EE201/MSE207 Lecture 10

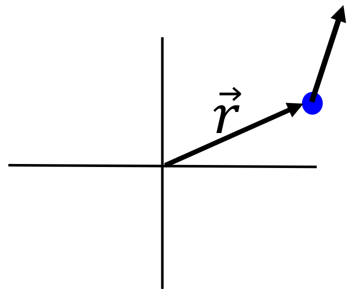
Angular momentum

(A lot of mathematics; I will try to explain as simple as possible)

Classical mechanics

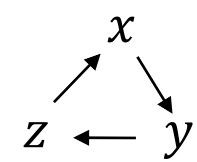
$$\vec{L} = \vec{r} \times \vec{p}$$

(angular momentum)



$$\vec{p} = m\vec{v}$$

$$L_x = y\,p_z - z\,p_y$$
$$L_y = z\,p_x - x\,p_z$$
$$L_z = x\,p_y - y\,p_x$$



Quantum mechanics

The same, but

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$
$$\hat{p}_y = -i\hbar\frac{\partial}{\partial y}$$
$$\hat{p}_z = -i\hbar\frac{\partial}{\partial z}$$

Therefore

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$
$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

To find quantized values of angular momentum, we can solve 3D differential equations to find eigenvalues and eigenstates. However, there is a simpler way, similar to the trick for oscillator.

Commutation relations

Simple to show:
(check yourself)

$$[\hat{L}_x, \hat{L}_y] = i\hbar\, \hat{L}_z$$
$$[\hat{L}_y, \hat{L}_z] = i\hbar\, \hat{L}_x$$
$$[\hat{L}_z, \hat{L}_x] = i\hbar\, \hat{L}_y$$

Most of the theory of angular momentum (and spin) can be derived from these commutation relations.

Will not write hats below for brevity

First observation: L_x, L_y, L_z are all incompatible (do not commute), so if L_x has a definite value, then L_y and L_z do not have definite values (except when all are zero, $\sigma_{Lx}\sigma_{Ly} \geq \frac{\hbar}{2}|\langle L_z\rangle|$).

However, $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with each component.

Simple to show:

$$\left.\begin{aligned} [L^2, L_x] &= 0 \\ [L^2, L_y] &= 0 \\ [L^2, L_z] &= 0 \end{aligned}\right\} \quad [L^2, \vec{L}\,] = 0$$

Therefore, L^2 (square of total angular momentum) and one component can have definite values at the same time (usually *z*-component is chosen)

Eigenvalues of L^2 and L_z

We want to find eigenvalues of operators L^2 and L_z (they are the quantized values for the angular momentum).

Need to solve: $\begin{cases} L^2 f = \lambda\, f \\ L_z f = \mu\, f \end{cases}$

(f is not very important)

Will show (next goal):

$$\lambda = \hbar^2 l(l+1)$$
$$\mu = \hbar\, m$$
$$l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$
$$m = -l, -l+1, \ldots l$$

Trick of ladder operators

Let us introduce operators $L_\pm = L_x \pm i L_y$

These operators are not Hermitian (unphysical), while L_x, L_y, L_z are of course Hermitian

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_x, L_y] = i\hbar L_y \pm i(-i\hbar)L_x = \pm\hbar L_\pm$$

Thus
$$\begin{cases} [L_z, L_\pm] = \pm\hbar\, L_\pm \\ [L^2, L_\pm] = 0 \end{cases}$$

Lemma If f is an eigenvector of both L^2 (with eigenvalue λ) and L_z (with eigenvalue μ), then $L_\pm f$ are also eigenvectors, with the same λ, but $\mu \to \mu \pm \hbar$.

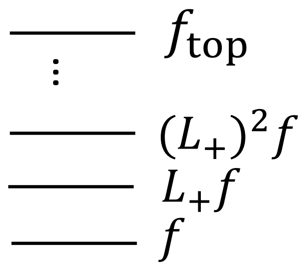
Proof

$$\begin{cases} [L_z, L_\pm] = \pm\hbar\, L_\pm \\ [L^2, L_\pm] = 0 \end{cases}$$

$$L^2(L_\pm f) = L_\pm(L^2 f) = \lambda L_\pm f$$

$$L_z(L_\pm f) = [L_z, L_\pm]f + L_\pm L_z f = \pm\hbar L_\pm f + \mu L_\pm f = (\mu \pm \hbar) L_\pm f$$

Now the same logic as for oscillator: apply L_+ many times, the process should stop at some point (since L_z^2 cannot exceed L^2), therefore $L_+ f_{\text{top}} = 0$.



$$\begin{cases} L_z f_{\text{top}} = \hbar l f_{\text{top}} & \text{(so far } l \text{ is arbitrary)} \\ L^2 f_{\text{top}} = \lambda f_{\text{top}} & \end{cases}$$

Let us show $\boxed{\lambda = \hbar^2 l(l+1)}$ (eigenvalues are related!)

Use formula $L^2 = L_- L_+ + L_z^2 + \hbar L_z$ (simple to derive explicitly), then

$$\lambda\, f_{\text{top}} = (\hbar l)^2 f_{\text{top}} + \hbar^2 l f_{\text{top}} = \hbar^2 l(l+1) f_{\text{top}}$$

Similarly, applying L_- many times, we should eventually get $L_- f_{\text{bottom}} = 0$.

Then similarly $L_z f_{\text{bottom}} = \hbar l' f_{\text{bottom}}$ (with arbitrary l'), and can show the relation between eigenvalues:

$$\lambda = \hbar^2 l'(l' - 1)$$

Comparing these two equations, we obtain $l' = -l$.

But they are related by integer number (integer number of applying $L_\pm$).

Therefore, l is integer or half-integer.

Thus, result:

Eigenvalues of L^2 (i.e., possible measured values) are $\lambda = \hbar^2 l(l+1)$, where $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$,

eigenvalues of L_z (i.e., possible measured values) are $\mu = \hbar\, m$, where $m = -l, -l+1, \ldots l-1, l$

If we denote the corresponding eigenvectors by f_l^m, then

$$\begin{cases} L^2 f_l^m = \hbar^2 l(l+1) f_l^m \\ L_z f_l^m = \hbar m\, f_l^m \end{cases}$$

Interesting that $L_z^2 \to \hbar^2 m^2 \leq \hbar^2 l^2$ is always smaller than $L^2 \to \hbar^2 l(l+1)$.

So, z-component cannot be parallel to $\vec{L}$.

(If it would, then $L_x = L_y = 0$, impossible.)

Eigenvectors of L^2 and L_z

We found eigenvalues of L^2 and L_z:

$$\begin{cases} L^2 f_l^m = \hbar^2 l(l+1) f_l^m \\ L_z f_l^m = \hbar m\, f_l^m \end{cases}$$

What are the eigenvectors f_l^m ?

This is not simple. Discuss only the result (not the derivation).

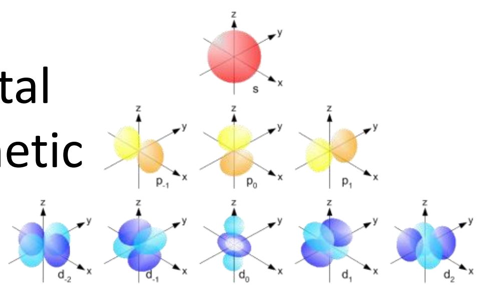
If the eigenvector f_l^m is represented by ψ-function in real space, $\psi(r,\theta,\varphi)$, then

l is integer (not half-integer) and $\psi(r,\theta,\varphi) = \underbrace{R(r)}_{\text{any}}\, \underbrace{Y_l^m(\theta,\varphi)}_{\text{spherical harmonics}}$

is the eigenfunction, $L^2\psi = \hbar^2 l(l+1)\psi$

$$L_z\psi = \hbar m\, \psi$$

Thus, azimuthal quantum number l means that square of the total angular momentum is $L^2 = \hbar^2 l(l+1)$ (s, p, d, f orbitals). Magnetic quantum number m means that $L_z = \hbar m$ (degeneracy $2l+1$).



However, half-integer l values really exist: spin (consider next).

Spin

Existence of half-integer values of l is experimental fact (Stern-Gerlach, 1922: neutral silver atoms travel though inhomogeneous magnets field, split into two beams)

Cannot explain as $\psi(r,\theta,\varphi) \Rightarrow$ something else. It is an angular momentum, which does not correspond to motion in $(x,y,z) \Rightarrow$ intrinsic angular momentum (spin).

The theory of spin is essentially the same as theory of angular momentum, just no $\psi(r,\theta,\varphi)$, only operator formalism.

$$[\hat{S}_x, \hat{S}_y] = i\hbar\, \hat{S}_z$$
$$[\hat{S}_y, \hat{S}_z] = i\hbar\, \hat{S}_x$$
$$[\hat{S}_z, \hat{S}_x] = i\hbar\, \hat{S}_y$$

Denote eigenstates of S^2 and S_z as $|s, m\rangle$ (total spin: s; z-component: m)

This is jargon, actually $\sqrt{S^2} = \hbar\sqrt{s(s+1)}$, $S_z = \hbar m$

More correctly,

$$\begin{cases} S^2|s,m\rangle = \hbar^2 s(s+1)\,|s,m\rangle \\ S_z|s,m\rangle = \hbar m\,|s,m\rangle \end{cases}$$

Useful relation: $S_\pm|s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s, m\pm 1\rangle$

Total spin: $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

z-component: $m = -s, -s+1, \ldots\, s$

$s = 0$: pi-mesons, Higgs boson,
$s = {}^1/_2$: electron, proton, neutron, quarks,
$s = 1$: photon, W, Z, $s = {}^3/_2$: deltas, omega,
$s = 2$: graviton, etc.

The case $s = 1/2$ (spin one-half)

(simplest and most important)

$$S^2 = \hbar^2 \frac{1}{2}\frac{3}{2} = \frac{3}{4}\hbar^2 \qquad S_z:\ m\hbar = \pm\frac{\hbar}{2}$$

What is the wavefunction? It cannot depend on x, y, z.

Let us introduce ***something*** that satisfies formalism (commutation relations).

Spinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{spin-up } \chi_\uparrow} + b\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{spin-down } \chi_\downarrow}$ (χ is chosen to resemble ψ)

Normalization: $|a|^2 + |b|^2 = 1$

Spin operators

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$\sigma_x, \sigma_y, \sigma_z$ are called Pauli matrices

These operators satisfy desired commutation relations:

$$[S_x, S_y] = i\hbar\, S_z, \quad [S_y, S_z] = i\hbar\, S_x,$$
$$[S_z, S_x] = i\hbar\, S_y$$

Also, as desired

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)$$
$$= \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\hbar^2$$

Eigenvectors and eigenvalues

We know that S_z has eigenvalues $\pm\hbar/2$; what are the eigenvectors?

$$S_z\begin{pmatrix}a\\b\end{pmatrix}=\lambda\begin{pmatrix}a\\b\end{pmatrix}$$

$\pm\frac{\hbar}{2}$ (→ λ)

$$S_z=\frac{\hbar}{2}\quad\Longrightarrow\quad\chi=\begin{pmatrix}1\\0\end{pmatrix}\qquad\text{(spin-up)}$$

Check: $S_z\begin{pmatrix}1\\0\end{pmatrix}=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$

Similarly $$S_z=-\frac{\hbar}{2}\quad\Longrightarrow\quad\chi=\begin{pmatrix}0\\1\end{pmatrix}\qquad\text{(spin-down)}$$

However, z-direction is not different from other directions, therefore the same eigenvalues for S_x and S_y.

$$S_x\begin{pmatrix}a\\b\end{pmatrix}=\lambda\begin{pmatrix}a\\b\end{pmatrix}$$

$\pm\frac{\hbar}{2}$ (→ λ)

$$S_x=\frac{\hbar}{2}\quad\Longrightarrow\quad\chi=\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$S_x=-\frac{\hbar}{2}\quad\Longrightarrow\quad\chi=\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Check: $S_x\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}=\frac{\hbar}{2}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ (similar for $-\hbar/2$)

Similarly find eigenvectors for S_y

$$S_y \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$\lambda = \pm \frac{\hbar}{2}$

$$S_y = \frac{\hbar}{2} \quad \Longrightarrow \quad \chi = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$S_y = -\frac{\hbar}{2} \quad \Longrightarrow \quad \chi = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

not included into this course

Actually, S_x and S_y correspond to new bases of eigenstates, and everything can also be considered in these new bases

$$\begin{pmatrix} a \\ b \end{pmatrix}_{(z)} \longrightarrow \begin{pmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{pmatrix}_{(x)}$$

$$\begin{pmatrix} a \\ b \end{pmatrix}_{(z)} \longrightarrow \begin{pmatrix} \frac{a-ib}{\sqrt{2}} \\ \frac{a+ib}{\sqrt{2}} \end{pmatrix}_{(y)}$$

(we will consider *only* the standard z-basis)

Measurement of spin

$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ (in standard z-representation)

Measure S_z: get $S_z = \hbar/2$ with probability $P_{z+} = |a|^2$, after that $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

get $S_z = -\hbar/2$ with probability $P_{z-} = |b|^2$, after that $\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Measure S_x: get $S_x = \hbar/2$ with probability

$$P_{x+} = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 = \frac{1}{2}\,|a+b|^2 \quad \text{after that} \quad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S_x = -\hbar/2 \text{ with } P_{x-} = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 = \frac{1}{2}\,|a-b|^2 \quad \text{then} \quad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

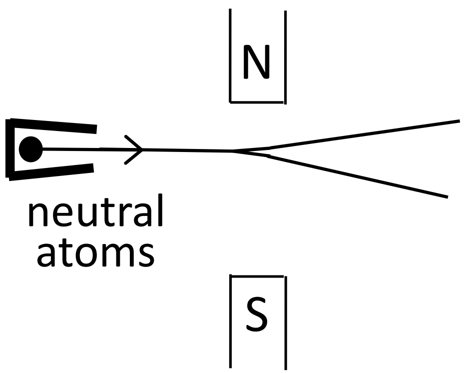
Measure S_y: get $S_y = \hbar/2$ with probability

$$P_{y+} = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 = \frac{1}{2}\,|a-ib|^2 \quad \text{then} \quad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$S_y = -\hbar/2 \text{ with } P_{y-} = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{+i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 = \frac{1}{2}\,|a+ib|^2 \quad \text{then} \quad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(not included into this course)

Experimental measurement of spin (Stern-Gerlach experiment, 1922)



inhomogeneous magnetic field produces force onto a magnetic moment

$$H = -\gamma \vec{B}\vec{S}$$

angular momentum (or spin) — $\vec{S}$
magnetic field — $\vec{B}$
gyromagnetic ratio — γ

If B is inhomogeneous (not constant), then $\vec{F} = -\nabla H = \gamma \nabla(\vec{B}\vec{S})$

So, force depends on $\vec{S}$.

If $\vec{B}(x,y,z) = (B_0 + \alpha z)\vec{k} - \underbrace{\alpha x \vec{\imath}}_{\text{not important}}$ (for magnetic field $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$),

then $\vec{F} = \gamma\alpha\left(S_z \vec{k} - \underbrace{S_x \vec{\imath}}\right)$

not important (oscillates because of Larmor precession about z-axis, so zero on average)

$$F_z = \gamma\alpha S_z$$

spin-up is deflected down ($\gamma < 0$),
spin-down is deflected up

(particle should be neutral because otherwise charge will circle in magnetic field)