EE201/MSE207 Lecture 11

Electron spin in magnetic field

Interaction energy
(as for compass needle)

$$H = -\vec{\mu}\vec{B}$$

$\vec{\mu}$ is magnetic dipole moment
$\vec{B}$ is magnetic field, dot-product

$$\vec{\mu} = \gamma_{\text{orbital}}\,\vec{L} \quad \text{or} \quad \vec{\mu} = \gamma_{\text{spin}}\,\vec{S}$$

γ is called gyromagnetic ratio

In classical physics:

$$\gamma_{\text{cl}} = \frac{q}{2m}$$

q ← charge ($-e$ for electron)

m ← mass

In quantum mechanics:

For orbital motion $\gamma_{\text{orbital}} = \gamma_{cl}$

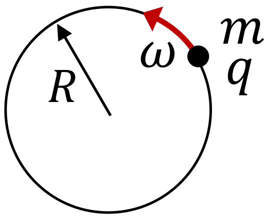
For spin $\gamma_{\text{spin}} = g\,\gamma_{cl}$

g ← *g*-factor

$g \approx$ 2.0023 for free electron,
different values in materials

In more detail

Classical physics



Angular momentum

$$L = mvR = mR^2\omega$$

Magnetic moment

$$\mu = IA = \frac{q}{T}\pi R^2 = q\,\frac{\omega}{2\pi}\pi R^2$$

current

area

Therefore $\gamma_{\mathrm{cl}} = \dfrac{\mu}{L} = \dfrac{q}{2m}$

Quantum, orbital motion

The same γ, $L_z = n\hbar \;\Rightarrow\; \mu_z = -n\dfrac{e\hbar}{2m}$

magnetic quantum number (usually m)

The smallest value (quantum of magnetic moment)

$$\boxed{\mu_B = \frac{e\hbar}{2m}}$$

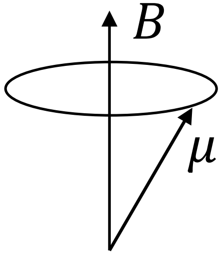
(Bohr magneton)

Quantum, spin

$S_z = \pm\hbar/2$, but still $\mu_z \approx \pm\mu_B$ (a little more, $\mu_z \approx \pm 1.001\,\mu_B$), so $\gamma_{\mathrm{spin}} \approx 2\gamma_{\mathrm{cl}}$ (i.e., $g \approx 2$).

(Different people use different notations; sometimes $g = -2$, sometimes γ is positive, sometimes g is called gyromagnetic ratio, etc.)

Evolution of spin in magnetic field



$$H = -\vec{\mu}\vec{B} = -\gamma\vec{B}\vec{S}$$

Classically, precession as in a gyroscope.

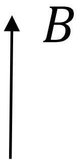
Larmor precession, $\omega = \gamma B$.

torque $\vec{\mu} \times \vec{B}$ $$\frac{d\vec{L}}{dt} = \gamma\vec{L} \times \vec{B}$$

$$\frac{dL_x}{dt} = \gamma L_y B_z \qquad \frac{dL_y}{dt} = -\gamma L_x B_z$$

(assume $\vec{B} = B_z\vec{k}$)

$$\frac{d^2L_x}{dt^2} = -\gamma^2 B_z^2 L_x \quad \Rightarrow \quad |\omega| = |\gamma B_z|$$



Quantum mechanics

Assume $\vec{B} = B\,\vec{k}$ $$\hat{H} = -\gamma B\hat{S}_z = -\gamma B\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

γ is negative, $-\gamma$ is positive

$$i\hbar\frac{d\chi}{dt} = \hat{H}\chi \quad \Rightarrow \quad \frac{d\chi}{dt} = -\frac{i}{\hbar}\hat{H}\chi$$

$$\chi(t) = \begin{pmatrix}a(t)\\ b(t)\end{pmatrix} \quad \Rightarrow \quad \begin{cases}\dfrac{da(t)}{dt} = \dfrac{i\gamma B}{2}\,a(t)\\[2ex] \dfrac{db(t)}{dt} = -\dfrac{i\gamma B}{2}\,b(t)\end{cases} \quad \Rightarrow \quad \chi(t) = \begin{pmatrix}a(0)\,e^{i\gamma Bt/2}\\ b(0)\,e^{-i\gamma Bt/2}\end{pmatrix}$$

$$|a|^2 + |b|^2 = 1$$

Both $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$ are eigenvectors

Average values

$$\chi(t) = \begin{pmatrix} a\, e^{i\gamma Bt/2} \\ b\, e^{-i\gamma Bt/2} \end{pmatrix} \qquad \begin{aligned} a &= a(0) \\ b &= b(0) \end{aligned}$$

Obviously $\langle S_z \rangle = \text{const}$, since $\left| a e^{i\gamma Bt/2} \right|^2 = |a|^2$ and also precession about z-axis. Nevertheless, let us check formally.

$$\langle S_z(t) \rangle = \langle \chi | S_z | \chi \rangle = \left(a^* e^{-i\gamma Bt/2}, \quad b^* e^{i\gamma Bt/2} \right) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a\, e^{i\gamma Bt/2} \\ b\, e^{-i\gamma Bt/2} \end{pmatrix} =$$

$$= \frac{\hbar}{2} \left(|a|^2 - |b|^2 \right)$$

Yes, does not depend on t.

Similarly

$$\langle S_x(t) \rangle = \langle \chi | S_x | \chi \rangle = \left(a^* e^{-i\gamma Bt/2}, \quad b^* e^{i\gamma Bt/2} \right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a\, e^{i\gamma Bt/2} \\ b\, e^{-i\gamma Bt/2} \end{pmatrix} =$$

$$= \frac{\hbar}{2} \left(a^* b\, e^{-i\gamma Bt} + a b^* e^{i\gamma Bt} \right) = \hbar\, \mathrm{Re}\left(a^* b\, e^{-i\gamma Bt} \right)$$

oscillations with frequency $\omega = \gamma B$

$$\langle S_y(t) \rangle = \left(a^* e^{-i\gamma Bt/2}, \quad b^* e^{i\gamma Bt/2} \right) \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a\, e^{i\gamma Bt/2} \\ b\, e^{-i\gamma Bt/2} \end{pmatrix} = \hbar\, \mathrm{Im}\left(a^* b e^{-i\gamma Bt} \right)$$

Dynamics of average values

$$\langle S_z(t)\rangle = \frac{\hbar}{2}\left(|a|^2 - |b|^2\right),\quad \langle S_x(t)\rangle = \hbar\,\mathrm{Re}\left(a^* b\, e^{-i\gamma Bt}\right),\ \langle S_y(t)\rangle = \hbar\,\mathrm{Im}\left(a^* b e^{-i\gamma Bt}\right)$$

If $\begin{cases} a = \cos(\alpha/2) \\ b = \sin(\alpha/2) \end{cases}$ then

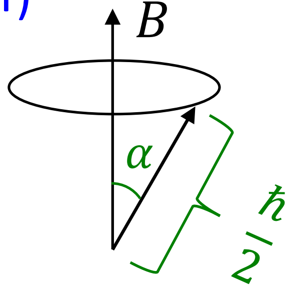
$$\langle S_x(t)\rangle = \frac{\hbar}{2}\sin\alpha\cos(\gamma Bt)$$

$$\langle S_y(t)\rangle = -\frac{\hbar}{2}\sin\alpha\sin(\gamma Bt)$$

$$\langle S_z(t)\rangle = \frac{\hbar}{2}\cos\alpha$$

Dynamics of $\langle\vec{S}(t)\rangle$ is <u>the same as in the classical case</u>, $\omega = \gamma B$.

A way to visualize
(as if)



However, remember that actually

$$\sqrt{S^2} = \sqrt{\frac{3}{4}}\,\hbar \approx 0.87\,\hbar$$

and if we measure S_x, S_y, or S_z, we always get $\pm\frac{\hbar}{2}$.

Measurement

$$\chi(t) = \begin{pmatrix} a\, e^{i\gamma Bt/2} \\ b\, e^{-i\gamma Bt/2} \end{pmatrix}$$

If we measure S_z, then we get $+\hbar/2$ with probability $\left|a\, e^{i\gamma Bt/2}\right|^2 = |a|^2$.

If we measure S_x, then we get $+\hbar/2$ with probability

$$P(S_x = +\hbar/2) = |\langle \chi_{x+}|\chi\rangle|^2 = \left|\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\begin{pmatrix} a\, e^{\frac{i\gamma Bt}{2}} \\ b\, e^{-\frac{i\gamma Bt}{2}} \end{pmatrix}\right|^2 =$$

$$= \frac{1}{2}\left|a e^{i\gamma Bt/2} + b\, e^{-i\gamma Bt/2}\right|^2 = \frac{1}{2}\left|a + b\, e^{-i\gamma Bt}\right|^2$$

Similarly, $P(S_x = -\hbar/2) = \frac{1}{2}\left|a - b\, e^{-i\gamma Bt}\right|^2$ frequency $\omega = \gamma B$

Special case: $a = b = 1/\sqrt{2}$, then

$$\begin{cases} P(S_x = +\hbar/2) = \cos(\gamma Bt/2) = \frac{1}{2} + \frac{1}{2}\cos(\gamma Bt) \\ P(S_x = -\hbar/2) = \sin(\gamma Bt/2) = \frac{1}{2} - \frac{1}{2}\cos(\gamma Bt) \end{cases}$$

(in general, amplitude can be smaller and also extra phase shift)

Quantum coherent oscillations: oscillations of probability (if measured)

If we measure S_x, the state will be collapsed onto x-axis ($\pm\hbar/2$), S_z and S_y components will be lost.

Another way to consider evolution (in x-basis)

(not included into this course)

z-basis $\longrightarrow$ x-basis

$$\widehat{H} = -\gamma B \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{(z)} \longrightarrow -\gamma B \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{(x)}$$

(Rabi oscillations in a qubit)

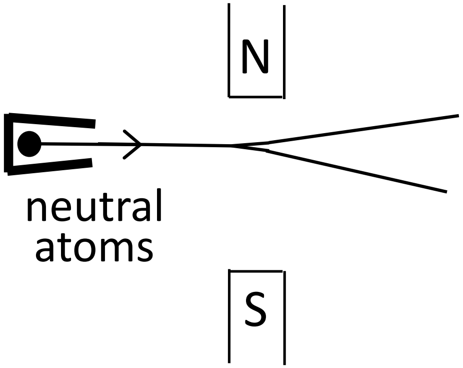
$$\chi_x = \begin{pmatrix} a_x \\ b_x \end{pmatrix} \qquad i\hbar \frac{d\chi}{dt} = \widehat{H}\chi$$

$$\begin{cases} i\,\hbar \dfrac{da_x}{dt} = -\gamma B \dfrac{\hbar}{2} b_x \\ i\,\hbar \dfrac{db_x}{dt} = -\gamma B \dfrac{\hbar}{2} a_x \end{cases} \Rightarrow$$

oscillations with frequency $\gamma B/2$,
probabilities will oscillate with freq. γB

(not included into this course)

Experimental measurement of spin
(Stern-Gerlach experiment, 1922)



inhomogeneous magnetic field produces force onto a magnetic moment

$$H = -\gamma \vec{B}\vec{S}$$

angular momentum (or spin)
magnetic field
gyromagnetic ratio

If B is inhomogeneous (not constant), then $\vec{F} = -\nabla H = \gamma \nabla (\vec{B}\vec{S})$

So, force depends on $\vec{S}$.

If $\vec{B}(x,y,z) = (B_0 + \alpha z)\vec{k} - \underbrace{\alpha x \vec{i}}_{\text{not important}}$ (for magnetic field $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$),

then $\vec{F} = \gamma\alpha\left(S_z\vec{k} - \underbrace{S_x\vec{i}}\right)$

not important (oscillates because of Larmor precession about z-axis, so zero on average)

$$F_z = \gamma\alpha S_z$$

spin-up is deflected down ($\gamma < 0$),
spin-down is deflected up

(particle should be neutral because otherwise charge will circle in magnetic field)

Addition of spins (similar for angular momenta)

Two particles $\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$ (vectors added) $S_z = S_z^{(1)} + S_z^{(2)}$ (scalars added)

However, not simple for the total spin $S^2 = \left(S^{(1)}\right)^2 + \left(S^{(2)}\right)^2 + 2\,\vec{S}^{(1)}\vec{S}^{(2)}$

Two particles with spin 1/2

$$\chi = \alpha_{\uparrow\uparrow}\,|\uparrow\uparrow\rangle + \alpha_{\uparrow\downarrow}\,|\uparrow\downarrow\rangle + \alpha_{\downarrow\uparrow}\,|\downarrow\uparrow\rangle + \alpha_{\downarrow\downarrow}\,|\downarrow\downarrow\rangle = \begin{pmatrix} \alpha_{\uparrow\uparrow} \\ \alpha_{\uparrow\downarrow} \\ \alpha_{\downarrow\uparrow} \\ \alpha_{\downarrow\downarrow} \end{pmatrix}$$

$|\uparrow\uparrow\rangle$ $m = {}^1/_2 + {}^1/_2 = 1\ (S_z = 1\cdot\hbar),\ s = 1$

$$\left.\begin{array}{ll} |\uparrow\downarrow\rangle & m = 0 \\ |\downarrow\uparrow\rangle & m = 0 \end{array}\right\}$$ these states are not eigenstates of the total spin S^2

$|\downarrow\downarrow\rangle$ $m = -{}^1/_2 - {}^1/_2 = -1\ (S_z = -1\cdot\hbar),\ s = 1$

Eigenstates of total spin: $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ $s = 0$ (called singlet), $m = 0$

$$\left.\begin{array}{l} |\uparrow\uparrow\rangle \\ (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \\ |\downarrow\downarrow\rangle \end{array}\right\}\ s = 1\ \text{(triplet)} \quad \begin{array}{l} m = 1 \\ m = 0 \\ m = -1 \end{array}$$

(easy to check)

Addition of arbitrary spins (or angular momenta)

Addition of arbitrary spins s_1 and s_2 is quite complicated.

Possible values range from $s_1 + s_2$ to $|s_1 - s_2|$ (integer ladder).

Eigenvectors are given by Clebsch-Gordan coefficients.

Example

quarks: $s = 1/2$

two quarks: $1/2 + 1/2 = 1 \text{ or } 0$ (mesons: vector and pseudoscalar)

three quarks: $(1 \text{ or } 0) + 1/2 = 1/2 \text{ or } 3/2$

proton,
neutron,
etc.

Delta,
Omega,
etc.