ME 221 lecture 3 10/4/22

Last time:

- interpretations of Rotation Matrices
  - → orientation
  - → mapping between frames
  - → rotation operators
- compositions
  - → moving frame : post-multiplication
  - → fixed frame : pre-multiplication.
- parameterization
  - → Euler Angles (moving frame)
  - → Roll, Pitch, Yaw (fixed frame)
    - → singularity
    - → gimbal lock

→ Axis-angle (Rodrigues Formula)

→ derive directly geometry

$$R(\hat{n}, \phi) = \Big[(1 - \cos\phi)\, \hat{u}\hat{u}^T + \cos\phi\, I + \tilde{u} \sin\phi \Big]$$

$$\tilde{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

note that any cross product is equivalent to matrix · vector

↑ skew symmetric

$$\hat{u} \times \vec{p} = \tilde{u}\, \vec{p}$$

Today :

- Exponetial Form
- Quaternion
- $\rightarrow$ Motion Kinematics
  - Homegeneous Transformation
  - Inverse other properties
  - Composition.

Exponential Coordinates

Murry Ch. 2.2.2

LiP Ch. 3.2

For axis-angle, we use two parameters : $\hat{u}, \phi$

What is $\underbrace{e^{\tilde{u}\phi}}_{\text{matrix exponential}}$?

let's do Taylor Expansion :

$$e^{\tilde{u}\phi} = I + \tilde{u}\phi + \frac{\phi^2}{2!}\tilde{u}^2 + \frac{\phi^3}{3!}\tilde{u}^3 \dots$$

$\tilde{u}^2 = \hat{u}\hat{u}^T - I$

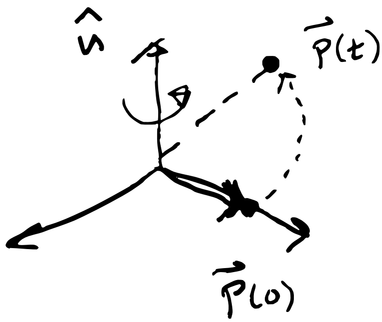
$\tilde{u}^3 = -\tilde{u}$

$$e^{\tilde{u}\phi} = I + \underbrace{\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \cdots\right)}_{\sin\phi}\tilde{u} + \underbrace{\left(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \frac{\phi^6}{6!} \cdots\right)}_{1-\cos\phi}\tilde{u}^2$$

$$e^{\tilde{u}\phi} = I + \tilde{u}\sin\phi + \tilde{u}^2(1-\cos\phi)$$

Rodigues Formula!

$$\boxed{e^{\tilde{u}\phi} = R(\hat{u}, \phi)}$$ * ← Wow!



if we rotate $\vec{p}$ at a <u>constant unit velocity</u> about $\hat{u}$

then we can described the velocity of $\vec{p}$ as:

$$\dot{\vec{p}}(t) = \hat{u} \times \vec{p}(t) = \tilde{u}\,\vec{p}(t)$$

matrix $\rightarrow$ $\dot{x} = Ax \longrightarrow x(t) = e^{At}x(0)$

$$\left\{ \begin{aligned} \frac{dx}{dt} &= Ax \\ \int \frac{dx}{x} &= \int A\,dt \\ \ln|x| &= At \\ x(t) &= e^{At} \end{aligned} \right.$$

$$\vec{p}(t) = e^{\tilde{u}t}\,\vec{p}(0)$$

$\tilde{u}$ is skew sym.

Therefore: $e^{\tilde{u}\phi}\vec{p}(0)$ $\rightarrow$ rotates vector $\vec{p}$ about axis $\hat{u}$ by angle ϕ ; is exactly equivalent to the axis-angle parameterization.

We revisit when we get to velocity

Quaternions:

* William Hamilton 1843
* extension of complex numbers

Def:

$$q = \underbrace{q_0}_{\text{pure scalar}} + \underbrace{\vec{q}}_{\text{vector}} = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$$

$$= q_0 + q_1 i + q_2 j + q_3 k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

mathematical def.

Properties:

① addition:

$$q + p = (q_0 + \vec{q}) + (p_0 + \vec{p})$$
$$= (q_0 + p_0)$$
$$+ (q_1 + p_1)\hat{i}$$
$$+ (q_2 + p_2)\hat{j}$$
$$+ (q_3 + p_3)\hat{k}$$

② multiplication:

$$qp = q \circ p$$
$$= (q_0 + \vec{q})(p_0 + \vec{p}_0)$$
$$= (q_0 p_0) + q_0 \vec{p} + p_0 \vec{q} + \underbrace{\vec{q}\,\vec{p}}_{\vec{q} \times \vec{p} - \vec{q} \cdot \vec{p}}$$

③ conjugate:

$$q^* = q_0 - \vec{q}$$
$$\therefore q q^* = (q_0 + \vec{q})(q_0 - \vec{q})$$
$$= |q|$$

(4) <u>Inverse</u> : $q^{-1} = \frac{1}{q} = \frac{q^*}{|q|^2}$

$\boxed{\begin{array}{l} * \text{ if } |q| = 1 \quad (\underline{\text{unit}} \text{ quaternion}) \\ q^{-1} = q^* \end{array}}$

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let $e(\hat{u}, \phi)$ be a unit quaternion

$$|e(\hat{u}, \phi)| = 1$$

$$e(\hat{u}, \phi) = e_0 + \vec{e}$$
$$= \cos\frac{\phi}{2} + \sin\frac{\phi}{2}\,\hat{u}$$

* We can use the axis-angle to construct rotation / orientation quaternions.

Consider a vector $\vec{r}$, we can construct a "pure vector quaternion"

$$r = 0 + \vec{r}$$

$$r' = \underbrace{e(\phi, \hat{u})\, r\, e^{*}(\phi, \hat{u})}_{\text{all quaternions}}$$

→ drop the scalar part to recover $\vec{r}\,'$.

→ $\vec{r}\,' = R(\phi, \hat{u})\, \vec{r}$

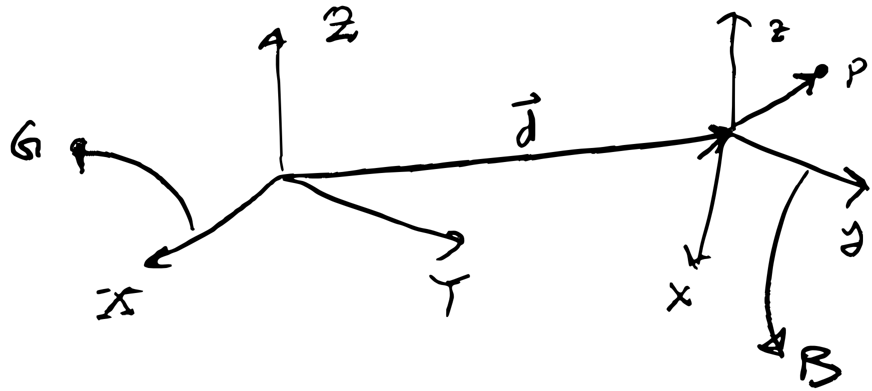
Who cares?

- almost <u>always</u> used in graphics & simulation
- interpolation of rotation matrices have numerical issues
  → not the case for quaternions.

→ down side: not very intuitive.

Motion Kinematics

JZ. Ch. 4
Lip. Ch. 3.3



How do you express point P in the global frame?

$$ {}^{G}\vec{r}_p = \,? $$

$$ = \underbrace{{}^{G}R_B \, {}^{B}\vec{r}_p + {}^{G}\vec{d}}_{\text{clunky}} \quad \checkmark $$

Conjecture: $\quad {}^{G}\vec{r}_p = {}^{G}T_B \, {}^{B}\vec{r}_p$

How?

Homogeneous Transformation Matrix

$$^{G}\vec{r}_p = {}^{G}T_B \, {}^{B}\vec{r}_p$$

HT-matrix

$$^{G}T_B = \left[\begin{array}{c|c} ^{G}R_B & ^{G}\vec{d} \\ \hline 0 \; 0 \; 0 & 1 \end{array}\right]$$

Transformation matrix

the one here makes it Homogeneous

4 x 4

Homogeneous coordinates

$$^{G}\vec{r}_p = \left[\begin{array}{c} X \\ Y \\ Z \\ \hline 1 \end{array}\right]$$

HT - matrix

① rotate
② translate → robotics

③ reflect
④ sheer
⑤ scale → graphics

"Special Eucledian" Group

$$SE(3): \quad T \in \mathbb{R}^3 \times SO(3)$$

Pure transformations

$$T = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} I & \vec{d} \\ 0 & 1 \end{bmatrix}$$

<u>Properties of HT-Matrices</u>

<u>Inverse</u>:

$$^{G}T_{B} = \underbrace{\begin{bmatrix} I & {}^{G}\vec{d} \\ 0 & 1 \end{bmatrix}}_{\text{translate}} \underbrace{\begin{bmatrix} {}^{G}R_{B} & 0 \\ 0 & 1 \end{bmatrix}}_{\text{rotate}}$$

$$^{B}T_{G} = {}^{G}T_{B}^{-1} = \begin{bmatrix} {}^{G}R_{B} & {}^{G}\vec{d} \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} {}^{G}R_{B}^{T} & -{}^{G}R_{B}^{T}\,{}^{G}\vec{d} \\ 0 & 1 \end{bmatrix}$$

Not orthogonal $\quad {}^{G}T_{B}^{-1} \neq {}^{G}T_{B}^{T}$

Why?

$$ {}^{G}\vec{r}_p = {}^{G}T_B \; {}^{B}\vec{r}_p $$

$$ {}^{G}\vec{r}_p = {}^{G}R_B \; {}^{B}\vec{r}_p + {}^{G}\vec{d} $$

solve for this

$$ {}^{G}\vec{r}_p - {}^{G}\vec{d} = {}^{G}R_B \; {}^{B}\vec{r}_p $$

$$ \underbrace{{}^{G}R_B^T}_{\text{rotation}} \; {}^{G}\vec{r}_p - \underbrace{{}^{G}R_B^T \; {}^{G}\vec{d}}_{\text{translation}} = {}^{B}\vec{r}_p $$

$$ {}^{B}\vec{r}_p = \begin{bmatrix} {}^{G}R_B^T & -{}^{G}R_B^T \; {}^{G}\vec{d} \\ 0 & 1 \end{bmatrix} {}^{G}\vec{r}_p $$

$$ \left({}^{G}T_B\right)^{-1} $$

Product :

$T_1 T_2 = T$ ← also a HT-matrix!

Associative :

$(T_1 T_2) T_3 = T_1 (T_2 T_3)$

$$\boxed{T_1 T_2 \neq T_2 T_1}$$

Compositions : Same as for Rotations:

① when applied to fixed frame : pre-multiply

② when applied to moving frame : post-multiply

We use both interpretations to check :

$$T = \begin{bmatrix} R & \vec{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

Moving Frame

① translate

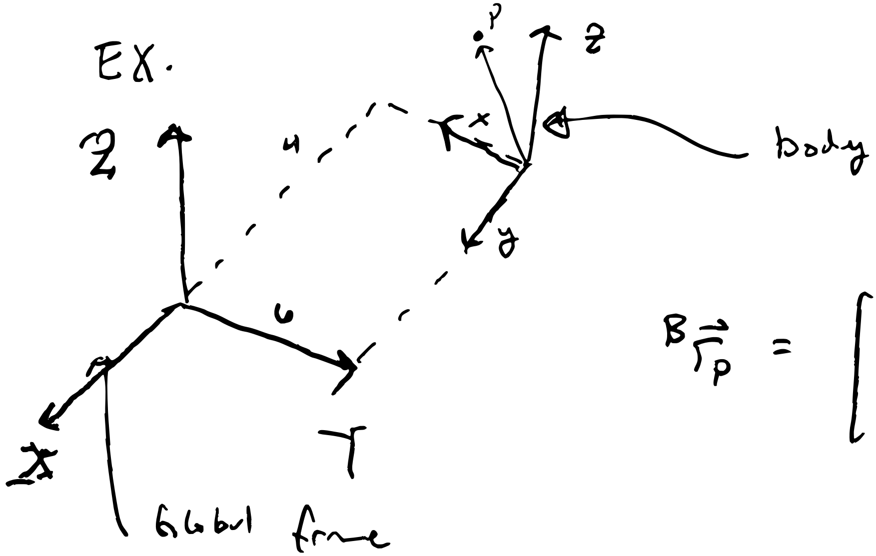
② rotate in new frame

Fixed Frame

① first rotate

② translate w/ respect fixed frame

EX.



$$ {}^{B}\vec{r}_p = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} $$

Find the coordinates of P

in the Global frame.

$$ {}^{G}T_B = \begin{bmatrix} {}^{G}R_B & {}^{G}\vec{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

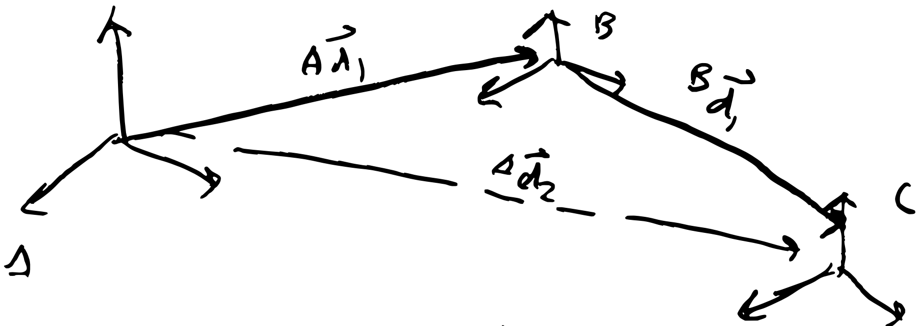
$$ {}^{G}R_B = R_z(-90) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} $$

$$ {}^{G}\vec{d} = \begin{bmatrix} -4 \\ 6 \\ 0 \end{bmatrix} $$

$$ {}^{G}T_G \begin{bmatrix} 2 \\ 0 \\ 1 \\ \hline 1 \end{bmatrix} $$

$$ = \begin{bmatrix} -4 \\ 4 \\ 1 \\ \hline 1 \end{bmatrix} $$

More on Compositions:



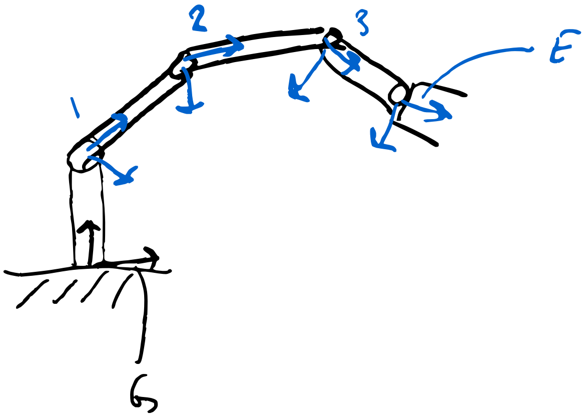
$$^AT_B = \begin{bmatrix} ^AR_B & ^A\vec{d}_1 \\ 0 & 1 \end{bmatrix}$$

$$^BT_C = \begin{bmatrix} ^BR_C & ^B\vec{d}_1 \\ 0 & 1 \end{bmatrix}$$

$$^AT_C = {^AT_{\cancel{B}}}\,{^{\cancel{B}}T_C} = {^AT_C}$$

⋮

Preview of "forward kinematics"



$${}^{G}T_{E} = {}^{G}T_{\cancel{1}}\,{}^{\cancel{1}}T_{\cancel{2}}\,{}^{\cancel{2}}T_{\cancel{3}}\,{}^{\cancel{3}}T_{E}$$