Last time: Velocity Transformation Matrix

The velocity transformation matrix is related to the derivative of the homogeneous transformation matrix

$$ {}^G\dot{T}_B = {}^G V_B \; {}^G T_B $$

or

$$ {}^G V_B = {}^G\dot{T}_B \, {}^G T_B^{-1} $$

recall that this is the end-effector twist, in matrix form:

$$ {}^G\tilde{\nu}_B = {}^G V_B = {}^G\dot{T}_B \, {}^G T_B^{-1} = \begin{bmatrix} {}^G\tilde{\omega}_B & {}^G\boldsymbol{v}_B \\ 0 & 0 \end{bmatrix} $$

rearranging into a vector:

$$ {}^G\boldsymbol{\nu}_B = \begin{bmatrix} {}^G\boldsymbol{\omega}_B \\ {}^G\boldsymbol{v}_B \end{bmatrix} $$

Jacobian Agenda

- Jacobian
- Direct Differentiation
- Forward Velocity Kinematics

- Jacobian Generating vectors
- Inverse Velocity Kinematics
- Other types of Jacobians
- Maneuverability Analysis

Do you remember **chain rule** for functions? Find:

$$\frac{d}{dt} f\left(x(t)\right) =$$

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What about for the multivariate case?

$$f(\boldsymbol{x}(t)) = f(x_1(t), x_2(t),\ \ldots,\ x_n(t))$$

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Deriving the Jacobian from Velocity Transformation Matrix

The velocity of **any body point** can be found in the Global frame if we know the homogeneous transformation matrix:

$$^{G}V_{B} = {}^{G}\dot{T}_{B}\ {}^{G}T_{B}^{-1}$$

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Recall from **forward kinematics** we can find the transformation from the Global base frame to the end effector from the DH-parameters (or PoE):

$$^{0}T_{E}(\boldsymbol{q}) = {}^{0}T_{1}(q_1)\ {}^{1}T_{2}(q_2)\ \ldots {}^{n-1}T_{E}(q_n)$$

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Then the velocity transformation matrix for the end-effector:

$$^{0}V_{E} = {^{0}\dot{T}_{E}(\mathbf{q})} \ {^{0}T_{E}^{-1}(\mathbf{q})}$$

Deriving the Jacobian from Velocity Transformation Matrix

Let's explore the velocity transformation matrix:

$$ {}^{0}\tilde{\boldsymbol{\nu}}_E = {}^{0}V_E = {}^{0}\dot{T}_E(\mathbf{q})\ {}^{0}T_E^{-1}(\mathbf{q}) $$

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\end{aligned}
$$

Deriving the Jacobian from Velocity Transformation Matrix

We can re-write in **vector form**:

$$ {}^{0}\boldsymbol{\nu}_E = \begin{bmatrix} {}^{0}\boldsymbol{\omega}_E \\ {}^{0}\boldsymbol{v}_E \end{bmatrix} = \boxed{\dot{\boldsymbol{X}} = J(\boldsymbol{q})\dot{\boldsymbol{q}}} $$

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with

$$ J(\boldsymbol{q}) = \left[\left(\frac{\partial\, {}^{0}T_E(\boldsymbol{q})}{\partial q_1}\, {}^{0}T_E^{-1}(\mathbf{q}) \right)^{\vee} \quad \dots \quad \left(\frac{\partial\, {}^{0}T_E(\boldsymbol{q})}{\partial q_n}\, {}^{0}T_E^{-1}(\mathbf{q}) \right)^{\vee} \right] $$

with the $^{\vee}$ symbol denoting the "vector form" (as opposed to the skew symmetric form)

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Very Important: The Jacobian provides a linear relationship between the joint velocities ($\boldsymbol{q}$) and the end effector's linear ($\boldsymbol{v}$) and angular ($\boldsymbol{\omega}$) velocities. Therefor we can use linear methods e.g.,

$$ Ax = b \rightarrow x = A^{-1}b \quad \text{(can be solved efficiently)} $$

How to calculate the Jacobian

$$\dot{\boldsymbol{X}} = \underbrace{J(\boldsymbol{q})}_{6 \times n} \dot{\boldsymbol{q}}$$

We can split up the Jacobian into two parts:

$$\dot{\boldsymbol{X}} = \begin{bmatrix} {}^0\boldsymbol{\omega}_E \\ {}^0\boldsymbol{v}_E \end{bmatrix} = \begin{bmatrix} J_R \\ J_D \end{bmatrix} \dot{\boldsymbol{q}}$$

- rarely ever calculated as we derived (direct differentiation)
- more systematic approach: Jacobian generating vectors (next time)
- difference between **analytic** vs **geometric** Jacobian (next time)

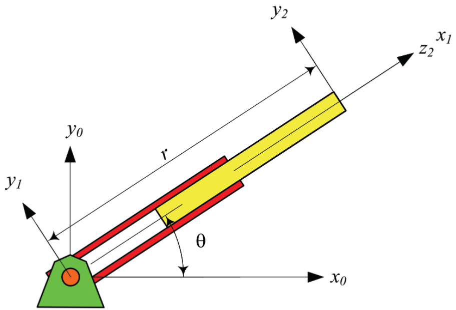
Direct differentiation: Ex.242, pg.446

If the robot is simple, it is easy to do direct differentiation:

Find forward kinematics:

$$^{0}T_2 = {}^{0}T_1\,{}^{1}T_2$$

$$^{0}T_1 =$$



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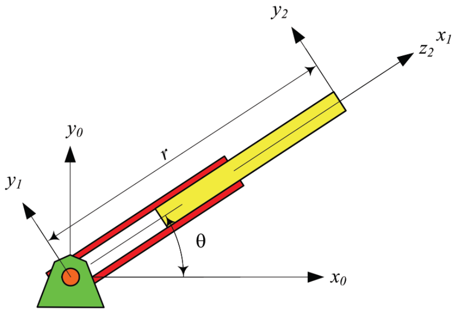
$$^{0}T_2 = {^{0}T_1}\,{^{1}T_2}$$

$$^{0}T_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{1}T_2 =$$



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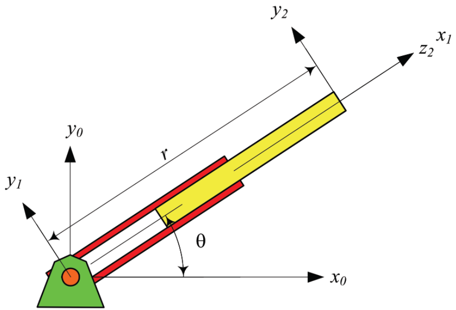
$$^{1}T_{2} = \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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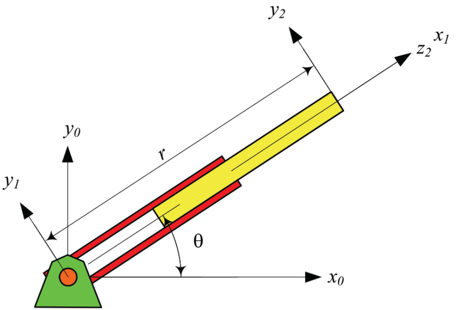

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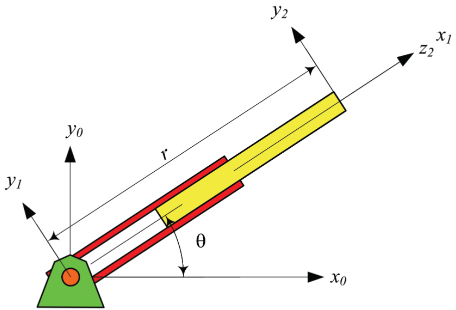

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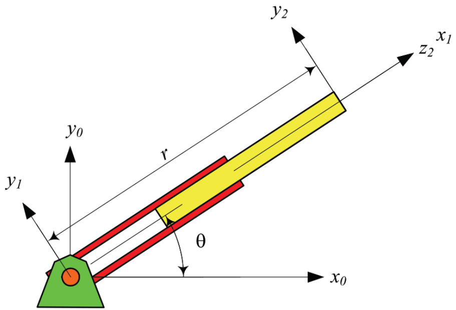
Therefore:

$$\begin{bmatrix} X \\ Y \end{bmatrix} =$$



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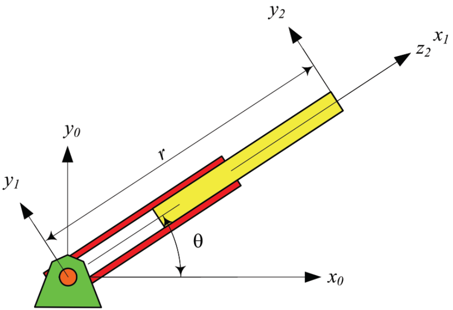


Direct differentiation

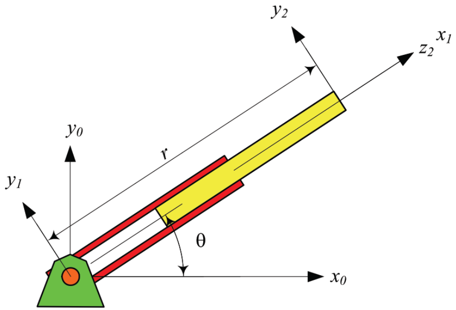
The forward kinematics maps joint variables to Cartesian space:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$\boldsymbol{x} = f(\boldsymbol{q})$$

with

$$\boldsymbol{x} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} r \\ \theta \end{bmatrix},$$



Direct differentiation

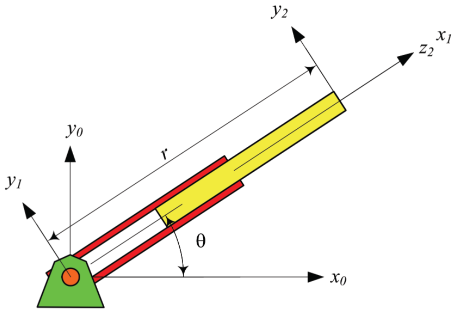
The displacement Jacobian is calculated by taking the time derivative of the displacement forward kinematics:

$$\dot{\boldsymbol{x}} = J_D(\boldsymbol{q})\dot{\boldsymbol{q}}$$

where

$$J_D(\boldsymbol{q}) = \begin{bmatrix} \frac{\partial f(\boldsymbol{q})}{\partial q_1} & \frac{\partial f(\boldsymbol{q})}{\partial q_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(\boldsymbol{q})}{\partial r} & \frac{\partial f(\boldsymbol{q})}{\partial \theta} \end{bmatrix}$$

$$=$$



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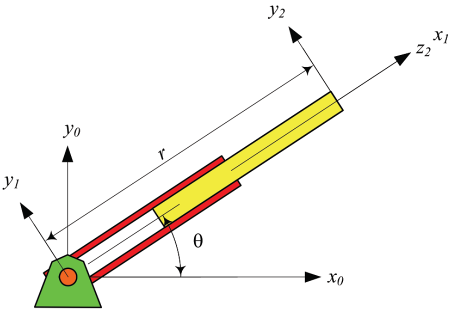
$$= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

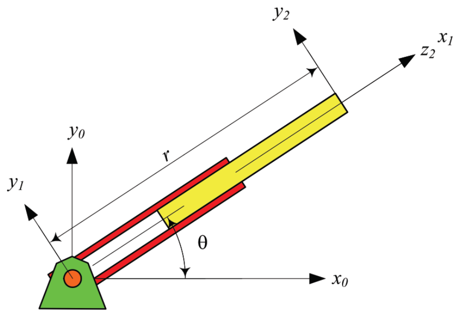


Direct differentiation

The rotational Jacobian is calculated find the angular velocity vector from the forward kinematics:

$$_0\tilde{\omega}_2 = {}^0\dot{R}_2\,{}^0R_2^T$$

$$^0T_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & r\cos\theta \\ \sin\theta & \cos\theta & 0 & r\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^0R_2 & {}^0\boldsymbol{d}_2 \\ \mathbf{0} & 1 \end{bmatrix}$$



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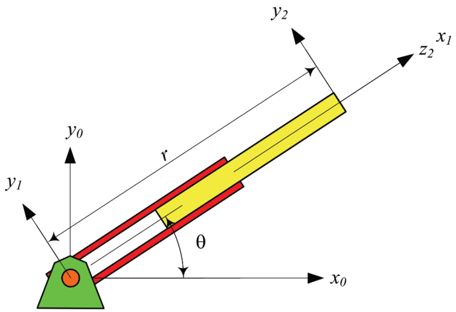

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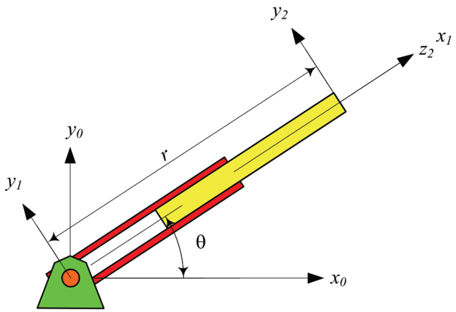
$$ {}^0\dot{R}_2 = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} $$

$$ {}^0R_2^T = $$



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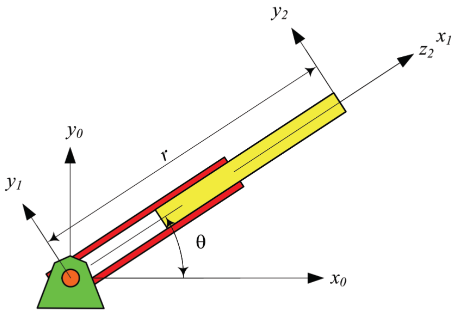

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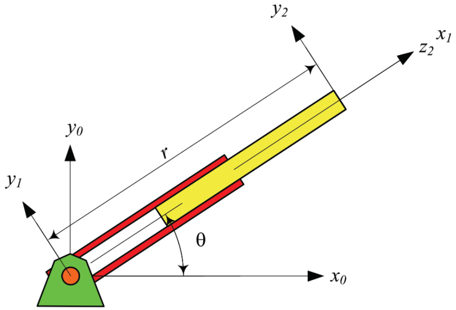

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$$
\begin{aligned}
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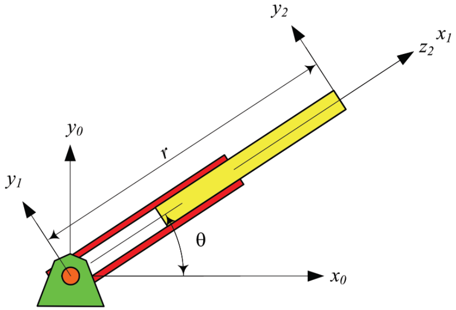
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}
\end{aligned}
$$



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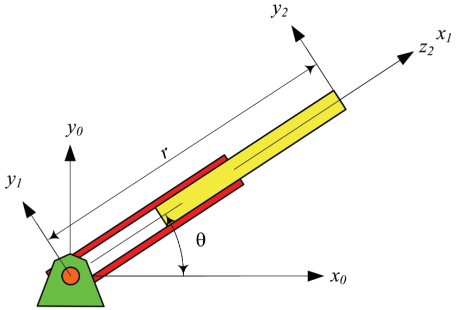
$$

Recall that for skew symmetric matrix:

$$
\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}
$$



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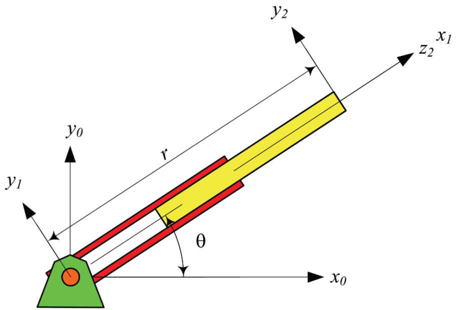

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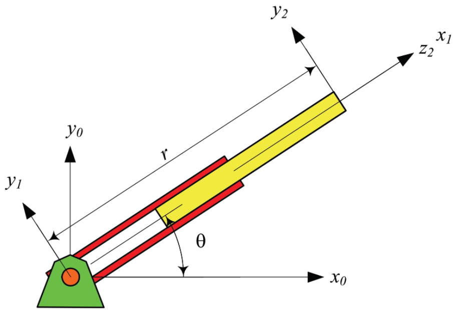

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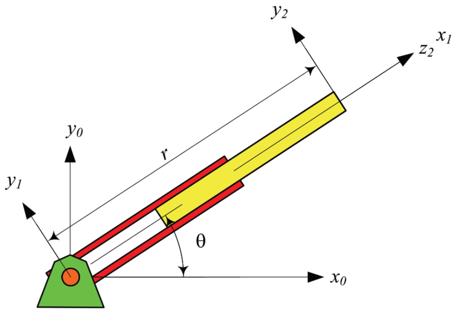
$$ {}_0\boldsymbol{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} $$

Or we can write as:

$$ \omega_3 = J_R \dot{\theta} $$

with $J_R = 1$



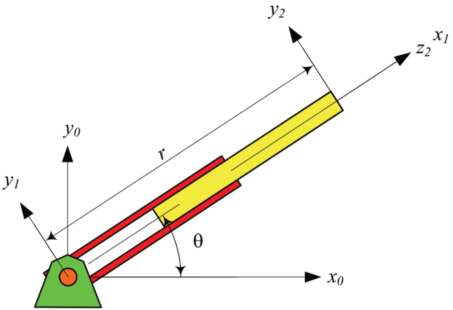
Direct differentiation

Let's put everthing together:

$$\dot{\boldsymbol{X}} = J(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \omega_3 \end{bmatrix} = \begin{bmatrix} J_D \\ J_R \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix}$$

$$=$$



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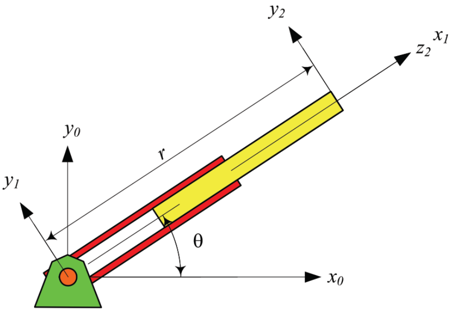

Forward velocity kinematics

Now, if we want to understand the end effector velocity for a given trajectory, we only need the starting joint variables $\boldsymbol{q}_0$ and the velocities $\dot{\boldsymbol{q}}$:

$$\boldsymbol{q}(t) = \int_0^T \dot{\boldsymbol{q}}(t)\, dt + \boldsymbol{q}_0$$

Then plug into:

$$\dot{\boldsymbol{X}} = J(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\boldsymbol{X}(t) = \int_0^T \dot{\boldsymbol{X}}(t)\, dt + \boldsymbol{X}_0$$