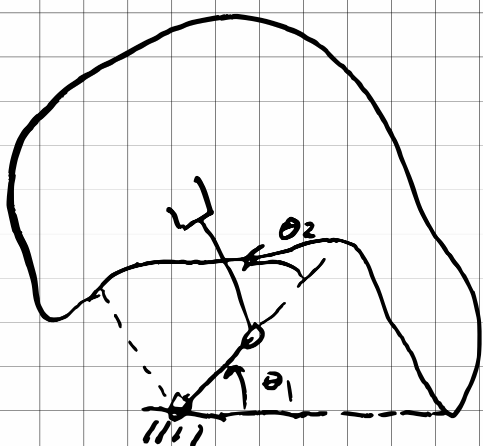
lecture 2 ME 221 9/29/22

Last time:

- Configuration vs. Task space
- workspace

mathematical structures

$q = [\theta_1 \cdots n]$ $\vec{x} = [x\ y\ z, \cdots]$



$q \in [\theta_1 \quad \theta_2]$

- Rotations Matrices

$R^T = R^{-1}$ $\det(R) = +1$

$R \in SO(3)$

$\longrightarrow$ all possible rotations in 3D.

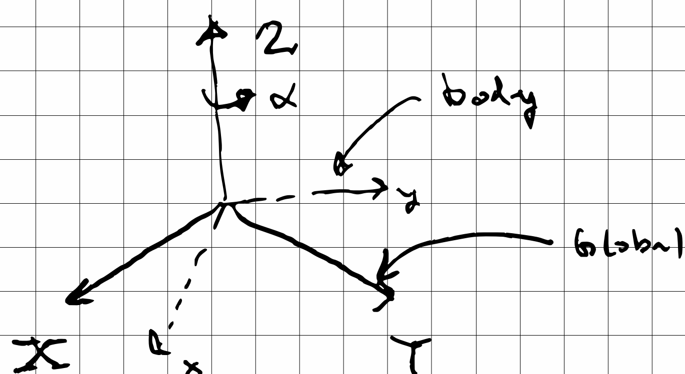
Today:

1. interpretations of R
2. compositions
3. Parameterization

Thee main interpretations of R.

1. represents orientation

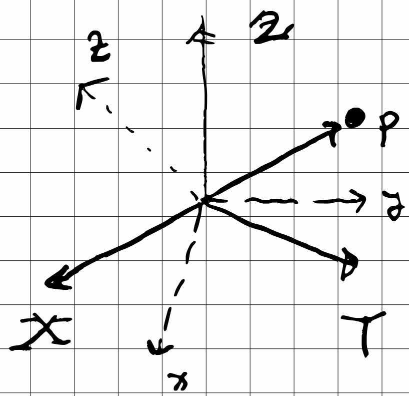
$R \to \underline{\underline{\text{is}}}$ a reference frame



$$^{G}R_{B} = R_{z}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$^{G}R_{B}$: B-frame in terms of G-frame

② Mapping between frames



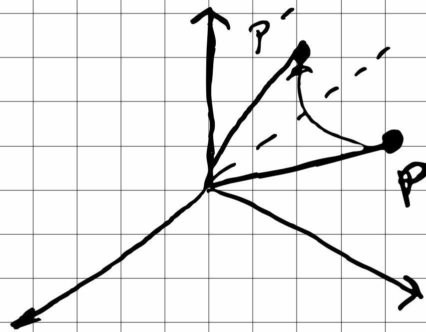
$$^{G}\vec{r}_p = {}^{G}R_B \, {}^{B}\vec{r}_p$$

R is mapping the point from one frame to the other

$$^{B}\vec{r}_p = {}^{B}R_G \, {}^{G}\vec{r}_p$$

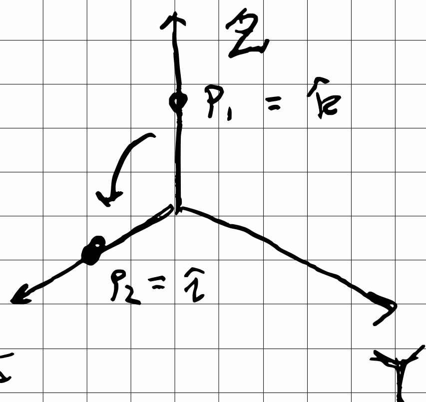
$$\boxed{{}^{G}R_B = {}^{B}R_G^{\,T}}$$

③ Rotation operator



$$\vec{P}' = R(\theta)\cdot\vec{P}$$

Ex.



Find the **R** that does that?

$$\vec{P}_2 = R\,\vec{P}_1$$

$$R = R_y(90^\circ) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

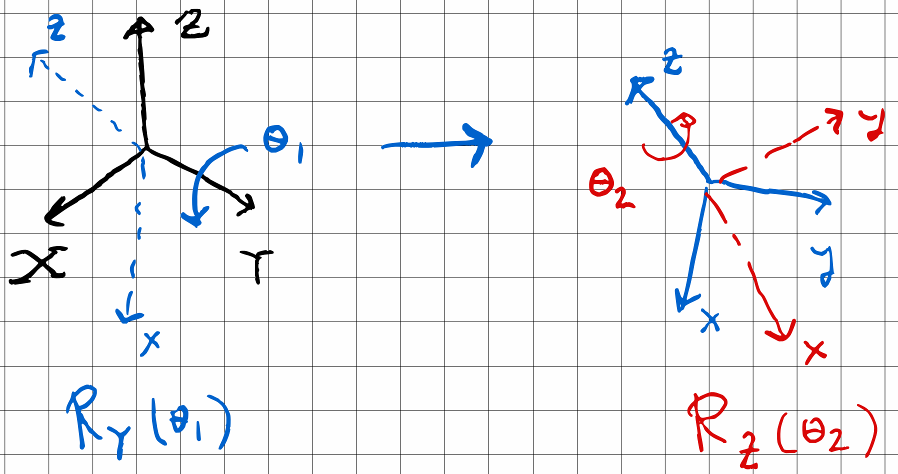
$$\vec{P}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{i}$$

Compositions

Rotation matrices can be <u>composed</u> (multiplied) to form new rotation matrices.

① Rotations with respect to moving frame

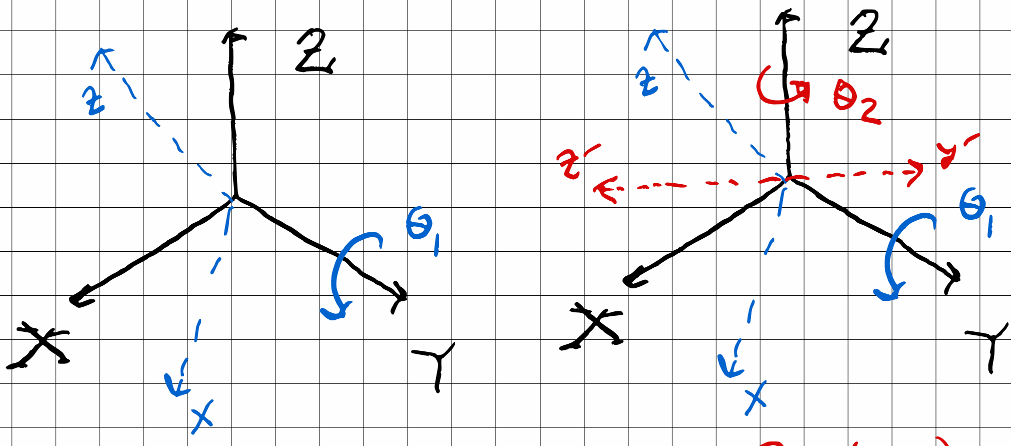


composition for moving frames:

post multiply

$$ {}^{1}R_2 = R_y(\theta_1)\, R_z(\theta_2) $$

(2) Rotations with respect to fixed frame:



$R_y(\theta_1)$ $\qquad$ $R_z(\theta_2)$

Compositions with fixed frame:

pre-multiply $\qquad {}^{1}R_2 = R_z(\theta_2)\, R_y(\theta_1)$

Example: Find the rotation matrix that take the initial orientation to the final orientation with following:

1. $R_x(\theta)$ moving frame
2. $R_z(\phi)$ moving frame
3. $R_z(\alpha)$ fixed frame
4. $R_y(\beta)$ moving frame
5. $R_x(\gamma)$ fixed frame

$R =$

$$R_x(\gamma) R_z(\alpha) R_x(\theta) R_z(\phi) R_y(\beta)$$

Parameterizations of Rotation

recall every R has 9 elements
but only $\underline{\underline{3}}$ dof.

→ parameterization allows us to
be <u>less</u> redundant
→ represent orientation w/
less than 9-elements

* any rotation can be composed by
at most three consecutive <u>Basic</u> rotation.

Major Parameterizations

- * Euler Angles (3-angles)
- * Roll, Pitch, Yaw (3-angles)
- Quaterion
  → extension of complex
  → 4-element, but 3 <u>dof</u>
- * Axis/angle representation
- exponential formulation

Euler Angles

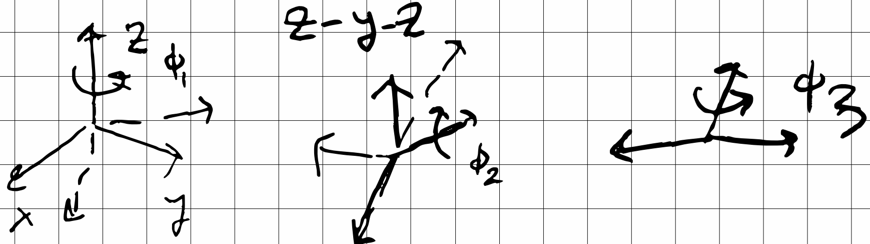
- 3 succesive rotations about the <u>moving</u> axis

- 12 possible parametrization

$$3 \cdot 2 \cdot 2 = \underline{12}$$

x y z
x y x
x z x
⋮

z · y · z



$$R_{zyz}(\phi_1, \phi_2, \phi_3) = R_z(\phi_1) \cdot R_y(\phi_2) \cdot R_z(\phi_3)$$

$$R_{zyz} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \leftarrow \text{given}$$

$$= \begin{bmatrix} x & x & c_1 s_2 \\ x & x & s_1 s_2 \\ -s_2 c_3 & s_2 s_3 & c_2 \end{bmatrix} \leftarrow \text{symbolic}$$

$$\phi_2 = \text{atan2}\left(\pm\sqrt{r_{13}^2 + r_{23}^2},\ r_{33} \right)$$

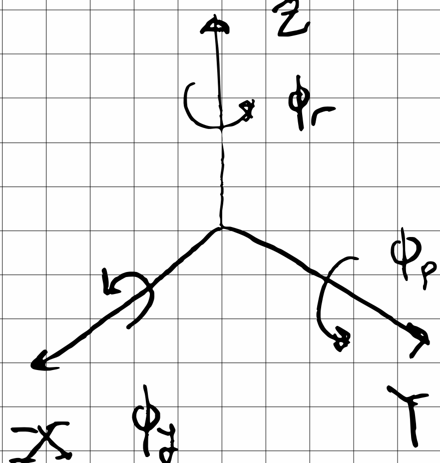
2-argument arctan

$$\phi_1 = \text{atan2}\left(\frac{r_{23}}{\sin\phi_2},\ \frac{r_{13}}{\sin\phi_2} \right)$$

$$\phi_3 = \text{atan2}\left(\frac{r_{32}}{\sin\phi_2},\ \frac{r_{31}}{\sin\phi_2} \right)$$

$\theta_2 = 0 \Rightarrow$ singularity

Roll, Pitch, Yaw



ϕ_r : roll

ϕ_p : pitch

ϕ_y : yaw

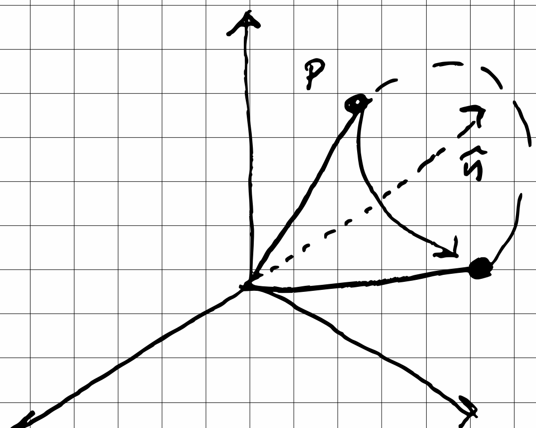
Sequence of rotations:

1. ϕ_y (yaw)
2. ϕ_p (pitch)
3. ϕ_r (roll)

$$R_{rpy}(\phi_y, \phi_p, \phi_r) = R_z(\phi_r) R_y(\phi_p) R_x(\phi_y)$$

Singularity $\cos\phi_p = 0$

Euler's theorem : Any rigid body displacement about a fixed frame is equivalent to a single rotation about some fixed axis

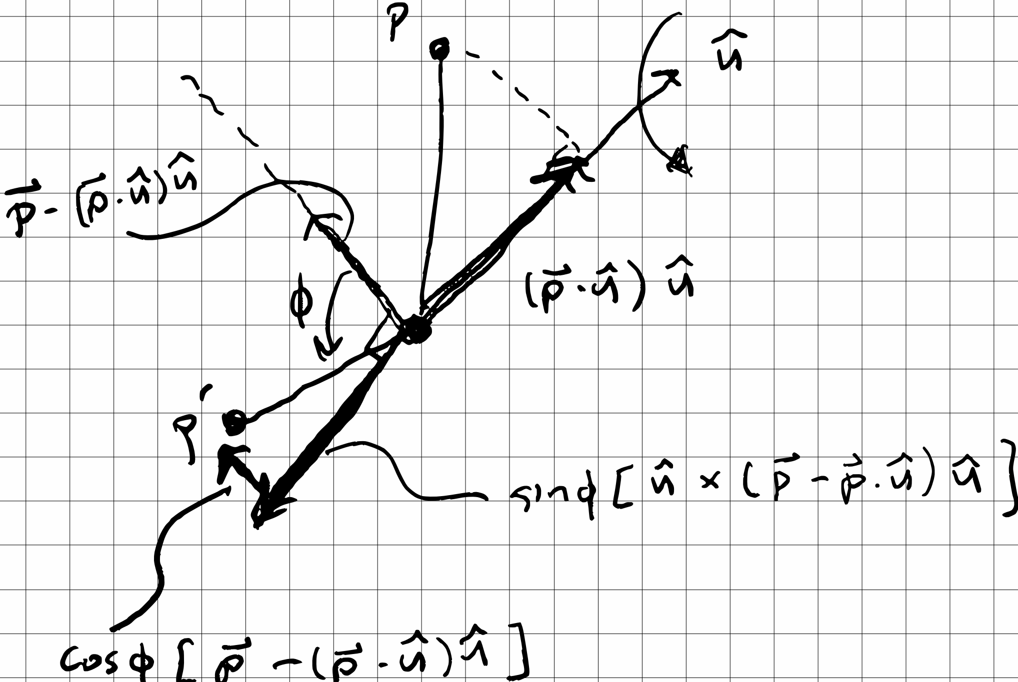


Is there an axis that remains fixed?

$$\hat{u} = R\,\hat{u} \implies R\,\hat{u} = \lambda\,\hat{u}$$

$\lambda = 1$

How can we create R from a given axis/angle.



$$\vdots$$

$$\tilde{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

$$p' = \left[(1-\cos\phi)\,\hat{u}\hat{u}^T + \cos\phi\, I + \tilde{u}\sin\phi\right]\vec{p}$$

$$\rightarrow \boxed{R(\hat{u}, \phi)}$$

Rodigues Formula

$$\begin{cases} \tilde{u} = \dfrac{1}{2\sin\phi}\left[R(\hat{u},\phi) - R(\hat{u},\phi)^T\right] \\ \cos\phi = \dfrac{1}{2}\left[tr\left(R(\hat{u},\phi) - 1\right)\right] \end{cases}$$