{R}_B(t) {}^B \mathbf{r}_p$$

$$\underbrace{{}^G R_B^T(t) {}^G \dot{\mathbf{r}}_p(t)}_{{}^B \dot{\mathbf{r}}_p} = \underbrace{{}^G R_B^T(t) {}^G \dot{R}_B(t)}_{{}^G \tilde{\boldsymbol{\omega}}_B} {}^B \mathbf{r}_p \quad \Bigg| \quad {}^G \dot{\mathbf{r}}_p(t) = \underbrace{{}^G \dot{R}_B(t) {}^G R_B^T(t)}_{{}^G \tilde{\boldsymbol{\omega}}_B} {}^G \mathbf{r}_p$$

- General rigid body motion with angular and linear velocity can be described:

$${}^G \mathbf{v}_p(t) = {}^G \dot{\mathbf{r}}_p(t) = {}^G \tilde{\boldsymbol{\omega}}_B ({}^G \mathbf{r}_p - {}^G \mathbf{d}_B) + {}^G \dot{\mathbf{d}}_B$$

- More conveniently we can use velocity transformation matrix:

$${}^G \mathbf{v}_p(t) = {}^G V_B {}^G \mathbf{r}_p, \quad {}^G V_B = \begin{bmatrix} {}^G \tilde{\boldsymbol{\omega}}_B & {}^G \mathbf{v}_B \\ 0 & 0 \end{bmatrix} = {}^G \tilde{\mathbf{t}}_B, \quad {}^G \mathbf{t}_B = \begin{bmatrix} {}^G \mathbf{v}_B \\ {}^G \boldsymbol{\omega}_B \end{bmatrix} \quad (\text{twist}).$$

- The velocity transformation matrix is related to the derivative of the homogeneous transformation matrix

$${}^G \dot{T}_B = {}^G V_B {}^G T_B$$

Jacobian Agenda

- Jacobian
 - Direct Differentiation
 - Forward Velocity Kinematics
-

- Jacobian Generating vectors
- Inverse Velocity Kinematics
- Other types of Jacobians
- Maneuverability Analysis

Do you remember **chain rule** for functions? Find:

$$\frac{d}{dt} f(x(t)) = \frac{d}{dx} f(x(t)) \frac{d}{dt} x(t) = \frac{d}{dx} f(x(t)) \dot{x}(t)$$

What about for the multivariate case?

$$f(\mathbf{x}(t)) = f(x_1(t), x_2(t), \dots, x_n(t))$$

$$\begin{aligned} \frac{d}{dt} f(\mathbf{x}) &= \frac{\partial}{\partial x_1} f(\mathbf{x}) \dot{x}_1 + \frac{\partial}{\partial x_2} f(\mathbf{x}) \dot{x}_2 + \dots + \frac{\partial}{\partial x_n} f(\mathbf{x}) \dot{x}_n \\ &= \sum_i^n \frac{\partial}{\partial x_i} f(\mathbf{x}) \dot{x}_i \end{aligned}$$

Deriving the Jacobian from Velocity Transformation Matrix

The velocity of **any body point** can be found in the Global frame if we know the homogeneous transformation matrix:

$${}^G V_B = {}^G \dot{T}_B {}^G T_B^{-1}$$

Recall from **forward kinematics** we can find the transformation from the Global base frame to the end effector from the DH-parameters (or PoE):

$${}^0 T_E(\mathbf{q}) = {}^0 T_1(q_1) {}^1 T_2(q_2) \dots {}^{n-1} T_E(q_n)$$

Then the velocity transformation matrix for the end-effector:

$${}^0 V_E = {}^0 \dot{T}_E(\mathbf{q}) {}^0 T_E^{-1}(\mathbf{q})$$

Deriving the Jacobian from Velocity Transformation Matrix

Let's explore the velocity transformation matrix:

$$\begin{aligned}({}^0\tilde{\mathbf{t}}_E) &= {}^0V_E = {}^0\dot{T}_E(\mathbf{q}) {}^0T_E^{-1}(\mathbf{q}) \\ &= \left(\sum_i^n \frac{\partial {}^0T_E(\mathbf{q})}{\partial q_i} \dot{q}_i \right) {}^0T_E^{-1}(\mathbf{q}) \\ &= \sum_i^n \left(\frac{\partial {}^0T_E(\mathbf{q})}{\partial q_i} {}^0T_E^{-1}(\mathbf{q}) \dot{q}_i \right)\end{aligned}$$

Deriving the Jacobian from Velocity Transformation Matrix

We can re-write in **vector form**:

$${}^0\mathbf{t}_E = \begin{bmatrix} {}^0\mathbf{v}_E \\ {}^0\boldsymbol{\omega}_E \end{bmatrix} = \boxed{\dot{\mathbf{X}} = J(\mathbf{q})\dot{\mathbf{q}}}$$

with

$$J(\mathbf{q}) = \left[\left(\frac{\partial {}^0T_E(\mathbf{q})}{{\partial q_1}} {}^0T_E^{-1}(\mathbf{q}) \right)^\vee \quad \dots \quad \left(\frac{\partial {}^0T_E(\mathbf{q})}{{\partial q_n}} {}^0T_E^{-1}(\mathbf{q}) \right)^\vee \right]$$

with the $^\vee$ symbol denoting the “vector form” (as opposed to the skew symmetric form)

Very Important: The Jacobian provides a linear relationship between the joint velocities (\mathbf{q}) and the end effector's linear (\mathbf{v}) and angular ($\boldsymbol{\omega}$) velocities. There for we can use linear methods e.g.,

$$Ax = b \rightarrow x = A^{-1}b \quad (\text{can be solved efficiently})$$

How to calculate the Jacobian

$$\dot{\mathbf{X}} = \underbrace{J(\mathbf{q})}_{6 \times n} \dot{\mathbf{q}}$$

We can split up the Jacobian into two parts:

$$\dot{\mathbf{X}} = \begin{bmatrix} {}^0\mathbf{v}_E \\ {}^0\boldsymbol{\omega}_E \end{bmatrix} = \begin{bmatrix} J_D \\ J_R \end{bmatrix} \dot{\mathbf{q}}$$

- rarely ever calculated as we derived (direct differentiation)
- more systematic approach: Jacobian generating vectors (next time)
- difference between **analytic** vs **geometric** Jacobian (next time)

Direct differentiation: Ex.242, pg.446

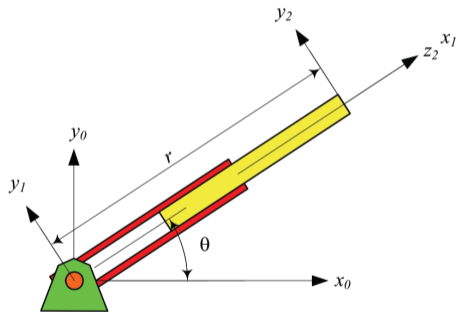
If the robot is simple, it is easy to do direct differentiation:

Find forward kinematics:

$${}^0T_2 = {}^0T_1 {}^1T_2$$

$${}^0T_1 = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Direct differentiation

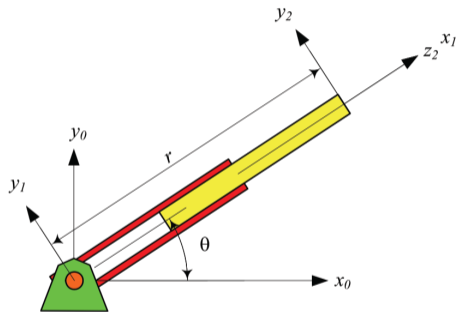
Find forward kinematics:

$${}^0T_2 = {}^0T_1 {}^1T_2$$

$${}^0T_2 = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & r \cos \theta \\ \sin \theta & \cos \theta & 0 & r \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix}$$



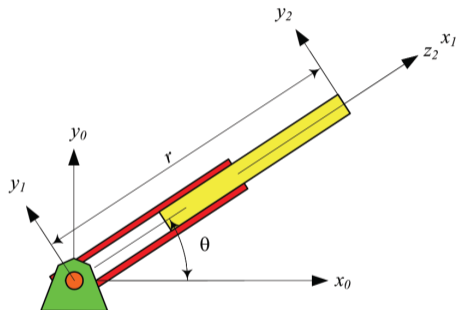
Direct differentiation

The forward kinematics maps joint variables to Cartesian space:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix}$$
$$\mathbf{x} = f(\mathbf{q})$$

with

$$\mathbf{x} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} r \\ \theta \end{bmatrix},$$



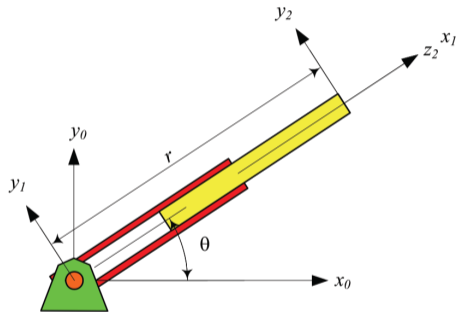
Direct differentiation

The displacement Jacobian is calculated by taking the time derivative of the displacement forward kinematics:

$$\dot{\mathbf{x}} = J_D(\mathbf{q})\dot{\mathbf{q}}$$

where

$$\begin{aligned} J_D(\mathbf{q}) &= \begin{bmatrix} \frac{\partial f(\mathbf{q})}{\partial q_1} & \frac{\partial f(\mathbf{q})}{\partial q_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(\mathbf{q})}{\partial r} & \frac{\partial f(\mathbf{q})}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \end{aligned}$$

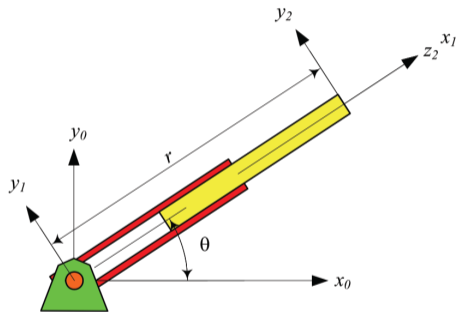


Direct differentiation

The rotational Jacobian is calculated
find the angular velocity vector from the
forward kinematics:

$${}^0\tilde{\omega}_2 = {}^0\dot{R}_2 {}^0R_2^T$$

$${}^0T_2 = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & r \cos \theta \\ \sin \theta & \cos \theta & 0 & r \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} {}^0R_2 & {}^0d_2 \\ \mathbf{0} & 1 \end{bmatrix}$$



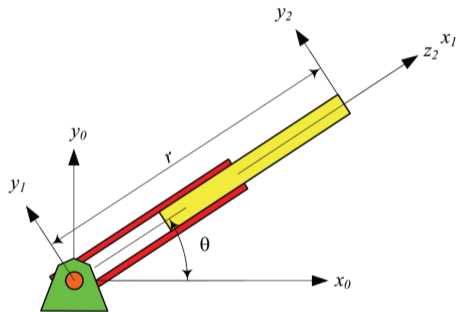
Direct differentiation

The rotational Jacobian is calculated
find the angular velocity vector from the
forward kinematics:

$${}^0\tilde{\omega}_2 = {}^0\dot{R}_2 {}^0R_2^T$$

$${}^0\dot{R}_2 = \begin{bmatrix} -\sin \theta & -\cos \theta & 0 \\ \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$${}^0R_2^T = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



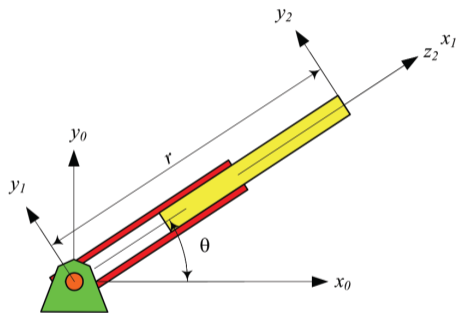
Direct differentiation

The rotational Jacobian is calculated
find the angular velocity vector from the
forward kinematics:

$$\begin{aligned} {}_0\tilde{\omega}_2 &= {}^0\dot{R}_2 {}^0R_2^T \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} \end{aligned}$$

Recall that for skew symmetric matrix:

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$



Direct differentiation

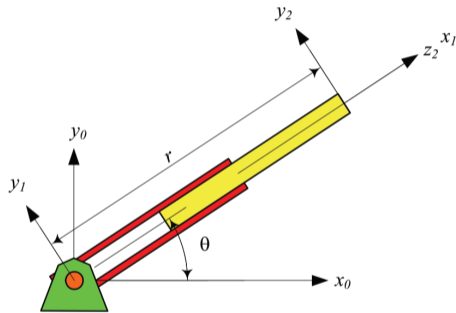
The rotational Jacobian is calculated
find the angular velocity vector from the
forward kinematics:

$${}^0\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

Or we can write as:

$$\omega_3 = J_R \dot{\theta}$$

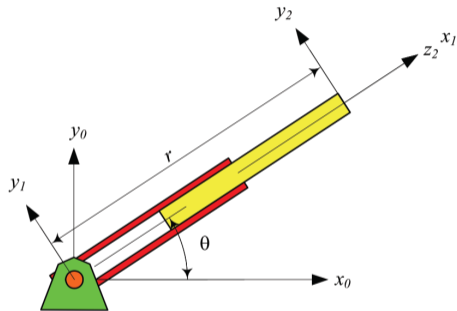
with $J_R = 1$



Direct differentiation

Let's put everything together:

$$\begin{aligned}\dot{\mathbf{X}} &= J(\mathbf{q})\dot{\mathbf{q}} \\ \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \omega_3 \end{bmatrix} &= \begin{bmatrix} J_D \\ J_R \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix}\end{aligned}$$



Forward velocity kinematics

Now, if we want to understand the end effector velocity for a given trajectory, we only need the starting joint variables \mathbf{q}_0 and the velocities $\dot{\mathbf{q}}$:

$$\mathbf{q}(t) = \int_0^T \dot{\mathbf{q}}(t) dt + \mathbf{q}_0$$

Then plug into:

$$\dot{\mathbf{X}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

$$\mathbf{X}(t) = \int_0^T \dot{\mathbf{X}}(t) dt + \mathbf{X}_0$$

