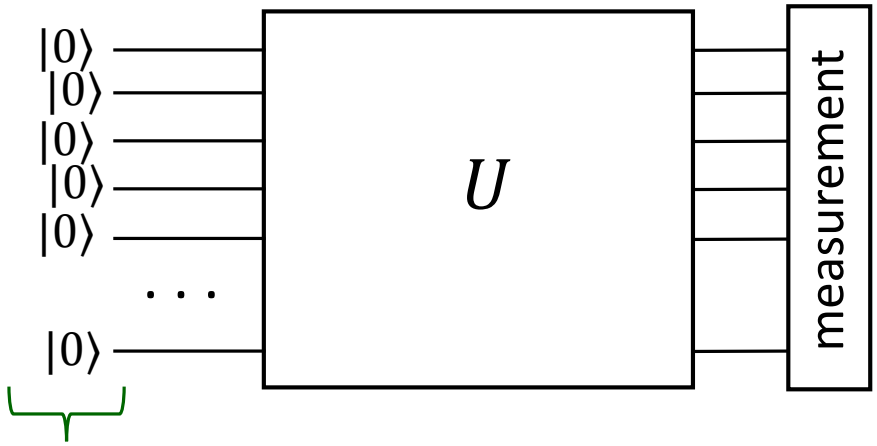
EE214/PHYS220 Quantum Computing

Lecture 3: QC structure and quantum gates

Simple structure of a quantum computer (without error correction)



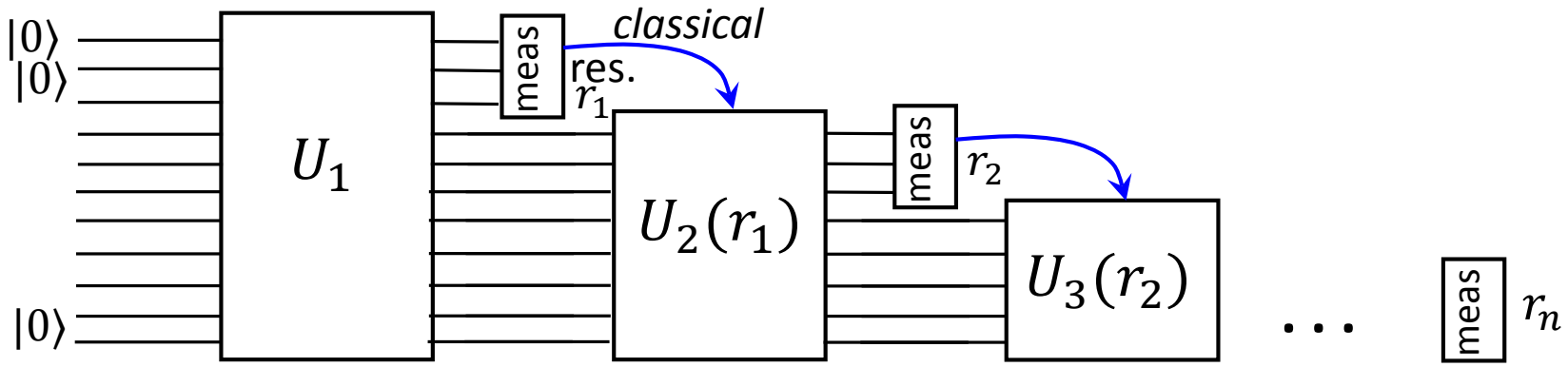
(quantum circuit diagram, read left-to-right)

Unitary operation U depends on what we need (e.g., a number to factor)

Idea: at the measurement stage some states are preferable (amplitudes of other states are ≈ 0), the result tells us what we need

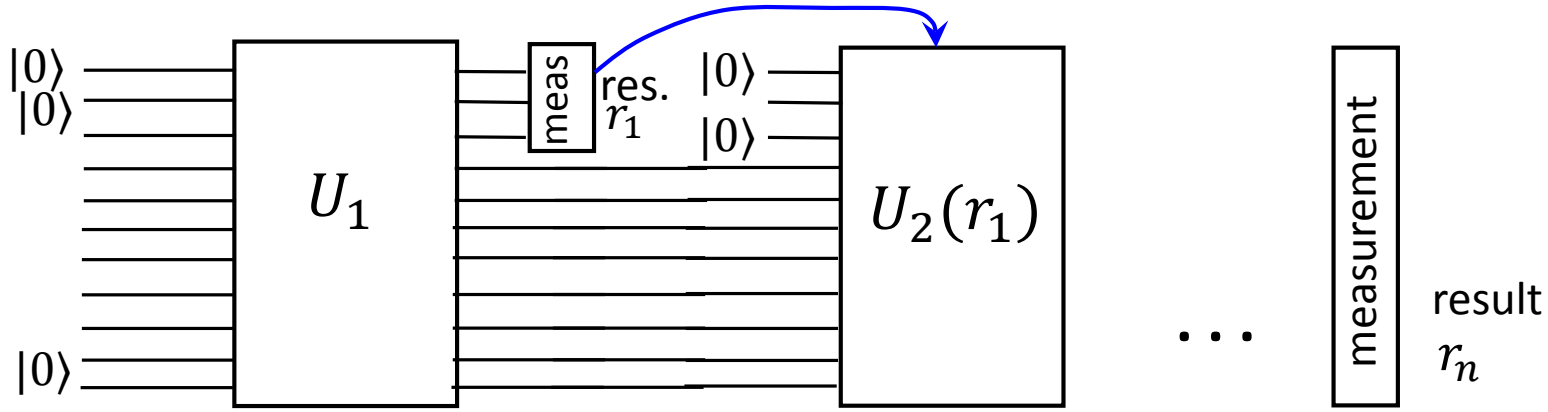
- We may need to measure only some qubits
- Still some randomness of the result (so mostly "hard to solve, easy to check" problems)

QC structure with error correction



Unitary operations U_i do not necessarily involve all qubits

Technical issue: physical qubits can be reused

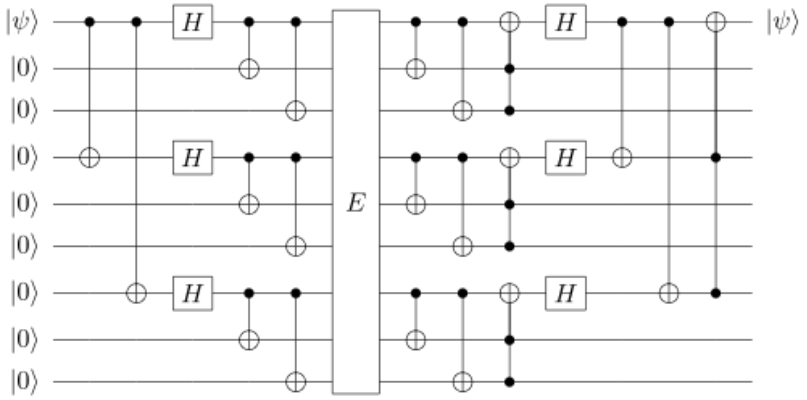


Unitary operations U_i can be decomposed into simpler gates (usually 1-qubit or 2-qubit, sometimes 3-qubit gates).

Unitary operations are reversible, so QC is related to reversible computing (classically permutations, often permutations in QC as well); measurement is irreversible.

Language of quantum circuit diagrams

(more notations later when we need them)



qubit idling: thin line ("wire") ——

several idling qubits —/— (Nielsen-Chuang's book) ═══ (Mermin's book)

measurement —[◠]— (N-C book)

[X] ← result
[M] (Mermin)

—[meas.]—

Read quantum circuit diagrams from left to right (⟶)

$|\psi\rangle$ —[U]— $U|\psi\rangle$

$|\psi\rangle$ —[U]—[V]— $VU|\psi\rangle$

So —[U]—[V]— = —[VU]—

One-qubit logic gates

"gate" = "operation" = "function" = "map" = "transformation"

Classically, 4 one-bit functions:

$$\underbrace{\begin{matrix}0\to 0\\ 1\to 1\end{matrix}}_{\mathbb{I}} \quad \underbrace{\begin{matrix}0\to 1\\ 1\to 0\end{matrix}}_{\text{NOT}} \quad \underbrace{\begin{matrix}0\to 0\\ 1\to 0\end{matrix}}_{\text{erase}} \quad \underbrace{\begin{matrix}0\to 1\\ 1\to 1\end{matrix}}_{\text{erase}'}$$

not reversible

($2^{(2^N)}$ N-bit→1-bit functions)

So, only 2 reversible 1-bit operations: NOT ($0 \leftrightarrow 1$) and unity operation

Quantum 1-qubit gate: any **unitary** 2×2 matrix

"Unitary" means $UU^\dagger = UU^\dagger = \mathbb{I}\ (= \hat{1})$

Actually, not U(2) group, but SU(2); "special" means $\det(U) = 1$
overall phase is not important for a 1-qubit gate
(though will be important for control-gates)

A unitary matrix has $8 - 4 = 4$ degrees of freedom
($UU^\dagger = \mathbb{I}$)

A matrix from SU(2) has 3 degrees of freedom, SU(2)↔SO(3) (3D rotation group)

A qubit state: direction of spin, a rotation is characterized by 3 Euler angles

Pauli matrices (digression)

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pauli matrices are Hermitian and unitary

Language of Quantum Computing to a significant extent is based on Pauli matrices

Together with unity matrix $\mathbb{I}$, they form an (almost) orthonornmal basis in the space of 2×2 matrices

$$\mathrm{Tr}\left(\sigma_i^\dagger \sigma_j\right) = 2^1 \delta_{ij}$$

Inner product for matrices is introduced as for vectors (matrix is "stretched" into vector):

$$\langle \alpha | \beta \rangle = \sum_n \alpha_n^* \beta_n$$

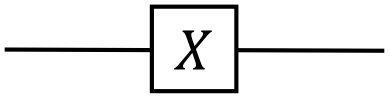
$$\langle \hat{A} \big| \hat{B} \rangle = \sum_{ij} A_{ij}^* B_{ij} = \sum_{ij} A_{ji}^\dagger B_{ij} = \mathrm{Tr}(A^\dagger B)$$

(called Frobenius inner product)

Most important 1-qubit gates

1. Bit flip (NOT, X-gate, Pauli-X)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X = \sigma_X = NOT$$



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \begin{matrix} \leftarrow |0\rangle \\ \leftarrow |1\rangle \end{matrix}$$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \beta|0\rangle + \alpha|1\rangle$$

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

2. Phase flip (Z-gate)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z = \sigma_Z$$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle$$

$$Z[\alpha|0\rangle + \beta|1\rangle] = \alpha|0\rangle - \beta|1\rangle$$

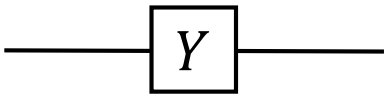
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

Most important 1-qubit gates (cont.)

3. Phase & bit flip (Y-gate)

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y = \sigma_Y$$



$$Y[\alpha|0\rangle + \beta|1\rangle] = -i\beta|0\rangle + i\alpha|1\rangle$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix}$$

$$Y|0\rangle = i|1\rangle$$
$$Y|1\rangle = -i|0\rangle$$

$$Y = iXZ = -iZX$$

Very often defined differently: $Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Mermin-web (Eqs. 1.48, 1.49)

In Mermin-book the usual definition (Eq. 1.51), except in Ch. 5 (error correction)

4. Hadamard

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\alpha+\beta}{\sqrt{2}} \\ \frac{\alpha-\beta}{\sqrt{2}} \end{pmatrix}$$

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Most important 1-qubit gates (cont.)

5. Phase gate

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = S$$

Notation from N-C book

$S = \sqrt{Z}$ since $Z = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}$

Do not confuse with Mermin's notation S_{ij} for SWAP

6. "$\pi/8$"-gate or T-gate

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = T$$

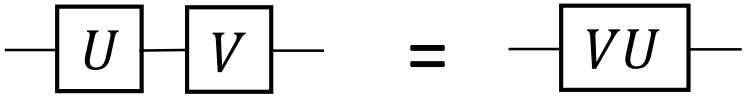
Notation from N-C book

$T = \sqrt{S}$

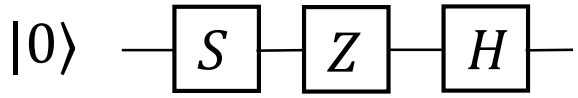
Called $\pi/8$ because equivalent to $\exp(-i\frac{\pi}{8}Z)$

Sequential gates

Possible confusion: left-to-right in quantum circuit diagrams,
right-to-left in matrix notations



Example



means

$$H\,Z\,S\,|0\rangle = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$H \qquad Z \qquad S \qquad |0\rangle$$

Summary for main 1-qubit gates

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ bit flip

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ phase flip

$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ phase & bit flip

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Hadamard

$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \sqrt{Z}$

$T = \sqrt{S}$

Some useful relations

$$X^2 = Y^2 = Z^2 = \mathbb{I}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H^2 = \mathbb{I}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$XY = -YX = iZ$$
$$YZ = -ZY = iX$$
$$ZX = -XZ = iY$$

The factor i is not important (overall phase), therefore sufficient to consider X and Z.

$$HXH = Z, \quad HZH = X$$

Check

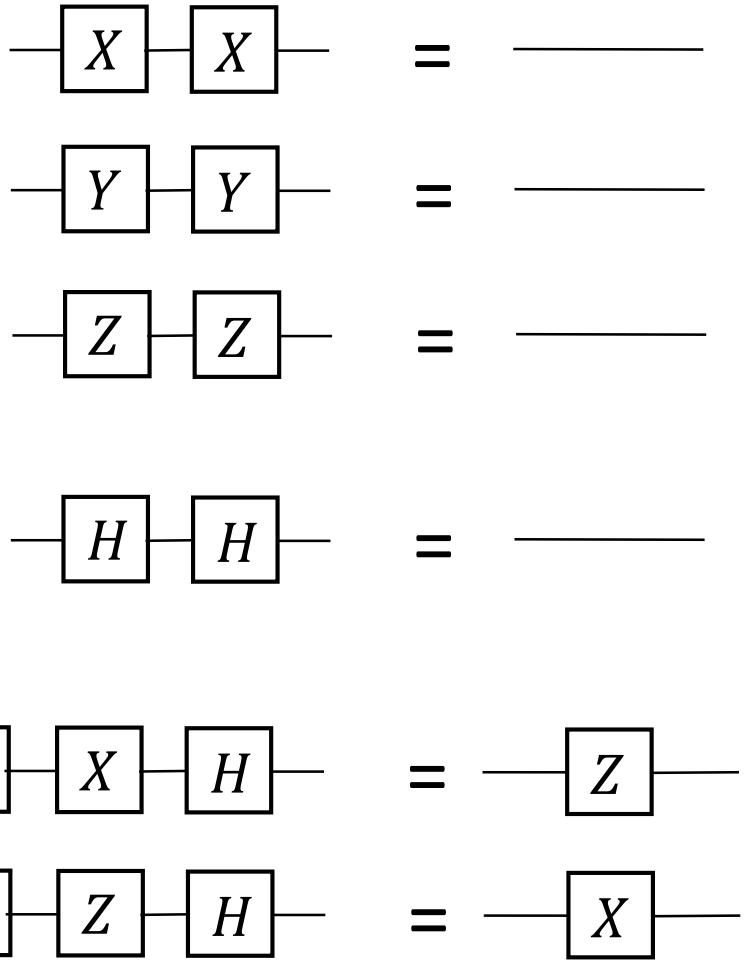
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

exchange rows

$XA = A_{\text{EXCHANGED ROWS}}$ $\quad AX = A_{\text{EXCHANGED COLUMNS}}$

The second equation $\quad HXH = Z \Rightarrow HHXHH = HZH$

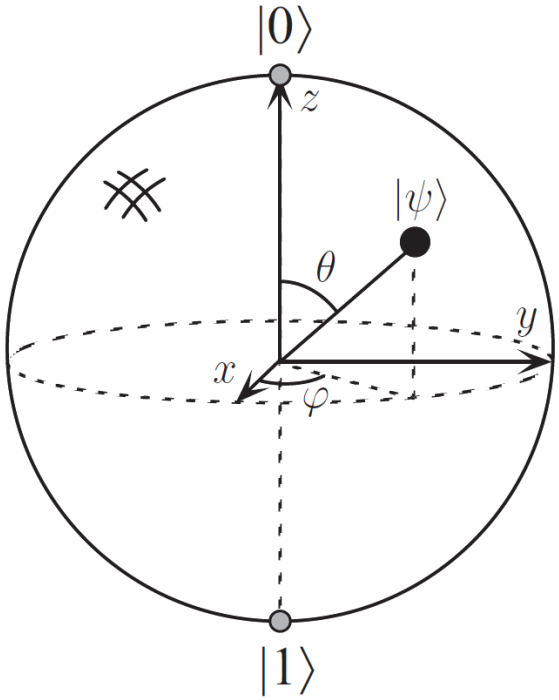
Same relations in the language of circuit diagrams



Bloch sphere (some physical meaning)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

irrelevant



θ: zenith (polar) angle, $0 \le \theta \le \pi$

φ: azimuth angle, $\mod 2\pi$

(often $|0\rangle$ at the bottom, $|1\rangle$ at the top)

Corresponds to direction of a spin in real space

Z axis → $|0\rangle$

$-Z$ axis → $|1\rangle$

Y axis → $\dfrac{|0\rangle + i|1\rangle}{\sqrt{2}}$

$-Y$ axis → $\dfrac{|0\rangle - i|1\rangle}{\sqrt{2}}$

X axis → $\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

$-X$ axis → $\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

XZ plane → $\cos\frac{\theta}{2}|0\rangle \pm \sin\frac{\theta}{2}|1\rangle$

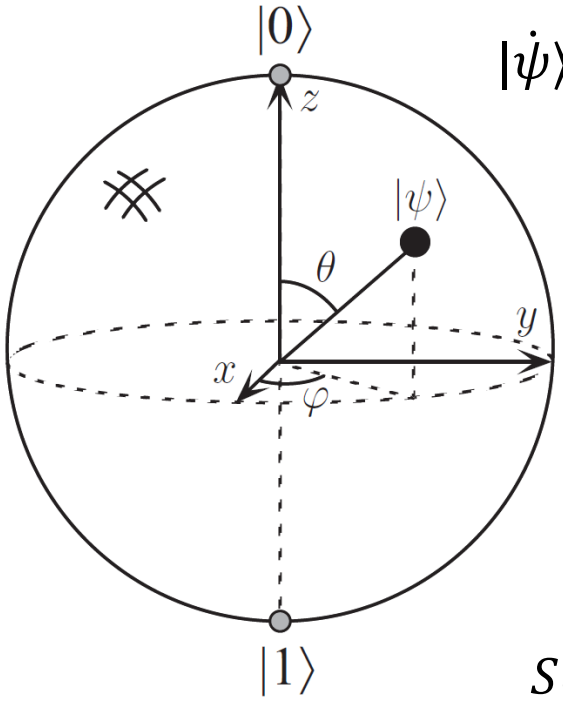
YZ plane → $\cos\frac{\theta}{2}|0\rangle \pm i\sin\frac{\theta}{2}|1\rangle$

equator → $\dfrac{|0\rangle + e^{i\varphi}|1\rangle}{\sqrt{2}}$

Orthogonal vectors in Hilbert space correspond to opposite directions on Bloch sphere

Unitary 1-qubit transformations

Physical evolution leads to a unitary transformation of wavefunctions



$$|\dot{\psi}\rangle = -\frac{i}{\hbar}\hat{H}\,|\psi\rangle \quad \Rightarrow \quad |\psi(t)\rangle = \hat{U}|\psi(0)\rangle, \;\; \hat{U} = e^{-(i/\hbar)\hat{H}t}$$

Since $\hat{H}$ is Hermitian, $\hat{U}$ is unitary

$$\hat{U}\hat{U}^{\dagger} = e^{-(i/\hbar)\hat{H}t}e^{(i/\hbar)\hat{H}^{\dagger}t} = \hat{1}$$

$U(2)$: group of unitary transformations in 2D

$SU(2)$: subgroup of $U(2)$ with $\det = 1$
(since overall phase in not important, S means special)

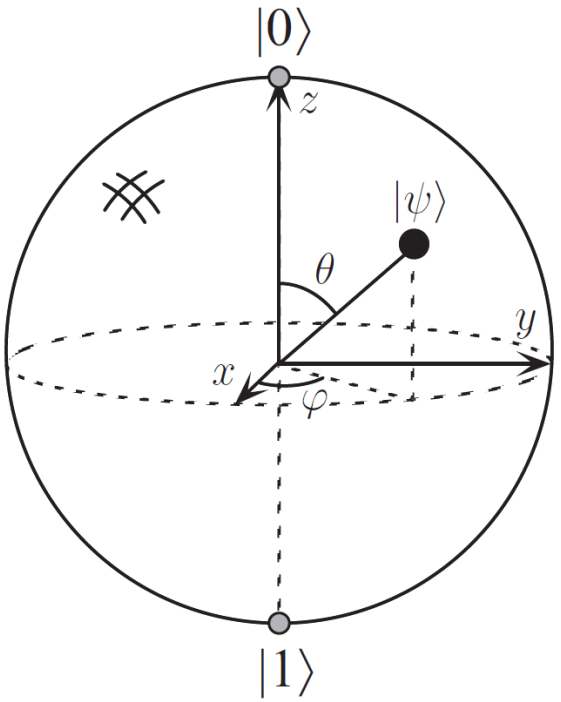
$$SU(2) \leftrightarrow SO(3)$$

(almost isomorphism; actually homomorphism $2 \to 1$, kernel $\pm\hat{1}$)

(special orthogonal in 3D, group of rotations in 3D, "special" means $\det = 1$, not -1)

1-qubit unitary operations correspond to rotations of the Bloch sphere

Main 1-qubit operations on the Bloch sphere



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ Rotation about X-axis by angle π (180°)

(rotation counterclockwise looking from the axis end, but not important since π)

X-axis does not move: $\dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \rightarrow \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

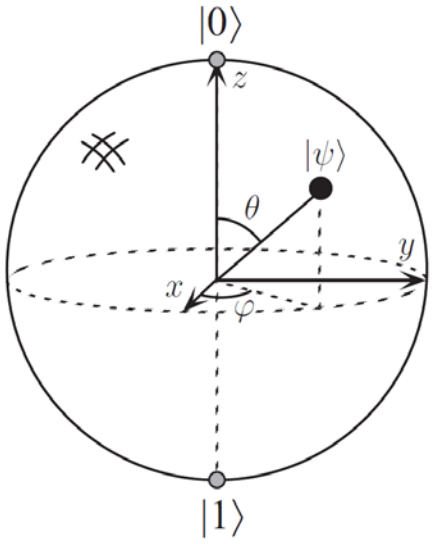
Note that $\dfrac{|0\rangle - |1\rangle}{\sqrt{2}} \rightarrow -\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

(be careful with overall phase)

Larmor rotation (precession) of a real spin by a magnetic field along X

(this is a physical picture, often used in QC; $\omega = \gamma B$, γ is gyromagnetic ratio)

Main 1-qubit operations on the Bloch sphere (cont.)



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Rotation about Z-axis by π (180°)

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Rotation about Y-axis by π (180°)

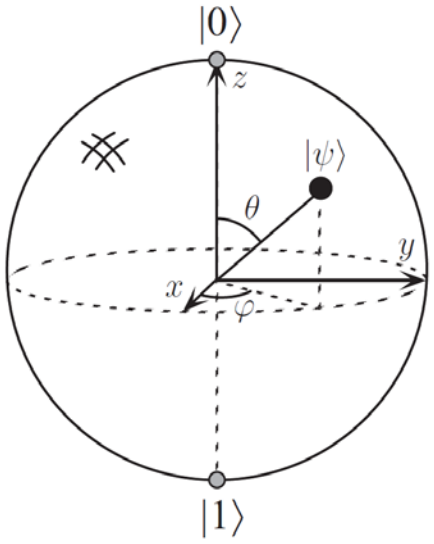
$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Rotation by π about axis in XZ plane, which is at angle $\pi/4$ from Z and X

Now it is obvious why $X^2 = Y^2 = Z^2 = H^2 = \hat{1}$, just a rotation by 2π

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$ Rotation about Z-axis by $\pi/2$ (90°)

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$ Rotation about Z-axis by $\pi/4$ (45°)

Main 1-qubit operations on the Bloch sphere (cont.)



$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotation by π about axis in XZ plane, which is at angle $\pi/4$ from Z and X

Another rotation realization for Hadamard

$$H = R_Y(\pi/2)\, R_Z(\pi)$$

(rotation about Z by π and rotation about Y by $\pi/2$)

Check:

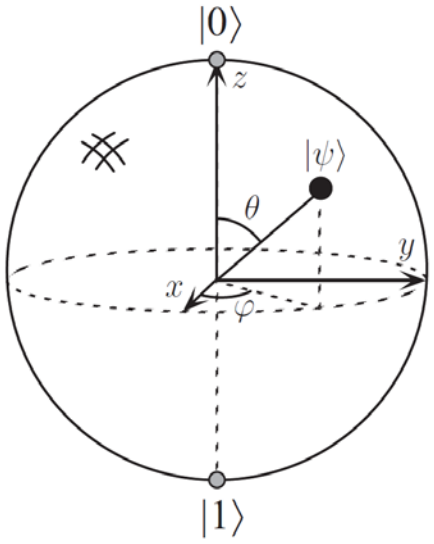
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \rightarrow \frac{-|0\rangle + |1\rangle}{\sqrt{2}}$$

Counterclockwise rotation about Y by $\pi/2$

Arbitrary 1-qubit unitary transformation



An arbitrary 1-qubit unitary can be parametrized as

$$U = e^{i\alpha} \exp\left(-i\frac{\theta}{2}(\vec{n}\vec{\sigma})\right)$$

where $\vec{n}\vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$,

$\alpha,\ \theta,\ n_x, n_y, n_z$ are real numbers, $n_x^2 + n_y^2 + n_z^2 = 1$

α is irrelevant (overall phase), so 3 real parameters

Counterclockwise rotation about axis $\vec{n}$ by angle θ
(this is why $\theta/2$ and "−" sign)

Useful relation: $$\exp\left(-i\frac{\theta}{2}(\vec{n}\vec{\sigma})\right) = \cos\frac{\theta}{2}\,\hat{1} - i\sin\frac{\theta}{2}\,(\vec{n}\vec{\sigma})$$

follows from $(\vec{n}\vec{\sigma})^2 = \hat{1}$

EE214/PHYS220 Quantum Computing

Two-qubit states and 2-qubit gates

Two-qubit states

$$|\psi\rangle = \underset{\alpha_1}{\alpha_{00}}|00\rangle + \underset{\alpha_2}{\alpha_{01}}|01\rangle + \underset{\alpha_3}{\alpha_{10}}|10\rangle + \underset{\alpha_4}{\alpha_{11}}|11\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

$$\begin{cases} |\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1 \\ \text{Overall phase is not important} \Rightarrow \text{can choose } \alpha_0 \text{ real} \end{cases}$$

$8 - 2 = 6$ degrees of freedom ($2 \cdot 2^k - 2$ degrees of freedom for k qubits)

Two-qubit states

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

$8 - 2 = 6$ degrees of freedom

Tensor-product states (outer-product, direct-product): each qubit is some state

$$(\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) =$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

$2 + 2 = 4$ degrees of freedom

A general 2-qubit state is a tensor-product state only if $\alpha_{00}\alpha_{11} = \alpha_{10}\alpha_{01}$.

Otherwise – entanged.

Notations for multi-qubit computational-basis states

$$|x_3\, x_2\, x_1\, x_0\rangle \equiv |x_3\rangle|x_2\rangle|x_1\rangle|x_0\rangle \equiv |x_3\rangle \otimes |x_2\rangle \otimes |x_1\rangle \otimes |x_0\rangle$$

Computational-basis state, represents classical state $\sum x_n 2^n$

$|x_3\rangle$ ——————— (most significant bit at the top)

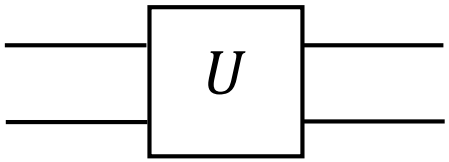
$|x_2\rangle$ ———————

$|x_1\rangle$ ———————

$|x_0\rangle$ ———————

However, people often say in opposite order: first qubit, second qubit, etc.

Two-qubit gates



Any unitary 4×4 matrix
(overall phase is not important)

Can be defined by transformation of the basis vectors:

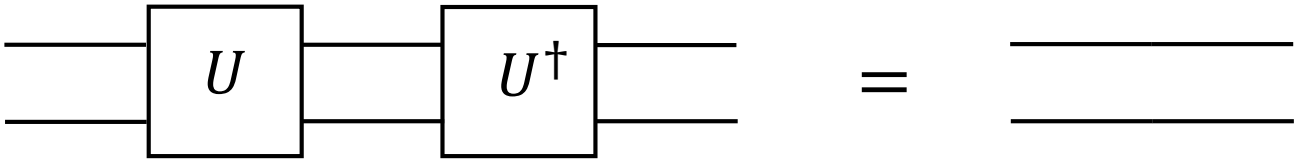
$|00\rangle \rightarrow$ … (4 complex numbers)
$|01\rangle \rightarrow$ … (4)
$|10\rangle \rightarrow$ … (4)
$|11\rangle \rightarrow$ … (4)

Then linearity

Degrees of freedom: $32 - 16 - 1 = 15$ (for k qubits $4^k - 1$)

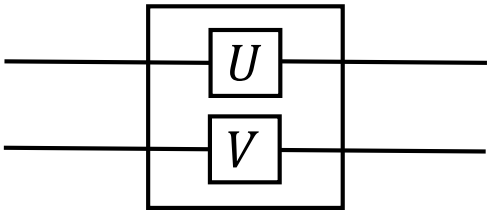
unitary overall phase

Reversible:



Examples of two-qubit gates

1. Trivial: tensor-product gates

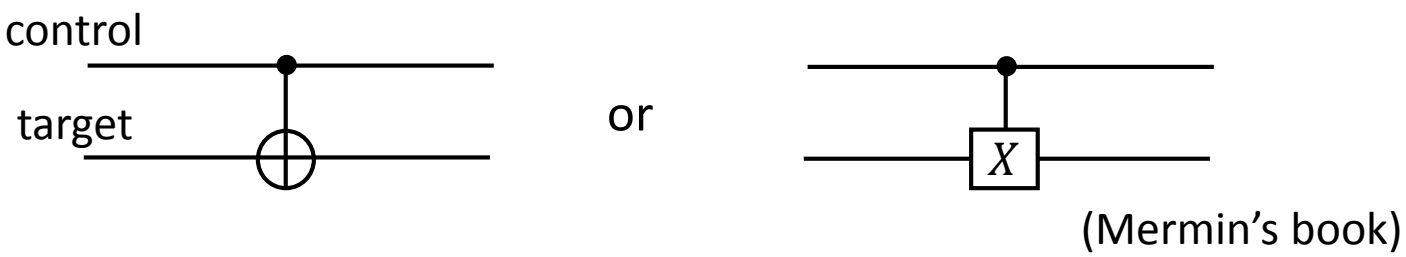


Math structure: tensor-product of matrices

$$U \otimes V = \begin{pmatrix} U_{00}\begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix} & U_{01}\begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix} \\ U_{10}\begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix} & U_{11}\begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix} \end{pmatrix}$$

2– 5. Many gates are of controlled type: one qubit controls the other one (consider next)

2. Controlled-NOT (CNOT)



(Mermin's book)

Generalizes classical CNOT: target bit is flipped if control bit is 1

control target

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle$

$|11\rangle \rightarrow |10\rangle$

Matrix

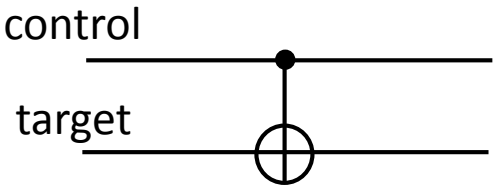
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

← 00
← 01
← 10
← 11

Unitary matrix; can be checked, but actually trivial, because a permutation of computational basis

$$\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle \longrightarrow \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_3|10\rangle + \alpha_2|11\rangle$$

CNOT (cont.)



Notation: CNOT_{ij} or C_{ij} (Mermin's book)

control target

$$\mathrm{CNOT}_{10}|x\rangle|y\rangle = |x\rangle\,|y \oplus x\rangle$$

for computational basis

qubit 1 qubit 0 addition modulo 2

(again, transformation for other states defined by linearity)

$$\mathrm{CNOT}_{01}|x\rangle|y\rangle = |x \oplus y\rangle\,|y\rangle$$

Actually, not possible to say that nothing happens to the control qubit; this is true only if it is $|0\rangle$ or $|1\rangle$. If control qubit is in a superposition state, it gets entangled with the target qubit. Then it will not have a state by itself, and it may depend on what happens next with the target qubit.

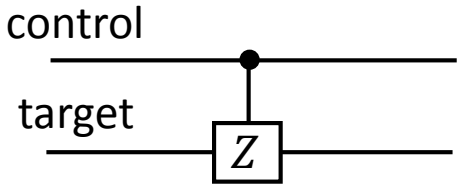
Example

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \longrightarrow$$

Control changes, while target the same!

$$\longrightarrow \frac{|00\rangle - |01\rangle + |11\rangle - |10\rangle}{2} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

3. Controlled-Z (CZ)



Phase-flip of target if control is $|1\rangle$

control target

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |10\rangle$

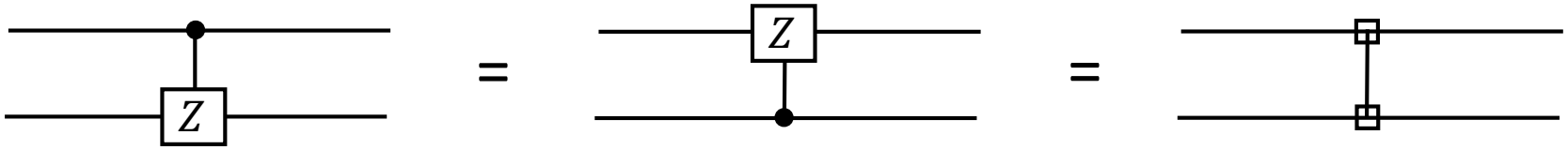
$|11\rangle \rightarrow -|11\rangle$

Matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

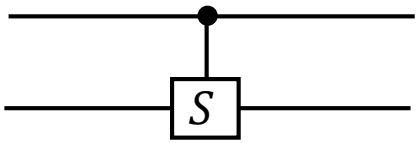
← 00
← 01
← 10
← 11

Somewhat surprisingly, symmetric



(Nielsen-Chuang)

4. Controlled-phase (C-phase)



Phase-S gate if control is $|1\rangle$

control target

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |10\rangle$

$|11\rangle \rightarrow i|11\rangle$

Matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

← 00

← 01

← 10

← 11

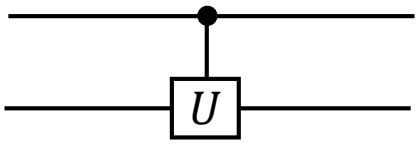
Also symmetric

Note that often controlled-phase means

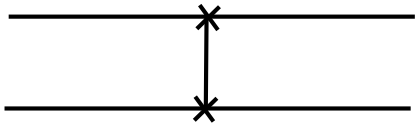
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi} \end{pmatrix}$$

Examples of two-qubit gates (cont.)

5. Any controlled-U



5. SWAP



(Nielsen-Chuang)

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |10\rangle$

$|10\rangle \rightarrow |01\rangle$

$|11\rangle \rightarrow |11\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notation SWAP_{ij}

S_{ij} (Mermin)

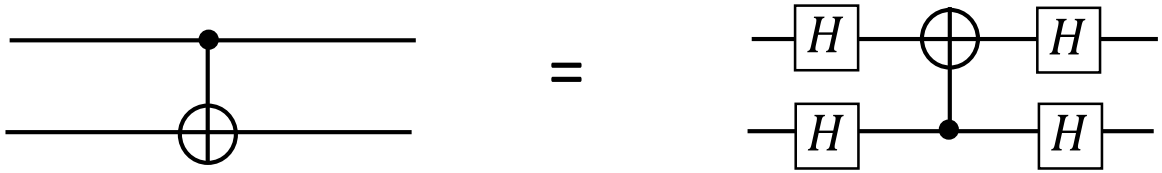
Symmetric

$$\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle \longrightarrow \alpha_0|00\rangle + \alpha_2|01\rangle + \alpha_1|10\rangle + \alpha_3|11\rangle$$

Useful relations between two-qubit gates

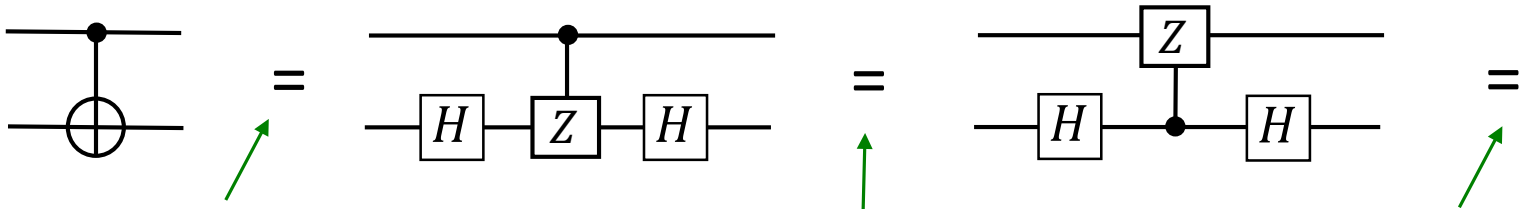
1. $\left(\mathrm{CNOT}_{ij}\right)^2 = \left(\mathrm{CZ}_{ij}\right)^2 = \left(\mathrm{SWAP}_{ij}\right)^2 = \hat{1}$

2. $\mathrm{CNOT}_{ij} = \left(H_i H_j\right) \mathrm{CNOT}_{ji} \left(H_i H_j\right)$



So, who controls whom is a matter of preference!

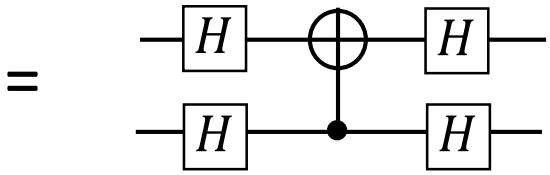
Proof



because $HZH = X$ (if control=1)
while $H^2 = \hat{1}$ (if control=0)

symmetric CZ

similar to the first step:
$HXH = Z,\ H^2 = \hat{1}$



Sufficient to prove only for basis states!

$$\mathrm{CNOT}_{ij} = (H_i H_j)\,\mathrm{CNOT}_{ji}\,(H_i H_j) \quad \text{(cont.)}$$

Another proof

$$\mathrm{CNOT}_{ij} = \underbrace{\tfrac{1}{2}(I_i + Z_i)}_{\begin{pmatrix}1&0\\0&0\end{pmatrix}\ \text{selects state } |0\rangle}\, I_j \;+\; \underbrace{\tfrac{1}{2}(I_i - Z_i)}_{\begin{pmatrix}0&0\\0&1\end{pmatrix}\ \text{selects state } |1\rangle}\, X_j =$$

$I \equiv \hat{1}$

rearrange

$$= I_i \tfrac{1}{2}(I_j + X_j) + Z_i \tfrac{1}{2}(I_j - X_j) =$$

(note that $X \leftrightarrow Z$ corresponds to $i \leftrightarrow j$)

now exchange order and multiply by $\hat{1}$ from both sides

$$= (H_i H_j)\underbrace{(H_i H_j)\left[\tfrac{1}{2}(I_j + X_j) I_i + \tfrac{1}{2}(I_j - X_j) Z_i\right](H_i H_j)}(H_i H_j) =$$

$(HXH = Z,\ HZH = X)$

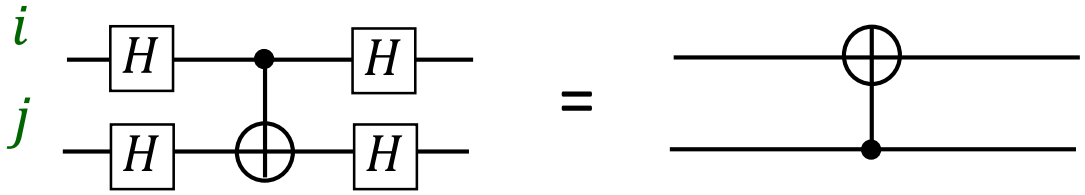
$$= (H_i H_j)\underbrace{\left[\tfrac{1}{2}(I_j + Z_j) I_i + \tfrac{1}{2}(I_j - Z_j) X_i\right]}_{\mathrm{CNOT}_{ji}}(H_i H_j) =$$

$$= (H_i H_j)\,\mathrm{CNOT}_{ji}\,(H_i H_j)$$

$\mathrm{CNOT}_{ij} = (H_i H_j)\,\mathrm{CNOT}_{ji}\,(H_i H_j)$ (cont.)

One more (direct) proof

Let us prove the opposite (equivalent) relation



$$|00\rangle \xrightarrow{H_i H_j} \frac{|0\rangle + |1\rangle}{\sqrt{2}}\,\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \longrightarrow$$

$$\xrightarrow{\mathrm{CNOT}_{ij}} \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) \xrightarrow{H_i H_j} |00\rangle$$

(the same)

(as should be)

$$|01\rangle \xrightarrow{H_i H_j} \frac{|0\rangle + |1\rangle}{\sqrt{2}}\,\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \longrightarrow$$

$$\xrightarrow{\mathrm{CNOT}_{ij}} \frac{1}{2}(|00\rangle - |01\rangle + |11\rangle - |10\rangle) = \frac{(|0\rangle - |1\rangle)\,|0\rangle - (|0\rangle - |1\rangle)\,|1\rangle}{2} =$$

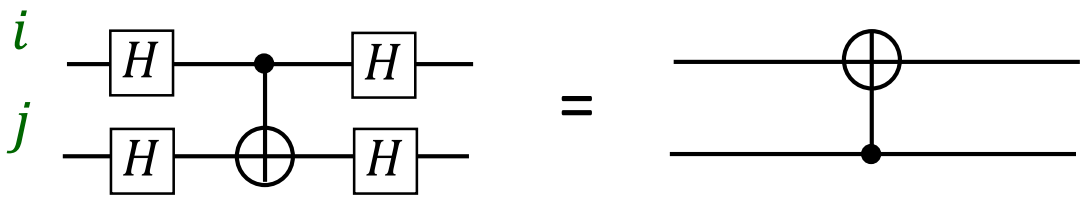
$$= \frac{|0\rangle - |1\rangle}{\sqrt{2}}\,\frac{|0\rangle - |1\rangle}{\sqrt{2}} \xrightarrow{H_i H_j} |11\rangle$$

since $H^2 = \hat{1}$

(as should be)

Two more initial states

$$\mathrm{CNOT}_{ij} = \left(H_i H_j\right) \mathrm{CNOT}_{ji} \left(H_i H_j\right) \quad \text{(cont.)}$$



(equivalent relation, $i \leftrightarrow j$)

Already showed: $|00\rangle \longrightarrow |00\rangle$, $|01\rangle \longrightarrow |11\rangle$

$$|10\rangle \xrightarrow{H_i H_j} \frac{0-1}{\sqrt{2}} \frac{0+1}{\sqrt{2}} \xrightarrow{\mathrm{CNOT}_{ij}} \frac{0\,(0+1)-1\,(1+0)}{2} = \frac{0-1}{\sqrt{2}} \frac{0+1}{\sqrt{2}} \xrightarrow{H_i H_j} |10\rangle$$

(for brevity do not write ket-notation)

$$|11\rangle \xrightarrow{H_i H_j} \frac{0-1}{\sqrt{2}} \frac{0-1}{\sqrt{2}} \xrightarrow{\mathrm{CNOT}_{ij}} \frac{0\,(0-1)-1\,(1-0)}{2} = \frac{0+1}{\sqrt{2}} \frac{0-1}{\sqrt{2}} \xrightarrow{H_i H_j} |01\rangle$$

We proved the relation for 4 initial basis states $\Rightarrow$ should hold for <u>any</u> initial state

Important example, it shows that CNOT is not a one-way action,
this is an interaction (has “quantum back-action”)

Useful relations between two-qubit gates (cont.)

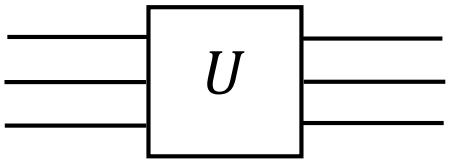
3. $\mathrm{SWAP}_{ij} = \mathrm{CNOT}_{ij}\ \mathrm{CNOT}_{ji}\ \mathrm{CNOT}_{ij}$

Proof

Again, consider only (computational) basis states for initial state

$$|x\rangle_i\,|y\rangle_j \xrightarrow{\mathrm{CNOT}_{ij}} |x\rangle_i\,|x\oplus y\rangle_j \xrightarrow{\mathrm{CNOT}_{ji}} |x\oplus x\oplus y\rangle_i\,|x\oplus y\rangle_j = |y\rangle_i\,|x\oplus y\rangle_j \xrightarrow{\mathrm{CNOT}_{ij}} |y\rangle_i\,|x\oplus y\oplus y\rangle_j = |y\rangle_i\,|x\rangle_j$$

Three-qubit gates



Any unitary 8×8 matrix
(overall phase is not important)

In general characterized by $4^3 - 1 = 63$ real numbers

Two most important 3-qubit gates: Toffoli and Fredkin

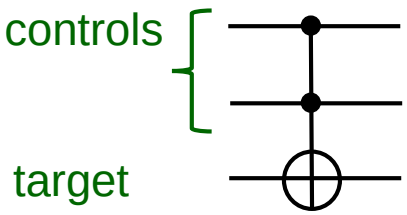
Paper “Reversible computing” by T. Toffoli, Lecture notes in computer science, vol. 85 (1980), ICALP 1980.

“Conservative logic” by Edward Fredkin and Tommaso Toffoli, Int. J. Theor. Phys. 21, 219, (1982),

Toffoli gate

Toffoli gate: controlled-controlled-NOT

Flip target if both controls are 1



$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & 0 & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & 0 & & & & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

$|110\rangle \leftrightarrow |111\rangle$

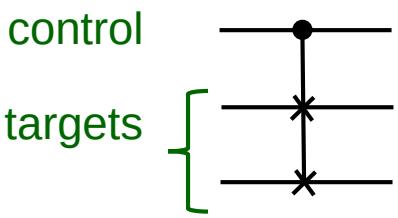
other basis states do not change

Can be constructed with CNOTs and 1-qubit gates (6 CNOTs needed)

Classical Toffoli gate is sufficient for classical reversible computation (to beat informational limit for energy dissipation in computation). Cannot be decomposed into 2-bit gates (in contrast to the quantum case), minimal gate for reversible computation. Realizes AND if target 0, NAND if target 1.

Fredkin gate

Fredkin gate: controlled-SWAP



$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & 0 & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 0 & 1 & \\ 0 & & & & & 1 & 0 & \\ & & & & & & & 1 \end{pmatrix}$$

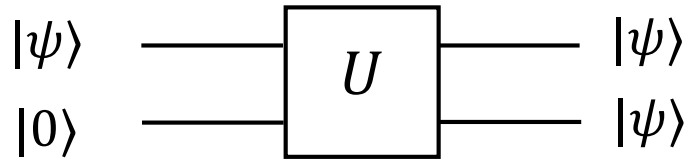
$|101\rangle \leftrightarrow |110\rangle$

other basis states do not change

Classical Fredkin realizes AND (in lower target) if lower target is initially 0.

No fan-out gate (no-cloning theorem)

Theorem: impossible to realize fan-out gate



Proof Assume $U\,|0\rangle|0\rangle = |0\rangle|0\rangle$

$U\,|1\rangle|0\rangle = |1\rangle|1\rangle$

Then $U\,(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha\,|0\rangle|0\rangle + \beta|1\rangle|1\rangle$, while for desired cloning

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \longrightarrow (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) =$$
$$= \alpha^2\,|0\rangle|0\rangle + \beta^2|1\rangle|1\rangle + \alpha\beta\,|0\rangle|1\rangle + \alpha\beta\,|1\rangle|0\rangle$$

(a different state!)

More general proof Assume $U\,|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$

$U\,|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle$

Unitary operation preserves inner product, therefore $\langle\phi|\psi\rangle = (\langle\phi|\psi\rangle)^2$.

This is possible only when $\langle\phi|\psi\rangle = 0$ or 1 (i.e., can clone only orthogonal states)