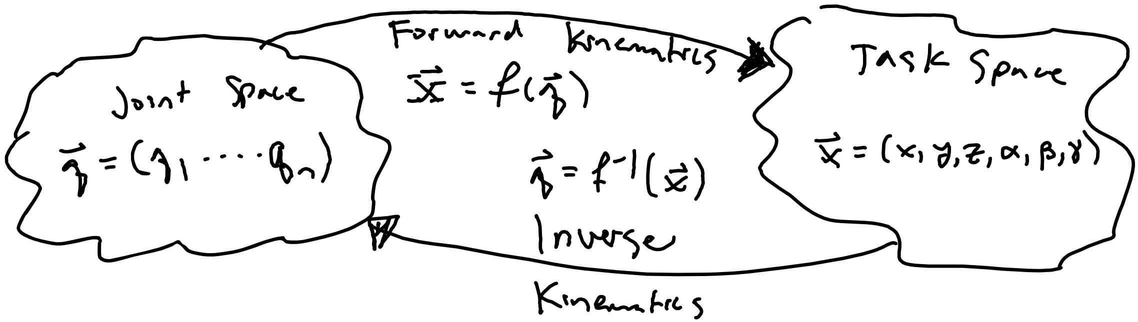
Lecture 5 ME 221 10/11/22

Last time: Forward Kinematics via DH method



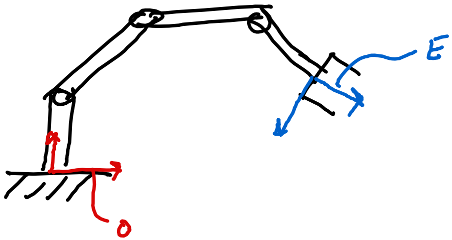
- DH-parameters are a systematic approach to assign coordinate frames to each joint:

$$\left.\begin{array}{ll} a_i : & \text{link length} \\ \alpha_i : & \text{link twist} \\ d_i : & \text{joint distance} \\ \theta_i : & \text{joint angle} \end{array}\right\} \text{completely define transformation } {}^{i-1}T_i$$

$q_i \leftarrow [d_i, \theta_i]$

$$^{i-1}T_i = f(a_i, \alpha_i, d_i, \theta_i)$$

(Hint for HW2 matlab)



$$\underbrace{^0T_E(\vec{q})}_{\vec{x} = f(\vec{q})} = {}^0T_1(q_1)\,{}^1T_2(q_2)\cdots{}^{n-1}T_E(q_n)$$

$$^0T_E = \begin{bmatrix} ^0R_E & \begin{matrix} ^0x_E \\ ^0y_E \\ ^0z_E \end{matrix} \\ 0 & 1 \end{bmatrix}$$

orientation of end-effector

position of the origin of end-effector frame

Murray Ch. 3.2
LSP Ch. 3.3 & Ch. 4

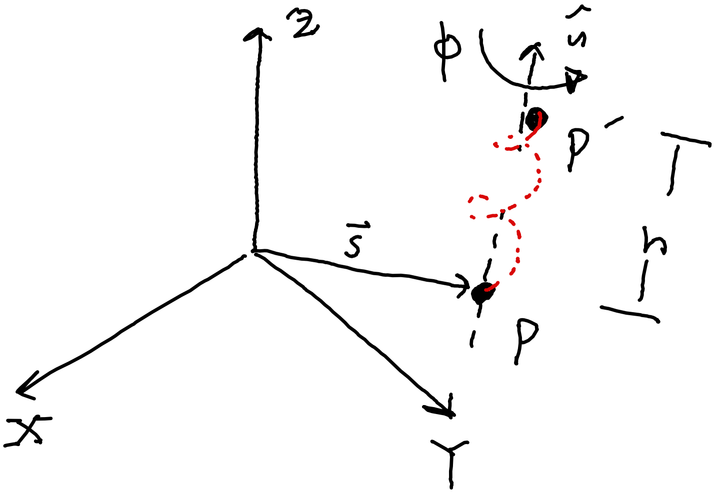
Today:
- screw coordinates
- product of exponentials $\rightarrow$ forward kinematics

Screw Coordinates: generalized approach to multi-body mechanics

Basic Idea: Chasles Theorem

Any displacement (rotation + translation) can be produced by a translation along a line followed by rotation.

Product of Exp. (PoE): a different method for forward kinematics



h: translation
ϕ: rotation
$\hat{u}$: axis of rotation
$\vec{s}$: location vector

p : pitch $= \frac{h}{\phi}$

For general screw motion:

$$\breve{S}(h, \phi, \hat{u}, \vec{s}) = \begin{bmatrix} R_{\hat{u},\phi} & \vec{s} - R_{\hat{u},\phi}\vec{s} + h\hat{u} \\ 0 & 1 \end{bmatrix}$$

$$\tilde{u} = \begin{bmatrix} 0 & u_3 & -u_1 \\ -u_3 & 0 & u_2 \\ u_1 & -u_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\tilde{u}\phi} & (I - e^{\tilde{u}\phi})\vec{s} + h\hat{u} \\ 0 & 1 \end{bmatrix}$$

Inverse Screw

$$\breve{S}^{-1}(h, \phi, \hat{u}, \vec{s}) = \begin{bmatrix} R^T_{\hat{u},\phi} & \vec{s} - R^T_{\hat{u},\phi}\vec{s} - h\hat{u} \\ 0 & 1 \end{bmatrix}$$

$$\left({}^{G}\breve{S}_{B}\right)^{-1} = {}^{B}\breve{S}_{G}$$

The screw formulation is a homogeneous Transformation matrix!

$${}^{A}T_{B} = {}^{A}\breve{S}_{B} \Rightarrow {}^{B}T_{A} = \left({}^{A}\breve{S}_{B}\right)^{-1} = {}^{B}\breve{S}_{A}$$

Would be nice to represent <u>whole</u> transformation as a matrix exponential!

Screw Axis

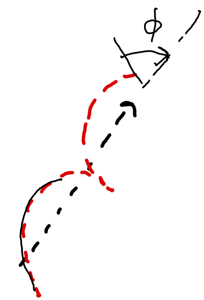
Plücker Coordinates

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \vec{\rho} + p\hat{u} \end{bmatrix}_{6\times 1} \qquad \begin{cases} \vec{\rho} = -\hat{u} \times \vec{s} \\ p = \dfrac{h}{\phi} \end{cases}$$

Matrix form of $\mathcal{S} \longrightarrow \tilde{\mathcal{S}} = \begin{bmatrix} \tilde{u} & \vec{\rho} + p\hat{u} \\ 0 & 0 \end{bmatrix}_{4\times 4}$

$$\Rightarrow \boxed{T = e^{\tilde{\mathcal{S}}\phi}}$$

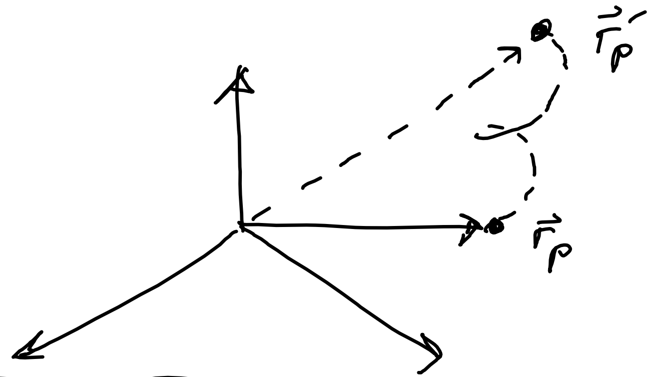
amount of displacment



$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \vec{\rho} + p\hat{u} \end{bmatrix}$$ Screw Axis

$$\tilde{\mathcal{S}} = \begin{bmatrix} \hat{u} & \vec{\rho} + p\hat{u} \\ 0 & 0 \end{bmatrix}$$ Screw Matrix

$$T = e^{\tilde{S}\phi}$$



$$\boxed{\vec{r}_p' = e^{\tilde{S}\phi}\, r_p}$$

Special case: infinite pitch (pure translation)

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \vec{\rho} + p\hat{u} \end{bmatrix}$$ Screw axis $\quad p = \frac{h}{\phi}$

$$\underbrace{\mathcal{S}\phi}_{\text{"displacement"}} \longrightarrow \mathcal{S}\phi = \begin{bmatrix} \phi\hat{u} \\ \phi\vec{\rho} + h\hat{u} \end{bmatrix} = \begin{bmatrix} \frac{h}{p}\hat{u} \\ \frac{h}{p}\vec{\rho} + h\hat{u} \end{bmatrix}$$

$$p \to \infty$$

$$\mathcal{S}\phi = \begin{bmatrix} 0 \\ h\hat{u} \end{bmatrix}$$ pure translation screw

We can use screw motion to describe **prismatic joints**!

$$\mathcal{S}q = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix} q$$

Special Case : zero pitch (pure rotation)

$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \vec{r} + p\hat{u} \end{bmatrix}$ screw axis $p = \frac{h}{\phi}$

$\vec{r} = -\hat{u} \times \vec{s}$

$\mathcal{S}\phi \longrightarrow \mathcal{S}\phi = \begin{bmatrix} \hat{u} \\ \vec{r} + p\hat{u} \end{bmatrix} \phi$

$p \to 0$

$\mathcal{S}\phi = \begin{bmatrix} \hat{u} \\ \vec{r} \end{bmatrix} \phi$ pure rotation

We will use this screw motion for revolute joints!

$\mathcal{S}\theta = \begin{bmatrix} \hat{u} \\ \vec{r} \end{bmatrix} \theta$

Product of Exponentials

Step 1 define base (global) frame, "Home" position, & end-effector frame. Find the homogeneous transformation matrix from global → End effector

$$M = {}^{0}T_E(\vec{q} = 0)$$

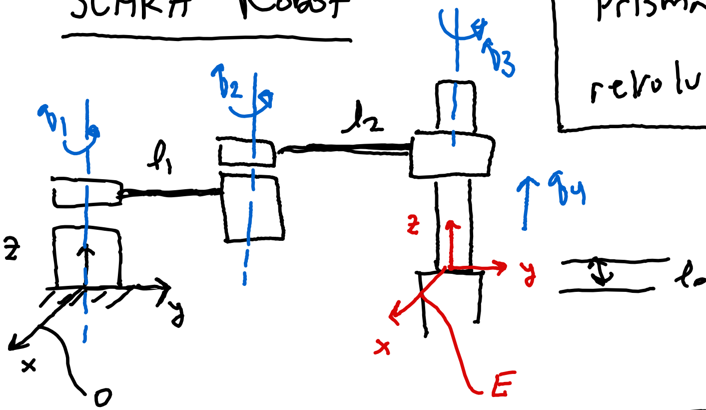
Step 2 define all axes of motion

Step 3 Find the screw axis of each joint

Step 4 Product of exponentials:

$${}^{0}T_E(\vec{q}) = e^{\tilde{S}_1 q_1} \, e^{\tilde{S}_2 q_2} \, e^{\tilde{S}_3 q_3} \cdots e^{\tilde{S}_n q_n} M$$

SCARA Robot



Prismatic $\mathcal{S} = \begin{bmatrix} 0 \\ \hat{n} \end{bmatrix}$

revolute $\mathcal{S} = \begin{bmatrix} \hat{n} \\ \vec{v} \end{bmatrix}$ $\quad \vec{v} = -\hat{n} \times \vec{s}$

$$M = {}^0T_E(\vec{q}=0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1+l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Joint 1

$\mathcal{S} = \begin{bmatrix} \hat{n} \\ \vec{v} \end{bmatrix}$

$\hat{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{s} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad -\hat{n} \times \vec{s} = 0$

$$\mathcal{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Joint 2

$\mathcal{S} = \begin{bmatrix} \hat{n} \\ \vec{v} \end{bmatrix}$

$\hat{n}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{s}_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$ $\quad -\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$

$\tilde{u} = \begin{bmatrix} 0 & u_3 & -u_1 \\ -u_3 & 0 & u_2 \\ u_1 & -u_2 & 0 \end{bmatrix}$ $\quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 1 & l_1 & 0 & 0 \end{bmatrix}^T$

Joint 3

$\mathcal{S} = \begin{bmatrix} \hat{n} \\ \vec{v} \end{bmatrix}$

$\hat{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} l_1+l_2 \\ 0 \\ 0 \end{bmatrix}$ $\quad \mathcal{S}_3 = \begin{bmatrix} 0 & 0 & 1 & l_1+l_2 & 0 & 0 \end{bmatrix}^T$

$\vec{s}_3 = \begin{bmatrix} 0 \\ l_1+l_2 \\ 0 \end{bmatrix}$

Joint 4.

$\mathcal{S} = \begin{bmatrix} 0 \\ \hat{n} \end{bmatrix}$

$\hat{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\mathcal{S}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$^{0}T_E = e^{\tilde{S}_1 \theta_1} e^{\tilde{S}_2 \theta_2} e^{\tilde{S}_3 \theta_3} e^{\tilde{S}_4 \theta_4} M$$

Some Advantages:

- ignore the in-between joints (transformations)
  → uses only two reference frames

"double edge"

$^{i-1}T_i$: we dont have expression for intermediate joints.