ME 221 lecture 1 9/27/22

Today:

L&P Ch. 2
- Degrees of freedom
- Joints
- Configuration space

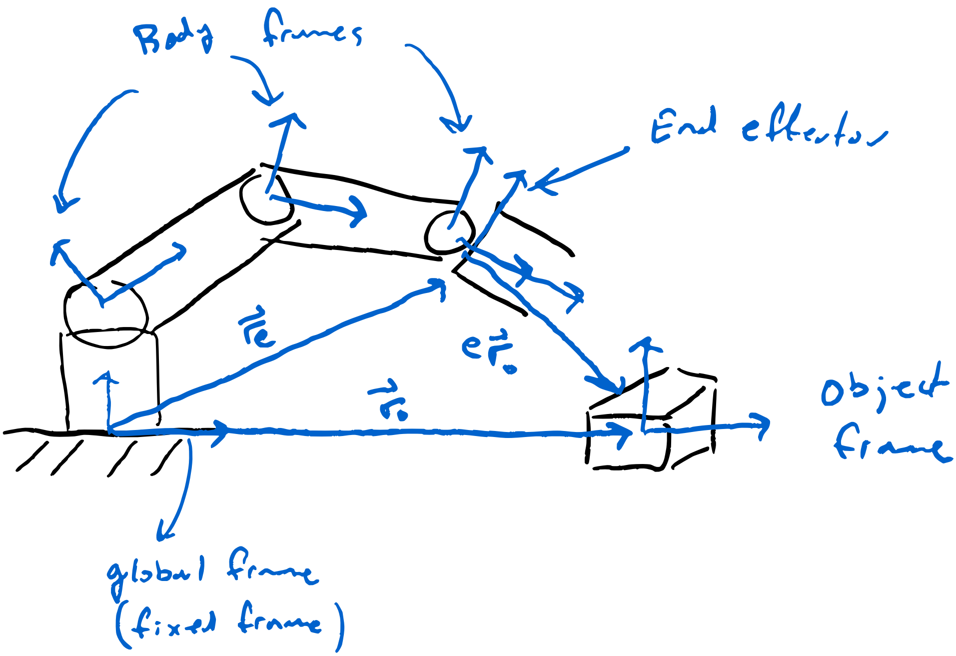
JZ Ch. 2, L&P Ch. 3
- Task space
- vector representation
- Rotations
- Orientation

Goal:

1. Basic concepts related the <u>configuration</u> of robot
2. Review coordinate frames & vectors
3. Introduce <u>Rotation matrices</u>

Motivation :

$\vec{r}$: vector



What are the mathematics that we need to represent object with respect to each other?

Before we go here, let's review some of the concepts from last time

<u>Configuration</u>: the complete specification of the position of every point of the robot.

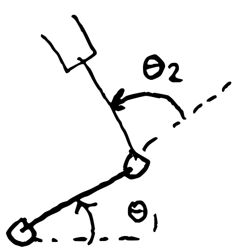
The minimum number (n) coordinates required to represent configuration is the degrees of freedom (DoF).

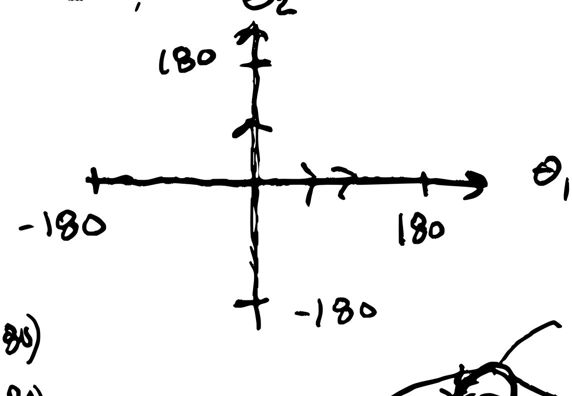
This <u>n-dimensional space</u> is called the <u>configuration space</u> (joint space)

<u>Example</u>:

$$\vec{q} = [\theta_1 , \theta_2]^T$$

$n = 2$,



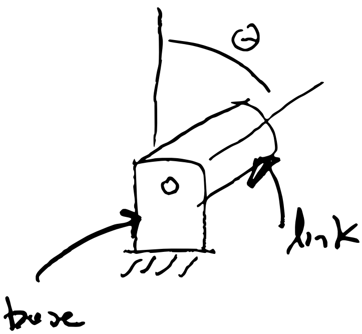


$\theta_1 \in [-180, 180)$

$\theta_2 \in [-180, 180)$

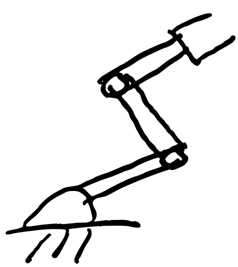
Degrees-of-freedom

$$\text{DoF} = \sum \text{DoF each body} - \sum \text{independent constraints}$$

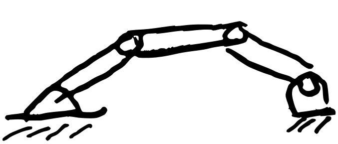


$$6 - 5 = 1$$

* **Serial mechanism**: any mechanism **without** a closed loop

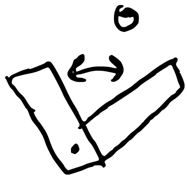


Parallel mechanism: any mechanism **with** a loop



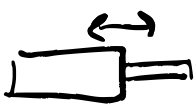
We will focus on two joints

① Revolute



How many constraints between two rigid bodies? 5

② Prismatic



How many constraints between two rigid bodies? 5.

Task Space: the natural space in which the robot's end-effector can expressed.

Configuration Space	Task Spaces
$\vec{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ d \\ \vdots \\ n \end{bmatrix}$	$\vec{X} = \begin{bmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$

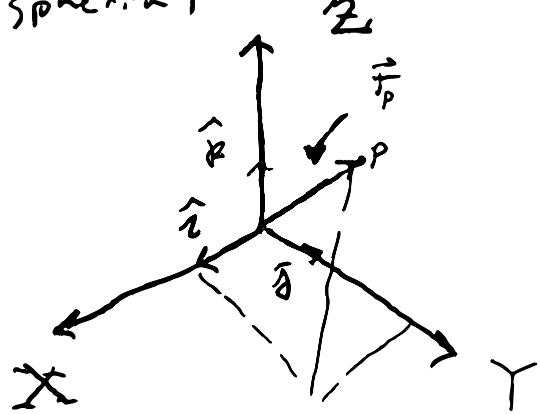
* **Workspace**: the space of all possible configurations of robot

Q: whats the difference?

Reference Frames

coordinate system: a set of values that describes a point in space

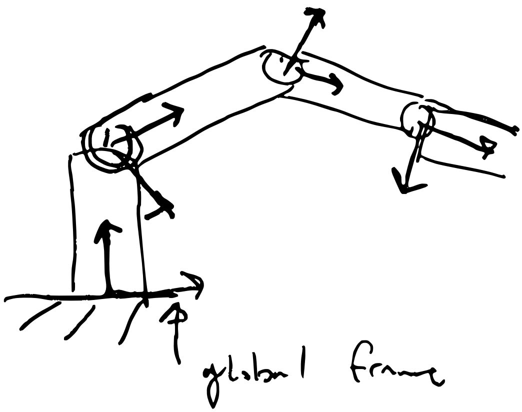
→ cartesian
polar
spherical



$$\vec{r}_p = \begin{bmatrix} r_{p_x} \\ r_{p_y} \\ r_{p_z} \end{bmatrix}$$

Reference frame: coordinate system to express a particular point in space relative to some other origin

We will use reference frames to understand the kinematics (spatial representation) of our robots.



Q: How to map between different reference frames?

A: Rotation matrices

Unit Vectors:

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

dot product:

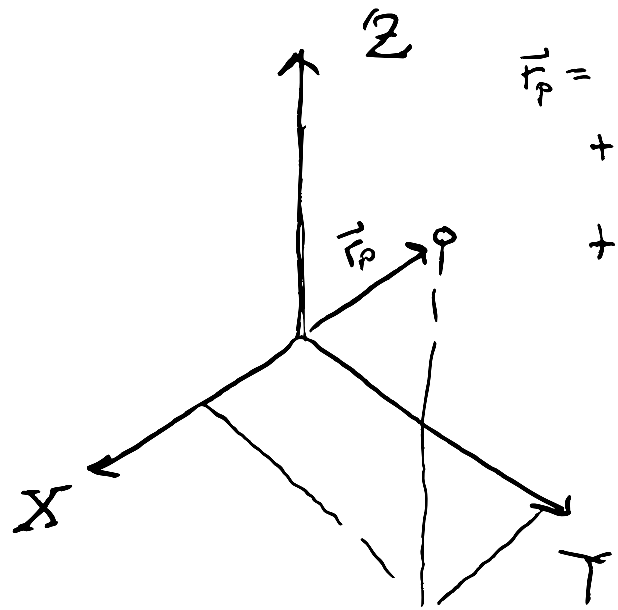
$$\hat{\imath} \cdot \hat{\jmath} = 0$$

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Convient to express a vector

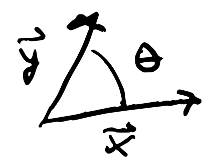
$$\vec{r}_p = \underbrace{r_{p_x} \hat{\imath} + r_{p_y} \hat{\jmath} + r_{p_z} \hat{k}}_{\text{every vector can be expressed as a linear combination of unit vectors!}}$$

every vector can be expressed as a linear combination of unit vectors!

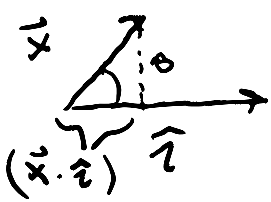


$$\vec{r}_p = (\vec{r}_p \cdot \hat{\imath})\,\hat{\imath} + (\vec{r}_p \cdot \hat{\jmath})\,\hat{\jmath} + (\vec{r}_p \cdot \hat{k})\,\hat{k}$$

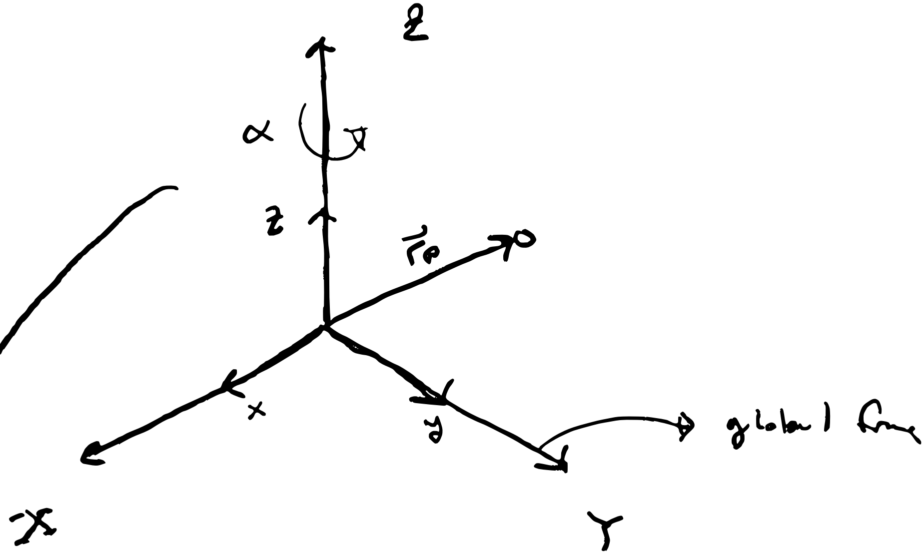
$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$
$$= \|\vec{x}\|\,\|\vec{y}\| \cos\theta$$



When use the dot product with a unit vector we get the <u>projection</u> of the vector onto the unit vector:

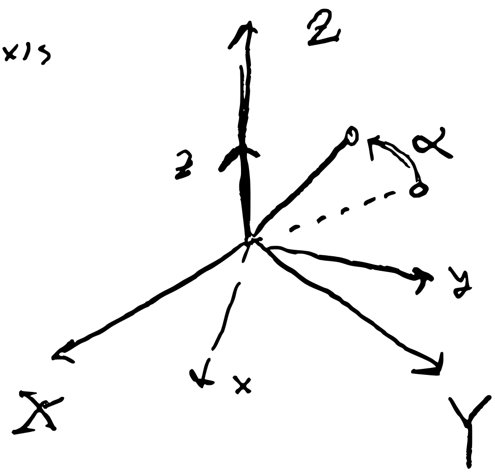


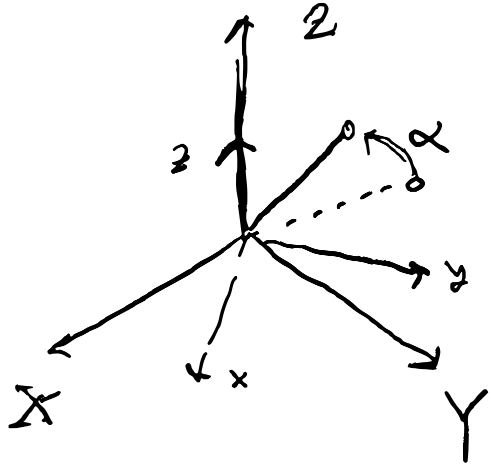
$$\vec{x} \cdot \hat{\imath} = \|\vec{x}\| \cos\theta$$



① Attach a new (body frame) to origin of the global frame

② rotate $\vec{r}_p$ (and the body) about Z-axis





How to express $\vec{r}_p$ in the global frame?

$$^B\vec{r}_p = x_p \hat{\imath} + y_p \hat{\jmath} + z_p \hat{k}$$

$$^G\vec{r}_p = \ ?$$

$$= X_p \hat{I} + Y_p \hat{J} + Z_p \hat{K}$$

$$X_p = ({}^B\vec{r}_p \cdot \hat{I}) = x_p \hat{\imath} \cdot \hat{I} + y_p \hat{y} \cdot \hat{I} + z_p \hat{k} \cdot \hat{I}$$

$$Y_p = ({}^B\vec{r}_p \cdot \hat{J}) = x_p \hat{\imath} \cdot \hat{J} + y_p \hat{y} \cdot \hat{J} + z_p \hat{k} \cdot \hat{J}$$

$$Z_p = ({}^B\vec{r}_p \cdot \hat{K}) = x_p \hat{\imath} \cdot \hat{K} + y_p \hat{y} \cdot \hat{K} + z_p \hat{k} \cdot \hat{K}$$

$$X_p = \left({}^{B}\vec{r}_p \cdot \hat{I}\right) = x_p \hat{\imath} \cdot \hat{I} + y_p \hat{\jmath} \cdot \hat{I} + z_p \hat{k} \cdot \hat{I}$$

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$$Z_p = \left({}^{B}\vec{r}_p \cdot \hat{K}\right) = x_p \hat{\imath} \cdot \hat{K} + y_p \hat{\jmath} \cdot \hat{K} + z_p \hat{k} \cdot \hat{K}$$

$$\begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\imath}\cdot\hat{I} & \hat{\jmath}\cdot\hat{I} & \cancelto{0}{\hat{k}\cdot\hat{I}} \\ \hat{\imath}\cdot\hat{J} & \hat{\jmath}\cdot\hat{J} & \cancelto{0}{\hat{k}\cdot\hat{J}} \\ \cancelto{0}{\hat{\imath}\cdot\hat{K}} & \cancelto{0}{\hat{\jmath}\cdot\hat{K}} & \hat{k}\cdot\hat{K} \end{bmatrix}}_{\text{directional cosines}} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

$$\hat{\imath}\cdot\hat{I} = \cos\alpha$$

$$\hat{\jmath}\cdot\hat{I} = -\sin\alpha \qquad R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\imath}\cdot\hat{J} = \sin\alpha$$

$$\hat{\jmath}\cdot\hat{J} = \cos\alpha$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

R_z, R_y, R_x : the Basic rotation matrices

__Any__ rotation about a fixed point can be realized w/ __at most__ three consecutive rotation w the Basic R. matrices.

Properties of Rotation Matrix

① Orthogonality

- unit norm condition

$$R = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{bmatrix}$$

$$\|\vec{r}_1\| = \|\vec{r}_2\| = \|\vec{r}_3\| = 1$$

__3 constraints__

- orthogonal condition

$$\vec{r}_1 \cdot \vec{r}_2 = 0$$
$$\vec{r}_1 \cdot \vec{r}_3 = 0$$
$$\vec{r}_3 \cdot \vec{r}_2 = 0$$

__3 constraints__

9 elements − 6 constraints = 3 dof

$$R^T = R^{-1} \Rightarrow RR^T = I$$

(2) $\det(R) = +1$ — Why?

$$RR^T = I$$

$$\det(RR^T) = \det(I)$$

$$\det(R)\det(R^T) = 1$$

$$\det(R)^2 = 1$$

$$\det(R) = \pm 1$$

We call the group of all possible rotations the special orthogonal group $SO(3)$

$$SO(3) = \{ R \in \mathbb{R}^3 \times \mathbb{R}^3 : RR^T = I,\ \det(R) = 1 \}$$

Lie group ("Lee")

Next time: interpretation R
Kinematics