EE201/MSE207 Lecture 8

Measurement and uncertainty principle

Determinate state

Theorem: If $\hat{A}|\Psi\rangle = \lambda|\Psi\rangle$, then measurement of A in state $|\Psi\rangle$ will certainly give result λ (therefore, such $|\Psi\rangle$ is called determinate state)

Proof: a) from postulate 5 of the previous lecture: $P_\lambda = |\langle f_\lambda|\Psi\rangle|^2$;
b) from postulate 4: $\langle A\rangle = \lambda$, $\langle A^2\rangle = \lambda^2$ $\Rightarrow$ no variance.

Also, from postulate 6, measurement will not change such determinate state.

In contrast, if $|\Psi\rangle$ is not an eigenvector of $\hat{A}$, then measurement of A can give different results, and changes (collapses) $|\Psi\rangle$.

Compatible and incompatible observables

Question: When a state can be a determinate state for two operators $\hat{A}$ and $\hat{B}$?

(why important: consider measurement sequence *A*, *B*, *A*)

Answer: If $\hat{A}$ and $\hat{B}$ commute, $[\hat{A}, \hat{B}] = 0$, then this is possible ("compatible");

if $[\hat{A}, \hat{B}] \neq 0$, then this is usually impossible ("incompatible").

More rigorously, two theorems from linear algebra (without proof).

Theorem 1

If two Hermitian operators commute, $[\hat{A}, \hat{B}] = 0$, then there exists a basis consisting of eigenvectors of both $\hat{A}$ and $\hat{B}$ simultaneously.

(These states are determinate states of both $\hat{A}$ and $\hat{B}$, i.e. $\sigma_A = \sigma_B = 0.$)

(σ is the standard deviation)

Theorem 2 (generalized uncertainty principle)

For Hermitian $\hat{A}$ and $\hat{B}$, if $[\hat{A}, \hat{B}] \neq 0$, then

$$\sigma_A^2 \, \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Example

$$[\hat{x}, \hat{p}] = i\hbar \quad \Rightarrow \quad \sigma_x^2 \, \sigma_p^2 \geq \left(\frac{\hbar}{2} \right)^2 \quad \Rightarrow \quad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

(Heisenberg uncertainty principle)

Energy-time uncertainty

You can often find inequality $$\Delta E\, \Delta t \geq \frac{\hbar}{2}$$

1) It is formally incorrect, but often gives correct intuition.
2) It is correct in the following sense.

Theorem

For any observable Q (which does not explicitly depend on time)

$$\sigma_E \frac{\sigma_Q}{|d\langle Q\rangle/dt|} \geq \frac{\hbar}{2}$$

(So, if anything changes significantly during Δt, then the energy spread should be large enough, $\Delta E\, \Delta t \geq \hbar/2$. In stationary state $\Delta E = 0$, therefore nothing changes.)

Proof

Straightforward from $\sigma_H \sigma_Q \geq \left(\frac{1}{2i}\langle[\hat{H},\hat{Q}]\rangle\right)^2$ and $$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle$$

$$\frac{d\langle Q\rangle}{dt} = \frac{d}{dt}\langle\Psi|\hat{Q}\Psi\rangle = \langle\frac{d\Psi}{dt}\Big|\hat{Q}\Psi\rangle + \langle\Psi\Big|\hat{Q}\frac{d\Psi}{dt}\rangle + \langle\Psi\Big|\frac{\partial\hat{Q}}{\partial t}\Psi\rangle =$$

$$= \langle\frac{-i}{\hbar}\hat{H}\Psi\Big|\hat{Q}\Psi\rangle + \langle\Psi\Big|\hat{Q}\frac{-i}{\hbar}\hat{H}\Psi\rangle = \frac{i}{\hbar}\left(\langle\Psi|\hat{H}\hat{Q}\Psi\rangle - \langle\Psi|\hat{Q}\hat{H}\Psi\rangle\right) = \frac{i}{\hbar}\langle\Psi|[\hat{H},\hat{Q}]\Psi\rangle$$

$$= (i/\hbar)\,\langle[\hat{H},\hat{Q}]\rangle$$

x-representation and p-representation

It is easy to check that both $\hat{x}$ and $\hat{p}$ are Hermitian (as for any observable).

We need to prove $\langle f | \hat{T} g \rangle = \langle \hat{T} f | g \rangle$.

For $\hat{T} = \hat{x}$ this is very simple: $\int f^*(x)\, x g(x)\, dx = \int [x\, f(x)]^* g(x)\, dx$.

For $\hat{T} = \hat{p}$ we need integration by parts: $\int f^*(x) \left(-i\hbar \frac{d}{dx}\right) g(x)\, dx =$

$$= \int i\hbar \frac{df^*(x)}{dx} g(x)\, dx = \int \left(-i\hbar \frac{df(x)}{dx}\right)^* g(x) dx.$$

Find eigenstates of $\hat{x}$ and $\hat{p}$ $\quad \hat{T} f(x) = \lambda\, f(x)$

Eigenstates of $\hat{x}$: $\quad x\, f(x) = \lambda\, f(x) \quad \Rightarrow \quad f(x) = A\, \delta(x - \lambda)$

Not normalizable, choose $A = 1$.

$f_\lambda(x) = \delta(x - \lambda)$, then $\langle f_\lambda | f_\mu \rangle = \delta(\mu - \lambda)$ $\quad$ orthonormal basis

Check: $\int \delta(x - \lambda)\, \delta(x - \mu)\, dx = \delta(\mu - \lambda)$ (since $x = \mu$)

Eigenstates of $\hat{p}$: $$-i\hbar \frac{d}{dx} g(x) = \lambda\, g(x) \quad \Rightarrow \quad g(x) = B\, e^{i\frac{\lambda}{\hbar}x}$$

$$g_\lambda(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{\lambda}{\hbar}x}, \quad \text{then } \langle g_\lambda | g_\mu \rangle = \delta(\mu - \lambda)$$ orthonormal basis

Check: $$\frac{1}{2\pi\hbar}\int e^{-i\frac{\lambda}{\hbar}x} e^{i\frac{\mu}{\hbar}x}\, dx = \frac{1}{2\pi\hbar}\int e^{i\frac{\mu-\lambda}{\hbar}x}\, dx = \delta(\mu - \lambda)$$

since $\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi\, \delta(k)$ and $\delta(ax) = \frac{\delta(x)}{|a|}$

Digression: proof of the formula $\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi\, \delta(k)$

Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}}\int e^{ikx} F(k)\, dk, \quad F(k) = \frac{1}{\sqrt{2\pi}}\int e^{-ikx} f(x)\, dx$

Therefore
$$F(k) = \frac{1}{\sqrt{2\pi}}\int dx\, e^{-ikx} \frac{1}{\sqrt{2\pi}}\int e^{ik'x} F(k')\, dk' = \int \underbrace{\left[\frac{1}{2\pi}\int e^{i(k'-k)x} dx\right]}_{\delta(k-k')} F(k')\, dk$$

Similarity of x-representation and p-representation

$\hat{x}$-eigenstates (x-basis) $\quad f_\lambda(x) = \delta(x-\lambda)$ $\qquad$ $\hat{p}$-eigenstates (p-basis) $\quad g_\lambda(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{\lambda}{\hbar}x}$

We can say that $\Psi(x)$ are actually components of vector $|\Psi\rangle$ in x-basis:

$$\Psi(x) = \int_{-\infty}^{\infty} \underbrace{\Psi(x')}_{\text{components}} \underbrace{\delta(x-x')}_{\text{basis vectors}} \, dx'$$

Similarly, we can write it in p-basis:

$$\Psi(x) = \int_{-\infty}^{\infty} \underbrace{\Phi(p)}_{\text{components}} \underbrace{\frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p}{\hbar}x}}_{\text{basis vectors}} \, dp \qquad \Big| \qquad \Phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i\frac{p}{\hbar}x} \, \Psi(x) \, dx$$

We can regard $\Phi(p)$ as a wavefunction in *p*-space

Actually, more often people use $\Phi(k)$ in k-space, where $k = p/\hbar$, then eigenstates (basis vectors) are $\frac{1}{\sqrt{2\pi}} e^{ikx}$.

Operators $\hat{x}$ and $\hat{p}$ in *p*-space

$$\hat{p}\Phi(p) = p\,\Phi(p)$$

Expected by analogy. Formal proof: $-i\hbar\dfrac{\partial\Psi(x)}{\partial x} = -i\hbar\dfrac{\partial}{\partial x}\displaystyle\int\Phi(p)\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}dp =$

$$= -i\hbar\int\Phi(p)\frac{1}{\sqrt{2\pi\hbar}}\frac{ip}{\hbar}\,e^{i\frac{p}{\hbar}x}dp = \int \underline{p\,\Phi(p)}\frac{1}{\sqrt{2\pi\hbar}}\,e^{i\frac{p}{\hbar}x}dp$$

$$\hat{x}\Phi(p) = i\hbar\frac{\partial}{\partial p}\,\Phi(p)$$

Proof: $x\Psi(x) = \displaystyle\int\Phi(p)\frac{1}{\sqrt{2\pi\hbar}}\underbrace{x\,e^{i\frac{p}{\hbar}x}}_{-i\hbar\frac{\partial}{\partial p}e^{i\frac{p}{\hbar}x}}dp =$

$$= (\text{by parts}) = \int \underline{i\hbar\frac{d\Phi(p)}{dp}}\,\frac{1}{\sqrt{2\pi\hbar}}\;e^{i\frac{p}{\hbar}x}dp$$

Commutator does not change

$$[\hat{x},\hat{p}] = \left[x, -i\hbar\frac{\partial}{\partial x}\right] = i\hbar$$

$$[\hat{x},\hat{p}] = \left[i\hbar\frac{\partial}{\partial p}, p\right] = i\hbar$$

Probabilities $|\Psi(x)|^2 dx$ and $|\Phi(p)|^2 dp$

$\Phi(p)$ is as good a wavefunction as $\Psi(x)$

For *x*-measurement, probability to find particle near x_0
is $\mathcal{P}(x_0)\, dx$ (postulate 5, rem. a), with

$$\mathcal{P}(x_0) = \left|\langle f_{x_0}|\Psi\rangle\right|^2 = \left|\int \delta(x - x_0)\, \Psi(x)\, dx\right|^2 = |\Psi(x_0)|^2$$

Similarly, for *p*-measurement, probability to find momentum near p_0
is $\mathcal{P}(p_0)\, dp$, with

$$\mathcal{P}(p_0) = \left|\langle g_{p_0}|\Psi\rangle\right|^2 = \left|\int \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{-i\frac{p_0}{\hbar}x}\, \Psi(x)\, dx\right|^2 = |\Phi(p_0)|^2$$

Minimum-uncertainty states

From Heisenberg uncertainty relation, we know that $\sigma_x \sigma_p \geq \hbar/2$

Which states satisfy the lower bound, $\sigma_x \sigma_p = \hbar/2$?

Answer:

$$\Psi(x) = A\, e^{-a(x-x_0)^2/2\hbar}\, e^{ip_0 x/\hbar} \qquad A = \left(\frac{a}{\pi\hbar}\right)^{1/4}$$

(in optics they are called “squeezed states”)

It has $\langle x \rangle = x_0$, $\langle p \rangle = p_0$, $\sigma_x = \sqrt{\frac{\hbar}{2a}}$, $\sigma_p = \sqrt{\frac{a\hbar}{2}}$,

In *p*-representation very similar form:

$$\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-i\frac{p}{\hbar}x}\, \Psi(x)\, dx = \left[e^{ip_0 x_0/\hbar} \frac{A}{\sqrt{a}} \right] e^{-(p-p_0)^2/(2\hbar a)}\, e^{-ix_0 p/\hbar}$$