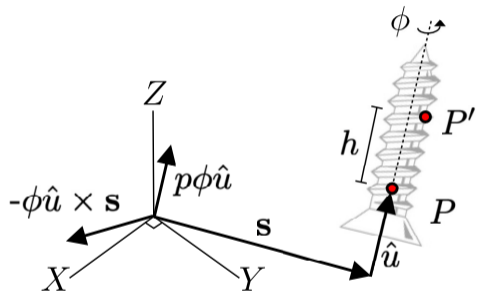


screw motion



h : translation

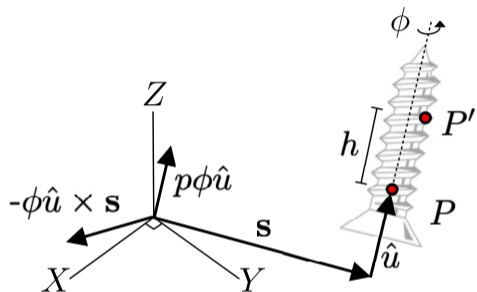
ϕ : rotation

$\hat{\mathbf{u}}$: axis of motion

\mathbf{s} : location vector

$$p : \text{pitch} = \frac{h}{\phi}$$

screw motion as a homogeneous transformation



For general screw motion:

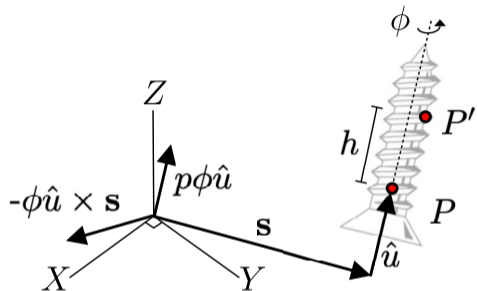
$$\begin{aligned}\check{s}(h, \phi, \hat{u}, \mathbf{s}) &= \begin{bmatrix} R_{\hat{u}, \phi} & \mathbf{s} - R_{\hat{u}, \phi} \mathbf{s} + h \hat{u} \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\tilde{u} \phi} & (I - e^{\tilde{u} \phi}) \mathbf{s} + h \hat{u} \\ \mathbf{0} & 1 \end{bmatrix}\end{aligned}$$

where \tilde{u} is the skew symmetric representation:

$$\tilde{u} = \begin{bmatrix} 0 & u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

It would be convenient to write entire transformation as a matrix exponential.

screw axis and matrix exponential form



Define screw axis using **Plücker Coordinates**:

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} + p\hat{u} \end{bmatrix}, \quad \boldsymbol{\rho} = -\hat{u} \times \mathbf{s}$$

$$p = \frac{h}{\phi}$$

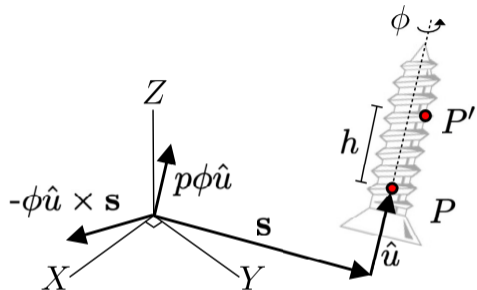
in “skew symmetric form”:

$$\tilde{\mathcal{S}} = \begin{bmatrix} \tilde{u} & \boldsymbol{\rho} + p\hat{u} \\ 0 & 0 \end{bmatrix}$$

Then, we can write the transformation $T = e^{\tilde{\mathcal{S}}\phi}$.

So the point P' in the illustration results from the screw motion: $P' = e^{\tilde{\mathcal{S}}\phi}P$.

screw axis infinite pitch: prismatic joint



$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} + p\hat{u} \end{bmatrix}, \quad \boldsymbol{\rho} = -\hat{u} \times \mathbf{s}, \quad p = \frac{h}{\phi}$$

Multiply by displacement ϕ :

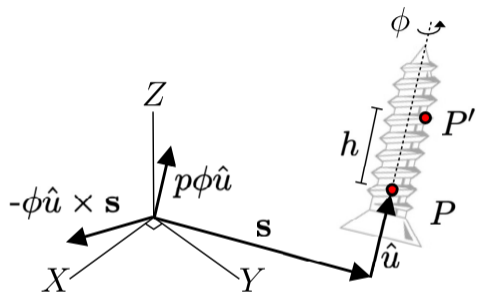
$$\mathcal{S}\phi = \begin{bmatrix} \phi\hat{u} \\ \phi\boldsymbol{\rho} + h\hat{u} \end{bmatrix} = \begin{bmatrix} \frac{h}{p}\hat{u} \\ \frac{h}{p}\boldsymbol{\rho} + h\hat{u} \end{bmatrix}$$

then $p \rightarrow \infty$

$$= \begin{bmatrix} 0 \\ h\hat{u} \end{bmatrix} \Rightarrow \mathcal{S}q = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix} q$$

Note we change variable from ϕ (which is related to h through p) to q to denote joint variable.

screw axis zero pitch: revolute joint



$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} + p\hat{u} \end{bmatrix}, \quad \boldsymbol{\rho} = -\hat{u} \times \mathbf{s}, \quad p = \frac{h}{\phi}$$

In this case it's clear with $p = 0$:

$$\begin{aligned} \mathcal{S}\phi &= \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} \end{bmatrix} \phi \\ \Rightarrow \mathcal{S}q &= \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} \end{bmatrix} q \end{aligned}$$

Again we change ϕ to q to denote that it is a joint variable.

summary of screw axis for robot joints

Prismatic Joint

$$\mathcal{S} = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix}$$

Revolute Joint

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \rho \end{bmatrix}, \quad \rho = -\hat{u} \times \mathbf{s}$$

In either case the transformation matrix from the screw motion can be computed:

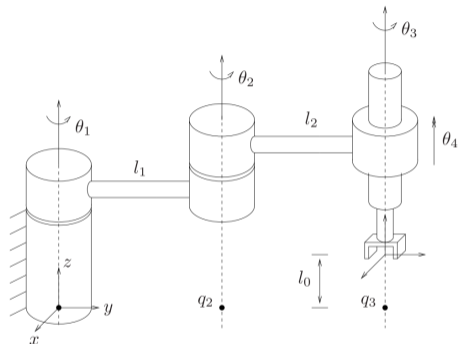
$$T = e^{\tilde{\mathcal{S}}q}$$

summary of product of exponentials for forward kinematics

- Define global fixed base frame and end-effector frame
- Find M , the homogeneous transformation matrix of the end-effector in the global frame at the **home** position ($q_i = 0$) for all i .
- Define all joint screw axis, \mathcal{S}_i in terms of the global frame
- Apply the product of exponentials formula:

$${}^0T_E = e^{\tilde{\mathcal{S}}_1 q_1} e^{\tilde{\mathcal{S}}_2 q_2} \dots e^{\tilde{\mathcal{S}}_n q_n} M$$

SCARA example



End-effector transformation:

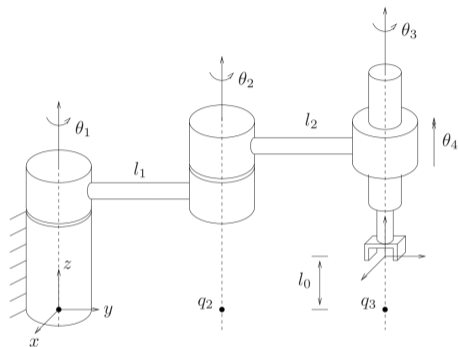
$$M = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \\ 1 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Screw axis \mathcal{S}_1

$$\mathcal{S}_1 = \begin{bmatrix} \hat{u}_1 \\ -\hat{u}_1 \times \mathbf{s}_1 \end{bmatrix}, \quad \hat{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_1 = [0 \ 0 \ 1 \ 0 \ 0 \ 0]^T$$

SCARA example



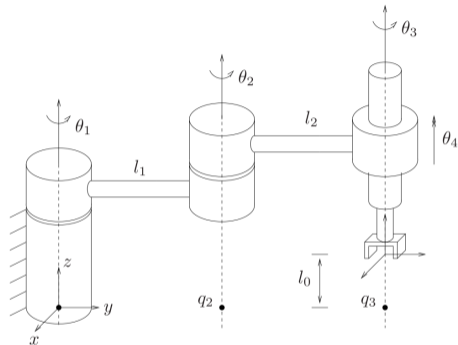
Screw axis \mathcal{S}_2

$$\mathcal{S}_2 = \begin{bmatrix} \hat{u}_2 \\ -\hat{u}_2 \times \mathbf{s}_2 \end{bmatrix}, \quad \hat{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ l_1 \\ 0 \\ 0 \end{bmatrix}$$

SCARA example



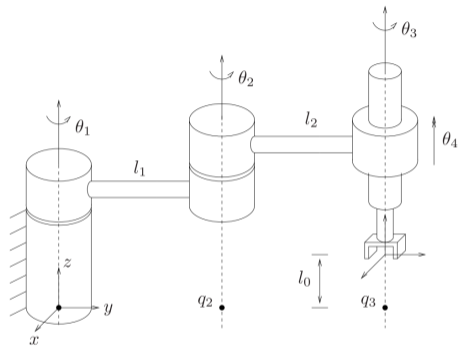
Screw axis \mathcal{S}_3

$$\mathcal{S}_3 = \begin{bmatrix} \hat{u}_3 \\ -\hat{u}_3 \times \mathbf{s}_3 \end{bmatrix}, \quad \hat{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

SCARA example



Screw axis \mathcal{S}_4

$$\mathcal{S}_4 = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix} \quad \hat{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{S}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the forward kinematics can be computed as:

$${}^0T_E = e^{\tilde{\mathcal{S}}_1 q_1} e^{\tilde{\mathcal{S}}_2 q_2} e^{\tilde{\mathcal{S}}_3 q_3} e^{\tilde{\mathcal{S}}_4 q_4} M$$