Lecture 7 ME 221 10/18/22

Last time: Overview of Inverse Kinematics

Problem: given an end-effector position + orientation

find the corresponding joint variables ($\vec{q}$)

$$T_d = \begin{bmatrix} {}^0R_E & {}^0\vec{d}_E \\ 0 & 1 \end{bmatrix}$$

Need to find $\vec{q}$ s.t. ${}^0T_E(\vec{q}) = T_d$

system of nonlinear equations

- We discussed a few "tips"
- focussed on numerical methods for solving IK
- The simplest "Newton Methods"

Newton Method (Root finding)

given: $\vec{x} = f(\vec{q}) \longleftrightarrow T_d = {}^{0}T_E(\vec{q})$

we want: $\vec{x} - f(\vec{q}) = 0$ (we to find the roots!)

$$f(\vec{q}) \approx f(\vec{q}_k) + \frac{\partial f(\vec{q}_k)}{\partial \vec{q}} (\vec{q} - \vec{q}_k)$$

$$\vec{q}_{k+1} = \vec{q}_k + J(\vec{q}_k)^{-1} (x_d - f(\vec{q}_k))$$

In general, we can pose the IK problem as an optimization:

$$\min_{\vec{q}} \ \| \vec{x}_d - f(\vec{q}) \| \quad \text{objective function}$$

subject to: $c_1(\vec{q}) \leq 0$] constraints

'generalized Inverse Kinematics' matlab

↳ you can specify 4-types of constraints.

Today:

- Time Derivatives & coordinate frames
- Angular Velocity Matrix
- Velocity Transformation Matrix
- Twists & wrenches

6

Time derivatives are frame dependent!

G - derivative	B - derivative
$\frac{{}^G d}{dt} \vec{r}_P$	$\frac{{}^B d}{dt} \vec{r}_P$

if $\vec{r}_P$ is in the same frame as the derivative, then unit vectors are constant.

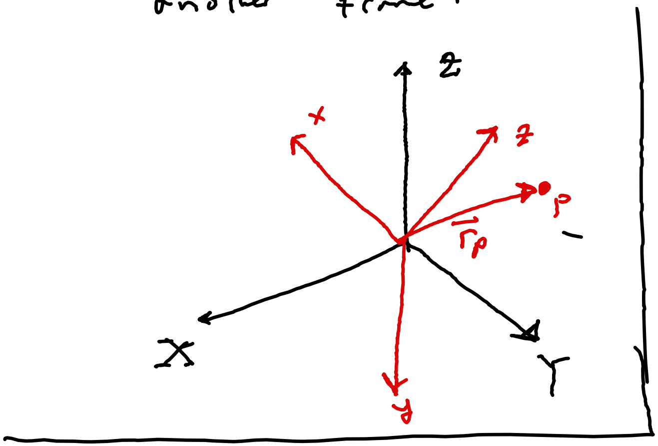
$$\frac{{}^G d}{dt} {}^G\vec{r}_P = \frac{{}^G d}{dt}\left(X\hat{I} + Y\hat{J} + Z\hat{K} \right)$$

$${}^G\dot{\vec{r}}_P = \dot{X}\hat{I} + \dot{Y}\hat{J} + \dot{Z}\hat{K}$$

$$\frac{{}^B d}{dt} {}^B\vec{r}_P = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$= {}^B\dot{\vec{r}}_P$$

We can also find time derivative with respect to another frame.



$$\frac{{}^{G}d}{dt}\,{}^{B}\vec{r}_P = \underbrace{{}^{B}\dot{\vec{r}}_P + {}^{B}_{G}\omega_B \times {}^{B}\vec{r}_P}_{{}^{B}_{G}V_P}$$

How to derive?

$${}^{B}\vec{r}_P = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$\frac{{}^{G}d}{dt}\,{}^{B}\vec{r}_P = \frac{{}^{G}d}{dt}\left(x(t)\,\hat{i}\right) + \frac{{}^{G}d}{dt}\left(y(t)\,\hat{j}\right) + \frac{{}^{G}d}{dt}\left(z(t)\,\hat{k}\right)$$

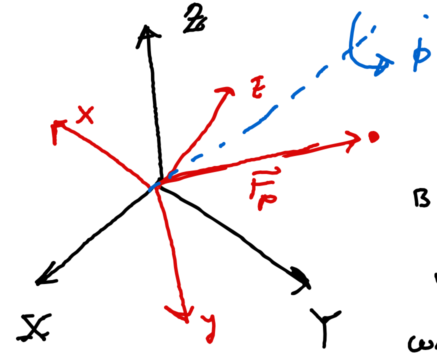
$$\frac{{}^{G}d}{dt}\left(x(t)\,\hat{i}\right) = \dot{x}(t)\,\hat{i} + x(t)\left(\frac{{}^{G}d}{dt}\,\hat{i}\right)$$

$$= \dot{x}(t)\,\hat{i} + {}^{B}_{G}\vec{\omega} \times \hat{i}$$

$$\boxed{\vec{v} = \omega \times r}$$

${}^{B}_{G}\vec{\omega}$: angular velocity of B relative to G

Angular Velocity Matrix / Vector



${}^B\vec{r}_p$

$\downarrow$

constant

$${}^G\vec{r}_p(t) = {}^G R_B(t)\, {}^B\vec{r}_p$$

What is the velocity in the global frame of point p?

$${}^G\vec{v}_p = \frac{{}^G d}{dt}\, {}^G\vec{r}_p = {}^G\dot{R}_B(t)\, {}^B\vec{r}_p$$

$${}^B\vec{r}_p = {}^G R_B(t)^T\, {}^G\vec{r}_p$$

$${}^G\vec{v}_p = \underbrace{{}^G\dot{R}_B(t)\, {}^G R_B^T(t)}_{{}_G\tilde{\omega}_B}\, {}^G\vec{r}_p$$

${}_G\tilde{\omega}_B$: skew symmetric angular velocity matrix

B w.r.t. G

$$\boxed{{}^G v_p(t) = {}_G\tilde{\omega}_B \times {}^G r(t)}$$

Why are angular Velocities skew symmetric?

<u>orthognality</u>

$${}^G R_B \, {}^G R_B^T = I$$

$$\frac{d}{dt}\left({}^G R_B \, {}^G R_B^T = I \right)$$

$${}^G \dot{R}_B \, {}^G R_B^T + {}^G R_B \, {}^G \dot{R}_B^T = 0$$

$${}^G \dot{R}_B \, {}^G R_B^T = - {}^G R_B \, {}^G \dot{R}_B^T$$

$$\boxed{\left({}^G R_B \, {}^G \dot{R}_B^T \right)^T = - {}^G R_B \, {}^G \dot{R}_B^T}$$

$$(B^T A) = (A^T B)^T$$

$A^T = -A$ skew symmetric

$${}^G R_B^T \, {}^G R_B = I$$

$$\vdots$$

$$\left({}^G R_B^T \, {}^G \dot{R}_B \right)^T = - {}^G R_B^T \, {}^G \dot{R}_B$$

All angular velocities are skew symmetric!

The group of <u>all</u> 3 x 3 skew symmetric matrices forms the *Lie Algebra* so(3) of the Lie Group SO(3). They represent all possible "generators" of rotations, e.g. angular velocity.

$$ {}^{G}\dot{\vec{r}}_P = {}^{G}\dot{R}_B(t)\ {}^{B}r_P $$

<u>left multiply by ${}^{G}R_B^T$</u>

$$ \underbrace{{}^{G}R_B^T\ {}^{G}\dot{\vec{r}}_P}_{{}^{B}\dot{\vec{r}}_P} = \underbrace{{}^{G}R_B^T\ {}^{G}\dot{R}_B}_{{}^{B}_{G}\tilde{\omega}_B}\ {}^{B}\vec{r}_P $$

$$ {}^{B}\dot{\vec{r}}_P = {}^{B}_{G}\tilde{\omega}_B\ {}^{B}\vec{r}_P $$

${}^{B}_{G}\tilde{\omega}_B$: instaneous angular velocity of B relative to G as seen in frame B

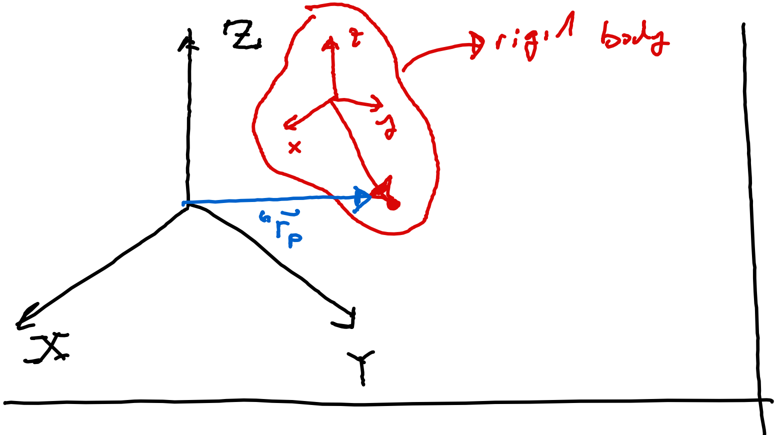
sub ${}^{B}\vec{r}_P = {}^{G}R_B^T(t)\ {}^{G}\vec{r}_P$

$$ {}^{G}\dot{\vec{r}}_P = \underbrace{{}^{G}\dot{R}_B(t)\ {}^{G}R_B^T(t)}_{{}^{G}_{G}\tilde{\omega}_B}\ {}^{G}\vec{r}_P $$

${}^{G}_{G}\tilde{\omega}_B$: instaneous angular velocity of B relative to G as seen by G.

So far our discussion has been restricted to pure rotations.

lets now consider ~~free~~ motion (rigid body motion)



$$ {}^G\vec{r}_P(t) = {}^G R_B(t)\, {}^B\vec{r}_P + {}^G d_B(t) $$

What is the velocity of point P in the global frame?

$$ {}^G\vec{V}_P = {}^G\dot{R}_B(t)\, {}^B\vec{r}_P + {}^G\dot{d}_B $$

$$ {}^B\vec{r}_P = {}^G R_B^T(t) \left[{}^G\vec{r}_P(t) - {}^G d_B(t) \right] $$

$$ \boxed{ {}^G\vec{V}_P(t) = {}_G\tilde{\omega}_B \left[{}^G\vec{r}_P(t) - {}^G d_B(t) \right] + {}^G\dot{d}_B } $$

$${}^{G}\vec{v}_P(t) = {}_G\tilde{\omega}_B \left[{}^{G}\vec{r}_P(t) - {}^{G}d_B(t) \right] + {}^{G}\dot{d}_B$$

$$\left({}^{G}\dot{T}_B \; {}^{G}T_B^{-1} \right) = \;?$$

$${}^{G}\dot{T}_B = \frac{d}{dt} \begin{bmatrix} {}^{G}R_B & {}^{G}d_B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{G}\dot{R}_B & {}^{G}\dot{d}_B \\ 0 & 0 \end{bmatrix}$$

$${}^{G}T_B^{-1} = \begin{bmatrix} {}^{G}R_B^T & -{}^{G}R_B^T \, {}^{G}d \\ 0 & 1 \end{bmatrix}$$

$${}^{G}\dot{T}_B \, {}^{G}T_B^{-1} = \begin{bmatrix} {}^{G}\dot{R}_B \, {}^{G}R_B^T & -{}^{G}\dot{R}_B \, {}^{G}R_T^T \, {}^{G}\vec{d} + {}^{G}\dot{\vec{d}} \\ 0 & 0 \end{bmatrix}$$

$$ {}^G\dot{T}_B\, {}^G T_B^{-1} = \begin{bmatrix} {}^G\dot{R}_B\, {}^G R_B^T & -{}^G\dot{R}_B\, {}^G R_T^T\, {}^G\vec{d} + {}^G\dot{\vec{d}} \\ 0 & 0 \end{bmatrix} $$

$$ = \begin{bmatrix} {}^G\tilde{\omega}_B & {}^G\dot{d}_B - {}^G\tilde{\omega}_B\, {}^G d_B \\ 0 & 0 \end{bmatrix} $$

$$ {}^G\vec{V}_B = \begin{bmatrix} {}^G\tilde{\omega}_B & {}^G\vec{v}_B \\ 0 & 0 \end{bmatrix} $$

same form

Screw Matrix

$$ \therefore \boxed{{}^G\vec{v}_P(t) = {}^G\vec{V}_B\, {}^G\vec{r}_P} $$

$$ {}^G V_B = {}^G\dot{T}_B\, {}^G T_B^{-1} $$

$$
{}^{G}V_B = \begin{array}{cc} 3\times 3 & 3\times 1 \\ \left[\begin{array}{c} {}^{G}\tilde{\omega}_B \\ 0 \end{array}\right. & \left.\begin{array}{c} {}^{G}\vec{v}_B \\ 0 \end{array}\right] \end{array}
$$

as vector

$$
{}^{G}\mathcal{V}_B = \begin{bmatrix} {}^{G}\vec{\omega}_B \\ {}^{G}\vec{v}_B \end{bmatrix}
$$

<u>twist</u>

* The group of all <u>twist</u> form a "Lie Algebra" se(3) of the Lie Group SE(3). They represent <u>all</u> infinitesimal generators of <u>screw motion</u>.

Wrenches

it is convenient to "stack" torque & forces onto a single vector:

$$\mathcal{F} = \begin{bmatrix} \vec{\tau} \\ \vec{f} \end{bmatrix} \begin{matrix} 3\times1 \\ 3\times1 \end{matrix}$$

Notes Cards

① Do you feel like you're learning?

② What do like about the course?
(What helps you learn).

③ What do no like about the course?
(What is inhibiting learning)

④ What would you change about the course?

→ slides instead of bad writing

→ homework is too hard.