Last time: Angular Velocity

- angular velocity is always skew symmetric because of the orthogonality condition. Easy to see when considering pure rotation:

$$ {}^G\dot{\boldsymbol{r}}_p(t) = {}^G\dot{R}_B(t)\ {}^B\boldsymbol{r}_p $$

$$ \underbrace{{}^G R_B^T(t)\,{}^G\dot{\boldsymbol{r}}_p(t)}_{{}^B\dot{\boldsymbol{r}}_p} = \underbrace{{}^G R_B^T(t)\,{}^G\dot{R}_B(t)}_{{}^B_G\tilde{\omega}_B}\ {}^B\boldsymbol{r}_p $$

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- More conveniently we can use velocity transformation matrix:

$$ {}^G\boldsymbol{v}_p(t) = {}^G V_B\ {}^G\boldsymbol{r}_p, \quad {}^G V_B = \begin{bmatrix} {}^G\tilde{\omega}_B & {}^G\boldsymbol{v}_B \\ 0 & 0 \end{bmatrix} = {}^G\tilde{\nu}_B, \quad {}^G\boldsymbol{\nu}_B = \begin{bmatrix} {}^G\boldsymbol{\omega}_B \\ {}^G\boldsymbol{v}_B \end{bmatrix} \quad \text{(twist)}. $$

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- The velocity transformation matrix is related to the derivative of the homogeneous transformation matrix

$$ {}^G\dot{T}_B = {}^GV_B\ {}^GT_B $$

Interpretation of angular velocity matrix

$$_G\tilde{\omega}_B = {}^G\dot{R}_B \, {}^GR_B^T$$

The matrix $_G\tilde{\omega}_B$ is associated with the angular velocity vector $_G\boldsymbol{\omega}_B = \hat{u}\dot{\phi}$, which is equal to an angular rate $\dot{\phi}$ about the instantaneous axis of rotation $\hat{u}$. In general, derivatives of rotation matrix parameterization (Euler angles or roll, pitch, yaw) are not equivalent to the angular velocity vector:

$$\frac{d}{dt}\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix} \neq \boldsymbol{\omega}$$

Midterm Review

- Basics
  - Degrees-of-Freedom
  - Joints/Links
  - Configuration/Task space
- Rotation Matrices and Orientation
- Homogeneous Transformation Matrices
- Forward Kinematics
  - Denavit–Hartenberg parameters
  - Product of Exponentials
- Inverse Kinematics
- Angular Velocity

Rotation Matrices

Three *basic* rotation matrices:

$$R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_y(\phi) = \cdots, \quad R_x(\phi) = \cdots$$

Can you derive a rotation matrix from first principles?

Rotation Matrices

Properties:

1. Orthogonality (Orthonormal)

- Unit norm

$$R = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \boldsymbol{r}_3 \end{bmatrix}$$

$$||\boldsymbol{r}_1|| = ||\boldsymbol{r}_2|| = ||\boldsymbol{r}_3|| = 1$$

- Orthogonal

$$\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = 0, \quad \boldsymbol{r}_1 \cdot \boldsymbol{r}_3 = 0, \quad \boldsymbol{r}_2 \cdot \boldsymbol{r}_3 = 0$$

The is means:

$$RR^T = R^T R = I$$

2. Determinant $= +1$ (Rotations preserve volume and orientation)

$$\det(R) = +1$$

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The is means:

$$RR^T = R^T R = I$$

2. Determinant $= +1$ (Rotations preserve volume and orientation)

$$\det(R) = +1$$

Rotation Matrices

Three main uses:

- Represent orientation
- Change the reference frame which a vector (rigid body) is represented in
- rotate a vector (rigid body)

Rotation Matrices

Compositions:

- Moving frame $\rightarrow$ post-multiply
- Fixed frame $\rightarrow$ pre-multiply

Rotation Matrices

Main Parameterizations:

- Euler Angles (moving frame)
- Roll, Pitch, Yaw (fixed frame)
- Axis/Angel ($\hat{u}/\phi$)
- Quaternion
- Matrix exponential ($e^{\tilde{u}\phi}$)

With each method, you can go back and forth between a rotation matrix and the underlying parameters (e.g., $R \leftrightarrow \alpha, \beta, \gamma$ or $R \leftrightarrow \hat{u}, \phi$).

Rotation Matrices: Examples

The following rotation matrix is applied to a reference frame that is initial coincident with a fixed global frame:

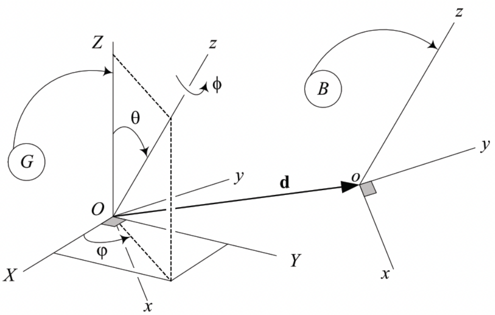
$$R = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

1. Draw the global frame and the final frame after the rotation.
2. Determine the x-y-z Euler angles that generates the rotation R.

Homogeneous Transformation Matrices

$$^{G}T_{B} = \begin{bmatrix} ^{G}R_{B} & \boldsymbol{d} \\ 0 & 1 \end{bmatrix}$$

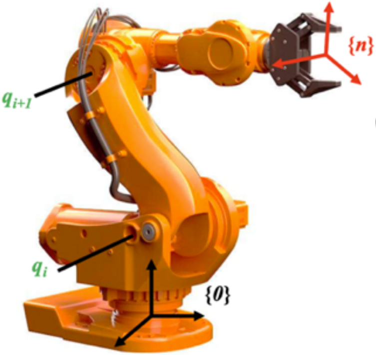
$$^{B}T_{G} = {^{G}T_{B}^{-1}} = \begin{bmatrix} ^{G}R_{B}^{T} & -{^{G}R_{B}^{T}}\boldsymbol{d} \\ 0 & 1 \end{bmatrix}$$

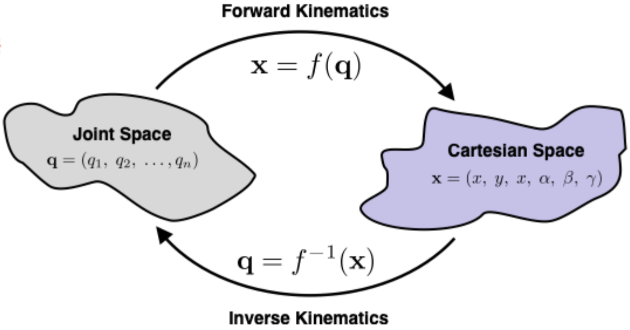


All same composition rules apply to Homogeneous Transformations.

Forward Kinematics

Big Picture:



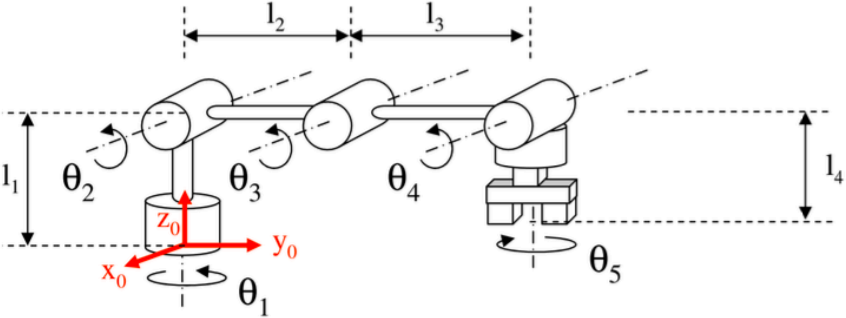


Forward Kinematics: DH-parameters

1. Assign all z-axes (every degree of freedom is along z_i)
2. Assign frame origins
3. Find the *Joint* and *Link* parameters
4. Generate transformations

$$^{i-1}T_i = D_{z_{i-1},d_i} R_{z-1,\theta_i} D_{x_{i-1},a_i} R_{x_{i-1},\alpha_i}$$

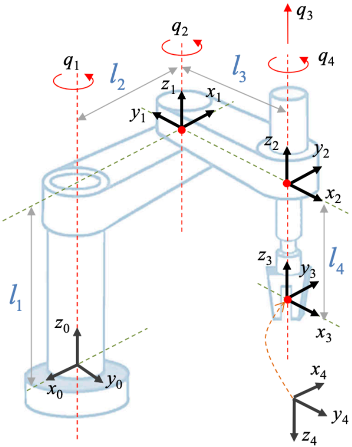
DH-Example



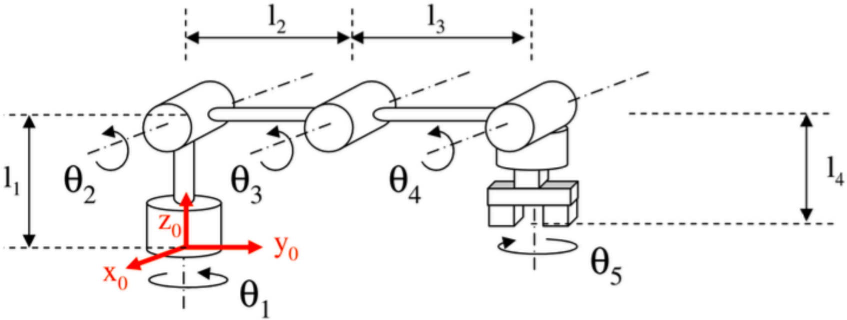
a. Assign reference frames according to DH-method
b. Find the DH-table

DH-Example - Frame assignment rules

- assign axis x_i in the direction of $z_{i-1} \times z_i$. If they are parallel assign along common normal between z_{i-1} & z_i
- assign axis y_i to complete the frame following right hand rule
- tool frame (end effector frame)
  - $x_n$orthogonal to z_{n-1}
  - z_n pointing outwards

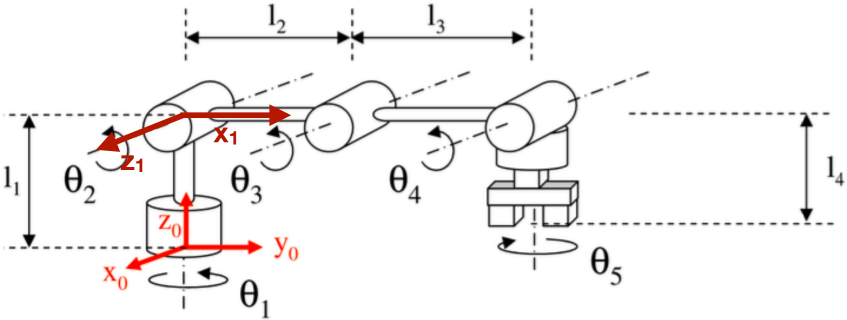


DH-Example



a. Assign reference frames according to DH-method (solution)

DH-Example



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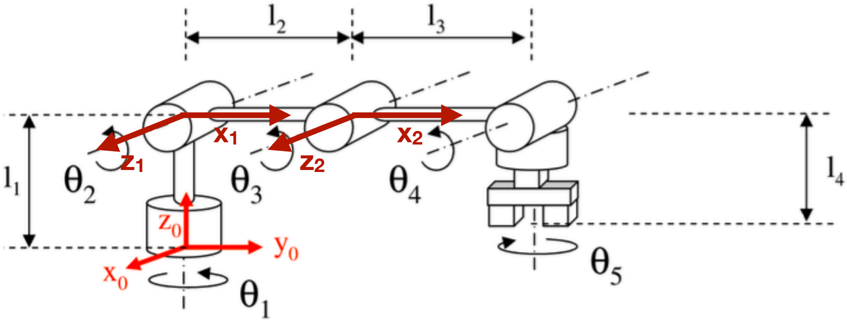
DH-Example



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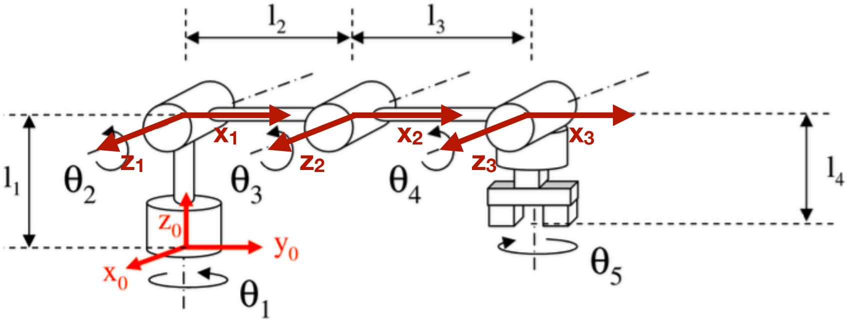
DH-Example



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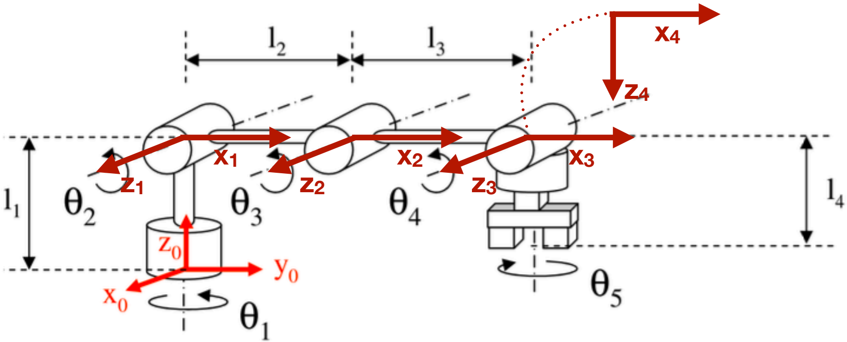
DH-Example



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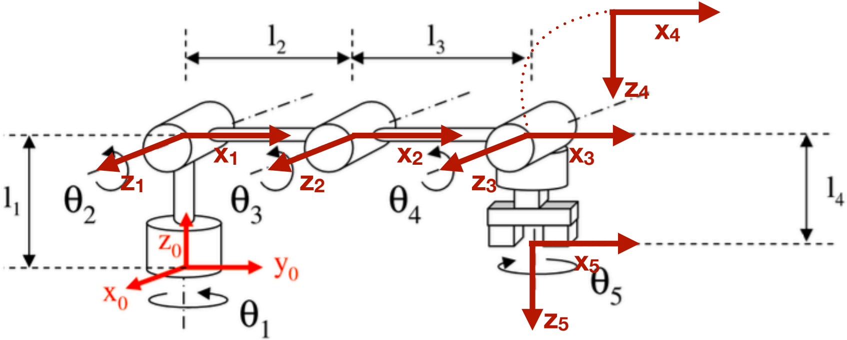
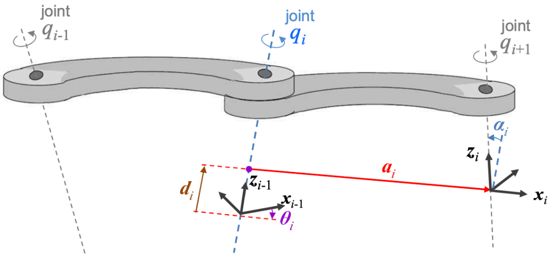


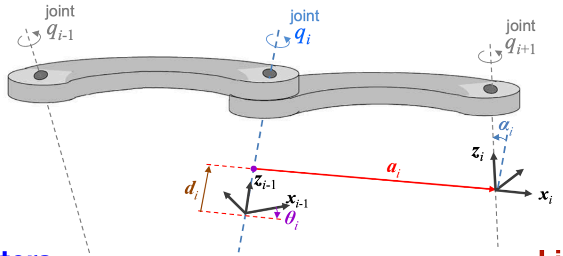
a. Assign reference frames according to DH-method (solution)

DH-Example - Parameter Rules



- d_i: distance from the origin of {i-1} to the intersection of z_{i-1} with x_i along z_{i-1}
- θ_i: rotation angle from x_{i-1} with x_i about z_{i-1}
- a_i: distance from the intersection of z_{i-1} with x_i along x_i
- α_i: angle from z_{i-1} with z_i about x_i

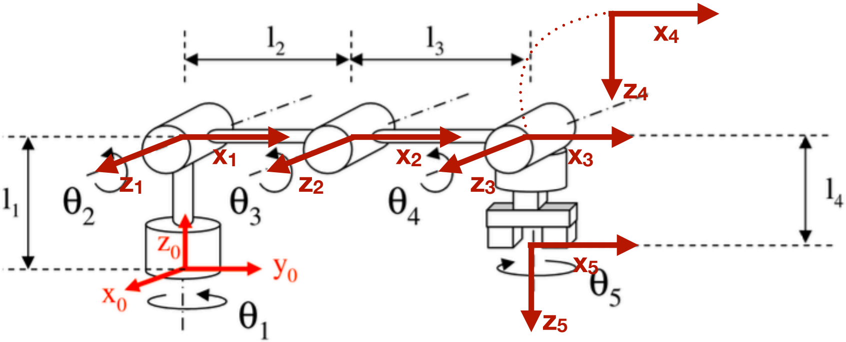
DH-Example - Parameter Rules



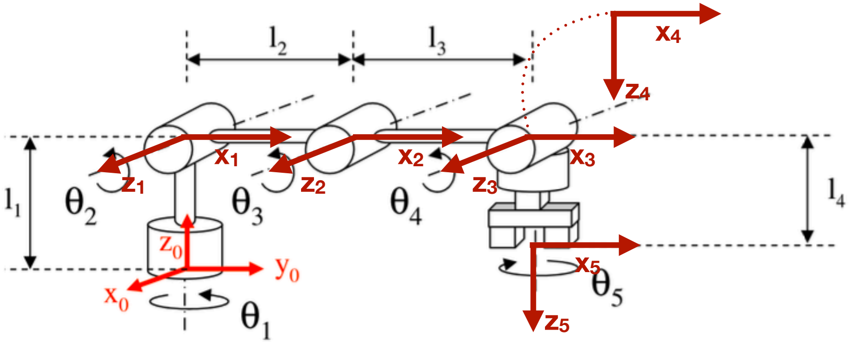
Joint Parameters

Link Parameters

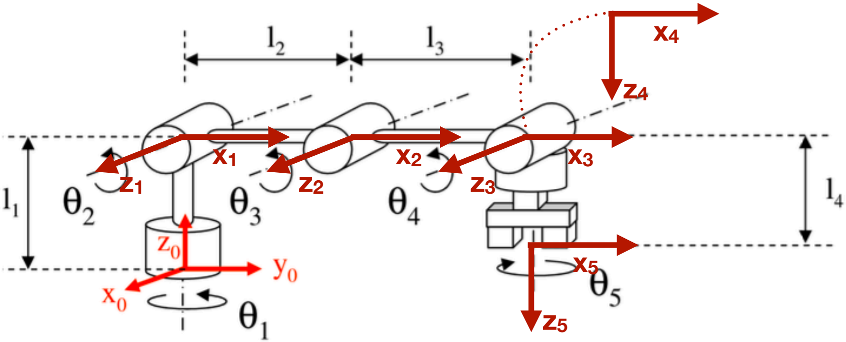
- d_i: distance from the origin of {i-1} to the intersection of z_{i-1} with x_i along z_{i-1}
- θ_i: rotation angle from x_{i-1} with x_i about z_{i-1}
- a_i: distance from the intersection of z_{i-1} with x_i along x_i
- α_i: angle from z_{i-1} with z_i about x_i



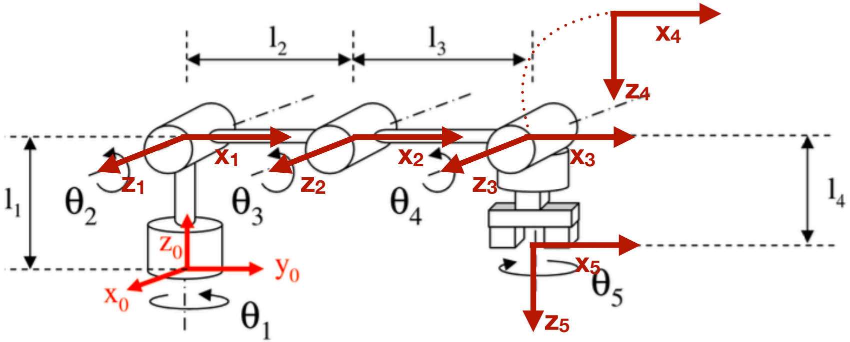
Joint i	d_i	theta_i	a_i	alpha_i
1				
2				
3				
4				
5				



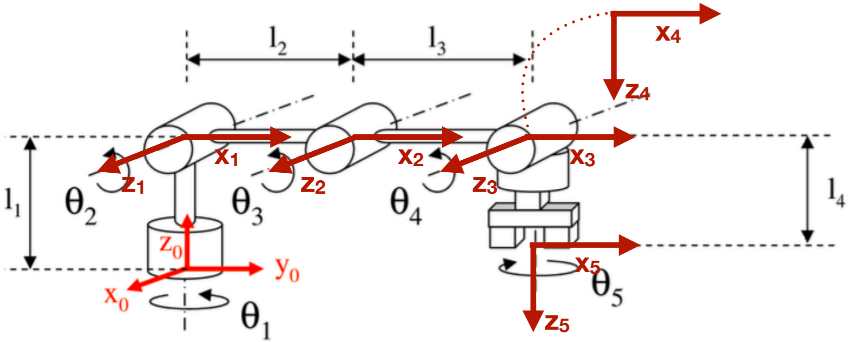
Joint i	d_i	theta_i	a_i	alpha_i
1	l1	90+q1	0	90
2				
3				
4				
5				



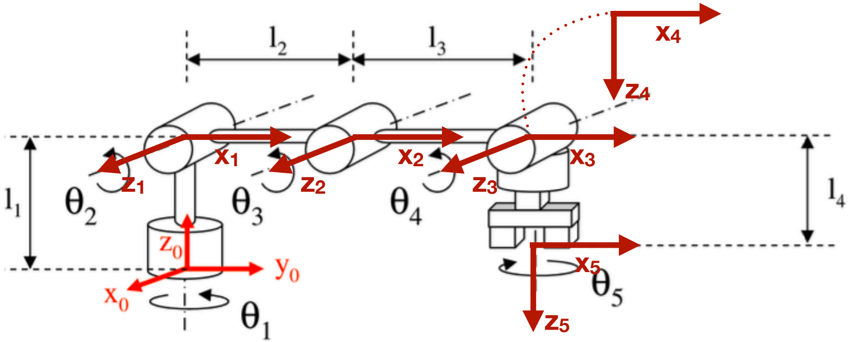
Joint i	d_i	theta_i	a_i	alpha_i
1	l1	90+q1	0	90
2	0	q2	l2	0
3				
4				
5				



Joint i	d_i	theta_i	a_i	alpha_i
1	l1	90+q1	0	90
2	0	q2	l2	0
3	0	q3	l3	0
4				
5				

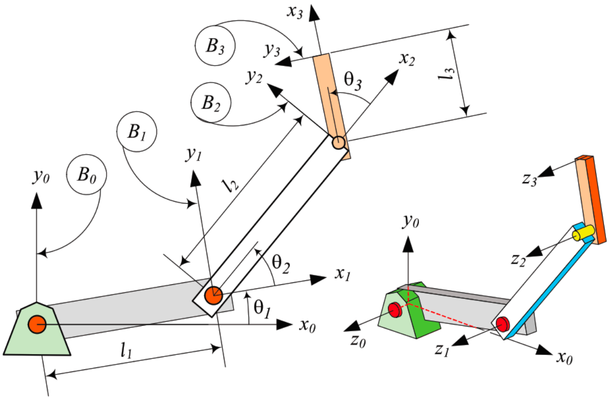


Joint i	d_i	theta_i	a_i	alpha_i
1	l1	90+q1	0	90
2	0	q2	l2	0
3	0	q3	l3	0
4	0	q4	0	90
5				



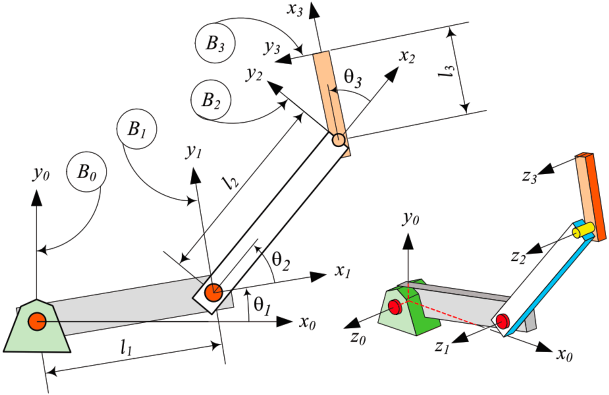
Joint i	d_i	theta_i	a_i	alpha_i
1	l1	90+q1	0	90
2	0	q2	l2	0
3	0	q3	l3	0
4	0	q4	0	90
5	l4	q5	0	0

Ex. 135 pg. 238



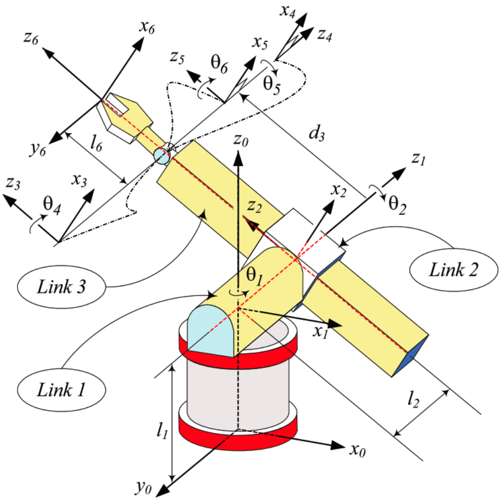
x_3
B_3
y_3
x_2
θ_3
l_3
y_2
B_2
B_1
y_1
l_2
y_0
B_0
θ_2
x_1
θ_1
x_0
l_1
z_3
y_0
z_2
z_0
z_1
x_0

Ex. 135 pg. 238



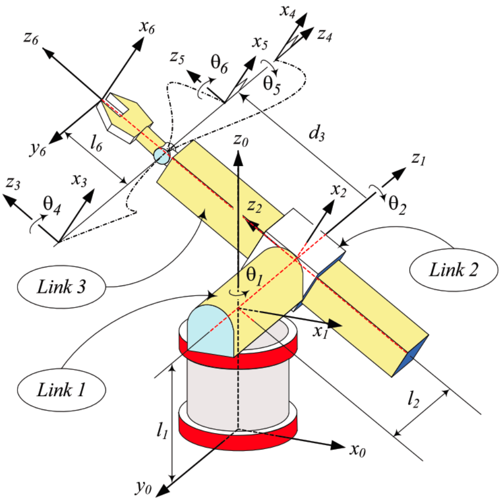
Frame No.	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2
3	l_3	0	0	θ_3

Ex.137 pg. 239



x4
z4
z6
x6
x5
z5
θ6
θ5
d3
z0
z1
y6
l6
x2
θ2
z3
x3
θ4
z2
Link 2
Link 3
θ1
x1
Link 1
l2
l1
x0
y0

Ex.137 pg. 239



Frame No.	a_i	α_i	d_i	θ_i
1	0	-90 deg	l_1	θ_1
2	0	90 deg	l_2	θ_2
3	0	0	d_3	0
4	0	-90 deg	0	θ_4
5	0	90 deg	0	θ_5
6	0	0	l_6	θ_6

Forward Kinematics: Product of Exponentials

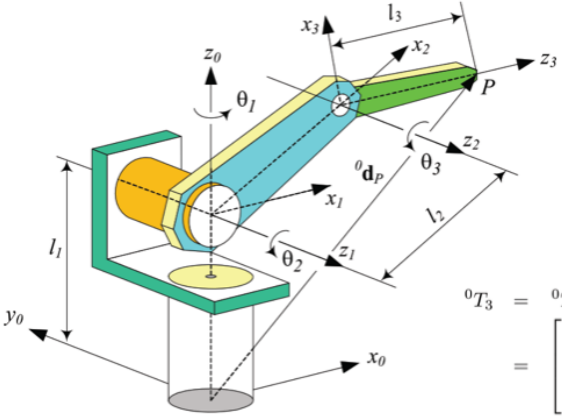
1. Define global and end-effector frames.
2. Find M, the homogeneous transformations matrix from the global frame to the base frame.
3. Define all screw axis.
4. Apply product of exponential

$$^{0}T_E = e^{\tilde{\mathcal{S}}_1 q_1} e^{\tilde{\mathcal{S}}_2 q_2} \cdots e^{\tilde{\mathcal{S}}_n q_n} M$$

Inverse Kinematics

- Multiplicity of solutions vs redundancy
- How to choose which solution?
- Basic ideas of numerical solutions
  - root finding
  - general optimization problem

Inverse Kinematics: Example (Jazar Ex.182 pg.328)



Find an expression for the x,y,z tip of the end-effector?

What about the orientation of the tip?

How would you solve the IK problem?

$$
\begin{aligned}
{}^0T_3 &= {}^0T_1\,{}^1T_2\,{}^2T_3 \\
&= \begin{bmatrix} c\theta_1 c(\theta_2+\theta_3) & s\theta_1 & c\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_1 c\theta_2 \\ s\theta_1 c(\theta_2+\theta_3) & -c\theta_1 & s\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_2 s\theta_1 \\ s(\theta_2+\theta_3) & 0 & -c(\theta_2+\theta_3) & l_1 + l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Angular Velocity: Example

Consider a rotation matrix composted using Euler angles:

$$^{B}R_{G} = R_z(\psi)R_x(\theta)R_z(\rho)$$

Find $^{B}\tilde{\omega}_{G}$