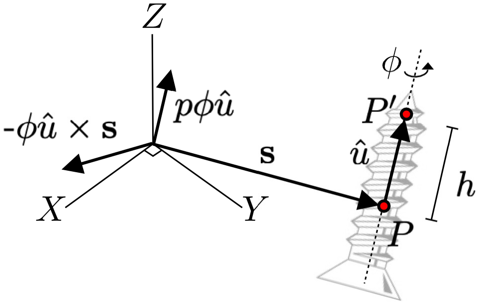
screw motion



h : translation

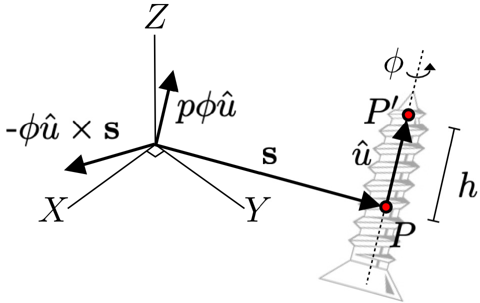
ϕ : rotation

$\hat{u}$: axis of motion

$\mathbf{s}$: location vector

p : pitch $= \dfrac{h}{\phi}$

screw motion as a homogeneous transformation



For general screw motion:

$$\breve{s}(h, \phi, \hat{u}, \mathbf{s}) = \begin{bmatrix} R_{\hat{u},\phi} & \mathbf{s} - R_{\hat{u},\phi}\mathbf{s} + h\hat{u} \\ \mathbf{0} & 1 \end{bmatrix}$$

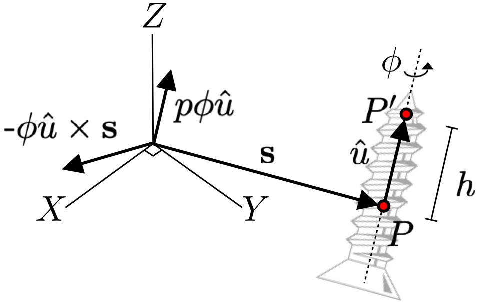
$$= \begin{bmatrix} e^{\tilde{u}\phi} & (I - e^{\tilde{u}\phi})\mathbf{s} + h\hat{u} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\tilde{u}$ is the skew symmetric representation:

$$\tilde{u} = \begin{bmatrix} 0 & u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

It would be convenient to write entire transformation as a matrix exponential.

screw axis and matrix exponential form



Define screw axis using **Plücker Coordinates**:

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} + p\hat{u} \end{bmatrix}, \quad \boldsymbol{\rho} = -\hat{u} \times \mathbf{s}$$

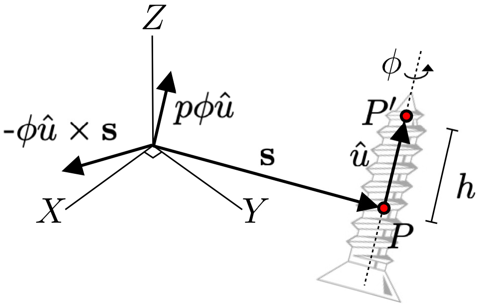
$$p = \frac{h}{\phi}$$

in "skew symmetric form":

$$\tilde{\mathcal{S}} = \begin{bmatrix} \tilde{u} & \boldsymbol{\rho} + p\hat{u} \\ 0 & 0 \end{bmatrix}$$

Then, we can write the transformation $T = e^{\tilde{\mathcal{S}}\phi}$.
So the point P' in the illustration results from the screw motion: $P' = e^{\tilde{\mathcal{S}}\phi} P$.

screw axis infinite pitch: prismatic joint



$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} + p\hat{u} \end{bmatrix}, \quad \boldsymbol{\rho} = -\hat{u} \times \mathbf{s}, \quad p = \frac{h}{\phi}$$

Multiply by displacement ϕ:

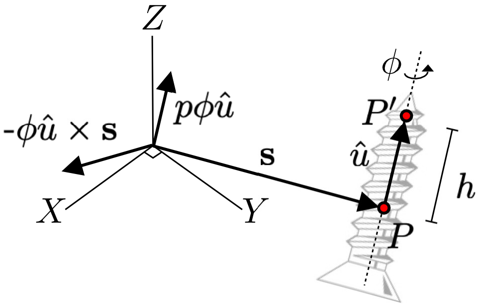
$$\mathcal{S}\phi = \begin{bmatrix} \phi\hat{u} \\ \phi\boldsymbol{\rho} + h\hat{u} \end{bmatrix} = \begin{bmatrix} \frac{h}{p}\hat{u} \\ \frac{h}{p}\boldsymbol{\rho} + h\hat{u} \end{bmatrix}$$

then $p \to \infty$

$$= \begin{bmatrix} 0 \\ h\hat{u} \end{bmatrix} \Rightarrow \mathcal{S}q = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix} q$$

Note we change variable from ϕ (which is related to h through p) to q to denote joint variable.

screw axis zero pitch: revolute joint



$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} + p\hat{u} \end{bmatrix}, \quad \boldsymbol{\rho} = -\hat{u} \times \mathbf{s}, \quad p = \frac{h}{\phi}$$

In this case it's clear with $p = 0$:

$$\mathcal{S}\phi = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} \end{bmatrix} \phi$$

$$\Rightarrow \mathcal{S}q = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} \end{bmatrix} q$$

Again we change ϕ to q to denote that it is a joint variable.

summary of screw axis for robot joints

Prismatic Joint

$$\mathcal{S} = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix}$$

Revolute Joint

$$\mathcal{S} = \begin{bmatrix} \hat{u} \\ \boldsymbol{\rho} \end{bmatrix}, \quad \boldsymbol{\rho} = -\hat{u} \times \mathbf{s}$$

In either case the transformation matrix from the screw motion can be computed:

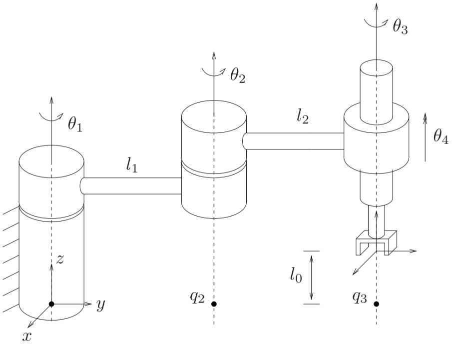
$$T = e^{\tilde{\mathcal{S}} q}$$

summary of product of exponentials for forward kinematics

- Define global fixed base frame and end-effector frame
- Find M, the homogeneous transformation matrix of the end-effector in the global frame at the **home** position $(q_i = 0)$ for all i.
- Define all joint screw axis, $\mathcal{S}_i$ in terms of the global frame
- Apply the product of exponentials formula:

$$^0T_E = e^{\tilde{\mathcal{S}}_1 q_1} e^{\tilde{\mathcal{S}}_2 q_2} \dots e^{\tilde{\mathcal{S}}_n q_n} M$$

SCARA example



End-effector transformation:

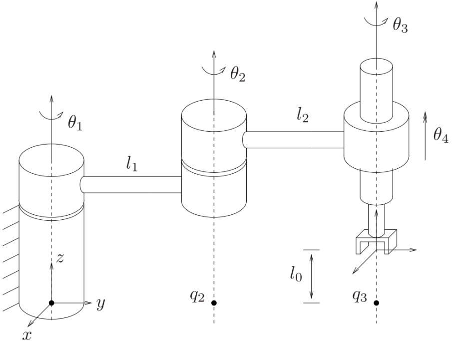
$$M = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Screw axis $\mathcal{S}_1$

$$\mathcal{S}_1 = \begin{bmatrix} \hat{u}_1 \\ -\hat{u}_1 \times \mathbf{s}_1 \end{bmatrix}, \quad \hat{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$$

SCARA example



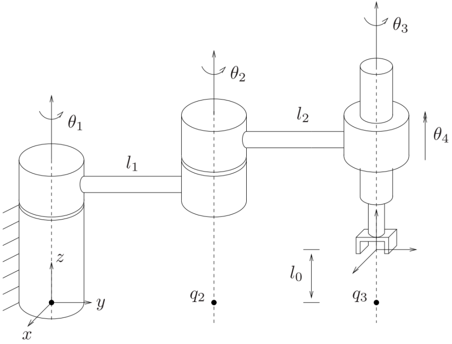
Screw axis $\mathcal{S}_2$

$$\mathcal{S}_2 = \begin{bmatrix} \hat{u}_2 \\ -\hat{u}_2 \times \mathbf{s}_2 \end{bmatrix}, \quad \hat{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ l_1 \\ 0 \\ 0 \end{bmatrix}$$

SCARA example



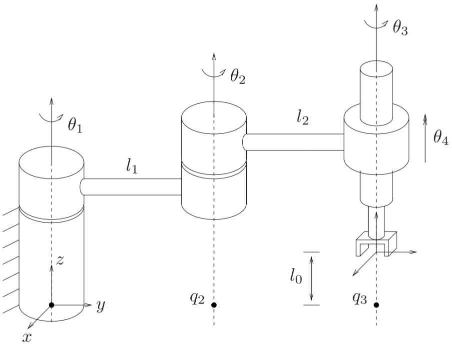
Screw axis $\mathcal{S}_3$

$$\mathcal{S}_3 = \begin{bmatrix} \hat{u}_3 \\ -\hat{u}_3 \times \mathbf{s}_3 \end{bmatrix}, \quad \hat{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

SCARA example



Screw axis $\mathcal{S}_4$

$$\mathcal{S}_4 = \begin{bmatrix} 0 \\ \hat{u} \end{bmatrix} \quad \hat{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{S}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the forward kinematics can be computed as:

$$^0T_E = e^{\tilde{\mathcal{S}}_1 q_1} e^{\tilde{\mathcal{S}}_2 q_2} e^{\tilde{\mathcal{S}}_3 q_3} e^{\tilde{\mathcal{S}}_4 q_4} M$$