ME 221 lecture 11 11/3/22

What we know about the Jacobian so far:

if we can write the forward kinematics:

$$\vec{x} = f(\vec{q})$$

then the Jacobian:

$$J(\vec{q}) = \left[\frac{\partial f(\vec{q})}{\partial q_1} \quad \cdots \quad \frac{\partial f(\vec{q})}{\partial q_n} \right]$$

$$\dot{x} = J(q)\,\dot{q}$$

We usually cannot do that directly

($R \rightarrow$ parameterized, so we would need to choose euler angles)

$\vec{V} = \dot{T}\, T^{-1}$ so you can rearranged :

$\triangle\; v = J(q)\, \dot{q}$

Another way to think about:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} J_R(\vec{q}) \\ \\ J_D(\vec{q}) \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

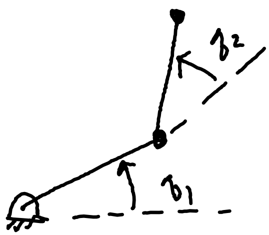
Forward kinematics $\quad {}^0T_E = \begin{bmatrix} {}^0R_E & {}^0\vec{d}_E \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = {}^0\vec{d}_E \implies J_D = \begin{bmatrix} \frac{\partial\, {}^0\vec{d}_E}{\partial q_1} & \cdots & \frac{\partial\, {}^0\vec{d}_E}{\partial n} \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underbrace{{}^0\dot{R}_E \, {}^0R_E^T}_{J(q)\,\dot{q}}$$

Summary: Jacobian is a linear map (depends on the current pose $\vec{q}$) between the end-effector velocity & the joint velocity.

Ex. 2R planar Robot



$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

$$\dot{x} = -l_1 s_1 \dot{q}_1 - l_2 s_{12} (\dot{q}_1 + \dot{q}_2)$$

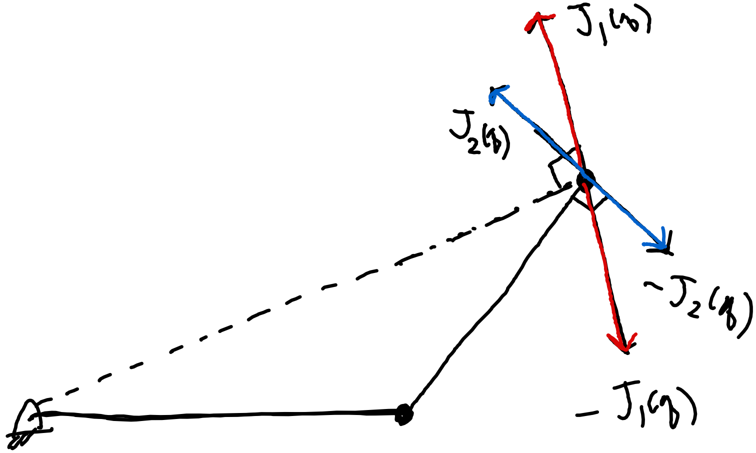
$$\dot{y} = l_1 c_1 \dot{q}_1 + l_2 c_{12} (\dot{q}_1 + \dot{q}_2)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left[\begin{array}{c|c} J_1(q) & J_2(q) \end{array} \right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J_1(q)\dot{q}_1 + J_2(q)\dot{q}_2$$

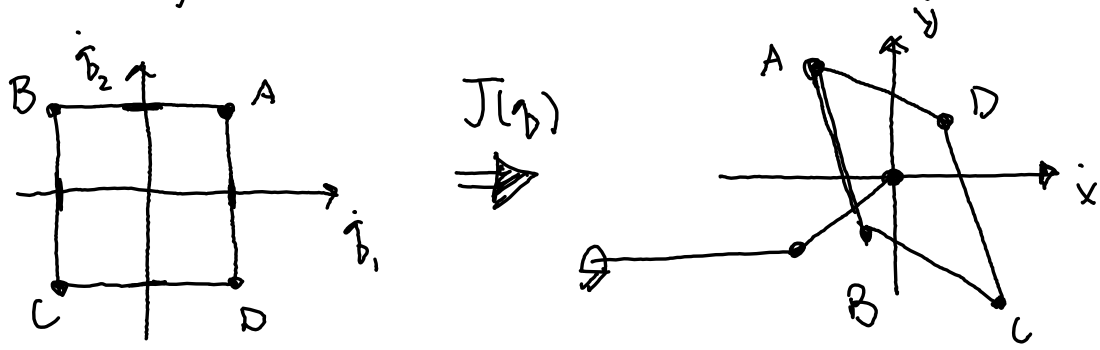
Vectors that tells us the contribution of each component of joint velocity to end-effector velocity

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J_1(q)\,\dot{q}_1 + J_2(q)\,\dot{q}_2$$



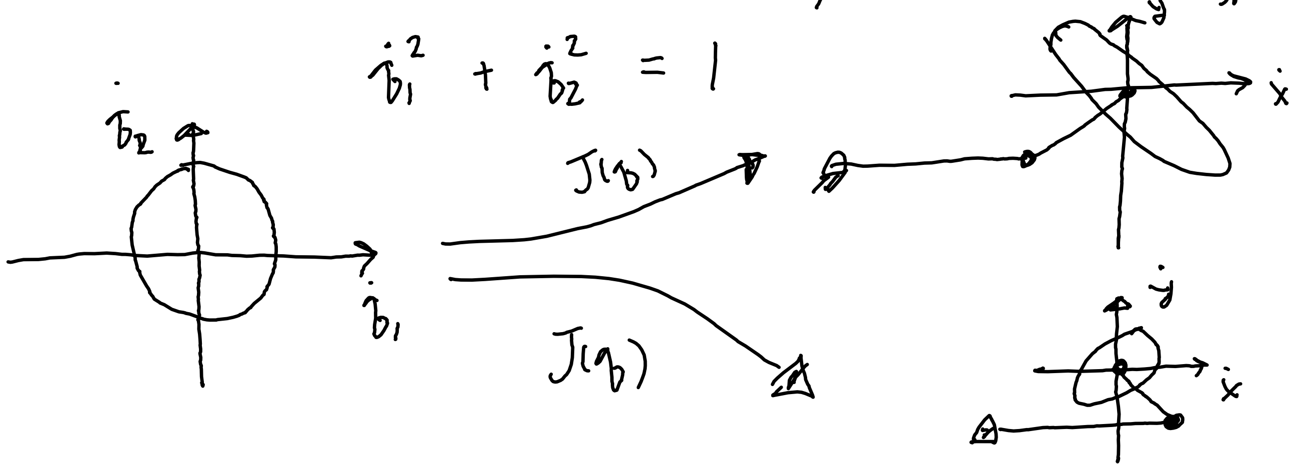
Jacobian tells us which direction we can move in.

- We can use Jacobian to map <u>bounds</u> on joint speed $\longrightarrow$ endeffector speed.



- lets use the same idea, for "unit" speeds.

$$\dot{q}_1^2 + \dot{q}_2^2 = 1$$



We call this ellipsoid: **the Manipulability Ellipsoid**

$\| \dot{\vec{q}} \| = 1$

$$1 = \dot{\vec{q}}^T \dot{\vec{q}}$$

$$= \left(J(q)^{-1} \dot{x} \right)^T \left(J(q)^{-1} \dot{x} \right)$$

$$= \dot{x}^T J^{-T} J^{-1} \dot{x}$$

$$= \dot{x}^T \underbrace{(J J^T)^{-1}}_{} \dot{x}$$

$x^T A x \geq 0$

$$\boxed{\dot{x}^T A^{-1} x = 1}$$

Eq. of ellipsoid

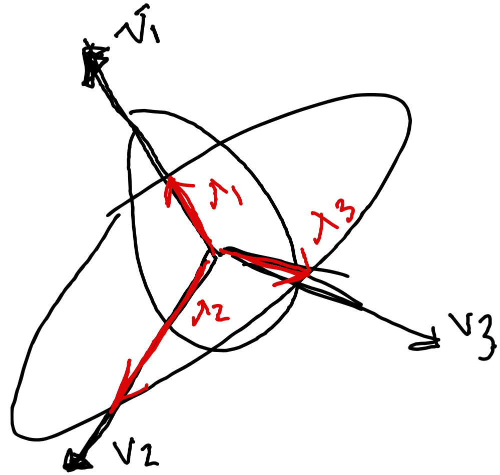
$A = \underbrace{J J^T}$ — positive def.

6×6, symmetric

$$\dot{x}^T A^{-1} \dot{x} = 1$$

$$A = V D V^T$$

$$= \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$



v_n: priciple semi-axes

$\sqrt{\lambda_n}$: length of each semi-axis

We can calculate the manipulability ellipsoid for each of the 3×3 sub Jacobians

$$J(\vec{q}) = \begin{bmatrix} J_R \\ J_D \end{bmatrix}$$

theres a few common <u>measures</u>:

$$V = \det(A) = \det(JJ^T) = \sqrt{\lambda_1 \cdots \lambda_n}$$

we want V to be maximum usually.

$$\mu_1(A) = \frac{\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}} \qquad \frac{\text{largest semi axis}}{\text{shortest semi-axis}} \geq 1$$

$\mu_1 \longrightarrow 1$ (isotropic)

$$\mu_2(A) = \frac{\lambda_{max}}{\lambda_{min}} \geq 1$$ can be used for sensitivity analysis

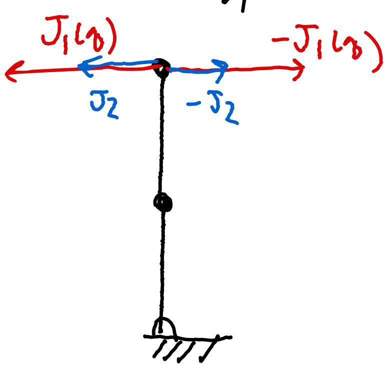
Singularities

$$\dot{x} = J(q)\,\dot{q} \iff \dot{q} = J(q)^{-1}\,\dot{x}$$

- a singular configuration occurs when the Jacobian looses rank.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

consider $q_1 = \pi/2$, $q_2 = 0$, $l_1 = l_2 = 1$



$$\begin{array}{cc} J_1(q) & J_2(q) \\ \end{array}$$

$$J = \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2\dot{q}_1 & -1\dot{q}_2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$q_1 = q_2 = 0 \quad , l_1 = l_2 = 1$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$