# Analysis of the radio-frequency single-electron transistor with large quality factor

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We have analyzed the response and noise-limited sensitivity of the radio-frequency single-electron transistor (rf-SET), extending the previously developed theory to the case of arbitrary large quality factor $Q$ of the rf-SET tank circuit. It is shown that while the rf-SET response reaches the maximum at $Q$ roughly corresponding to the impedance matching condition, the rf-SET sensitivity worsens monotonically with the increase of $Q$. In addition, we propose an operation mode in which an overtone of the incident rf wave is in resonance with the tank circuit. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1614840]

The problem of relatively small bandwidth of the conventional single-electron transistor,[1,2] (SET) due to its large output resistance, has been solved for many applications by the invention[3] of the radio-frequency SET (rf-SET), which in many instances has already replaced the traditional SET setup. The principle of the rf-SET operation is somewhat similar to the operation of the rf superconducting quantum interference device[4] and is based on the microwave reflection[3,5–8] from a tank circuit containing the SET (Fig. 1), which affects the quality factor ($Q$-factor) of the tank; another possibility is to use the transmitted wave.[9,10] The wide bandwidth of the rf-SET is due to the signal propagation by the microwave instead of charging the output wire, while the tank circuit provides a better match between the cable wave impedance $R_0 = 50$ $\Omega$ and much larger SET resistance ($\sim 10^5$ $\Omega$).

The rf-SET bandwidth over 100 MHz has been demonstrated[3] using the microwave carrier frequency $\omega/2\pi = 1.7$ GHz and relatively low $Q$-factor $Q \simeq 6$. However, in the present-day experiments, the bandwidth is typically about 10 MHz because of lower carrier frequency (to reduce amplifier noise) and higher $Q$-factor (as an example, the bandwidth of 7 MHz for $\omega/2\pi = 332$ MHz and $Q > 20$ has been reported in Ref. 6). Since the SET sensitivity is limited by the $1/f$ noise only at frequencies below a few kilohertz, the rf-SET typically operates in the frequency range of shot-noise-limited sensitivity of the SET.[11,12] The rf-SET charge sensitivity of $3.2 \times 10^{-6} e/\sqrt{\text{Hz}}$ (4.8$\hbar$ in energy units) at 2 MHz has been reported in Ref. 6 (this value still contains comparable contributions from the SET shot noise and amplifier noise). Such sensitivity and bandwidth are almost enough for a single-shot readout of a Cooper-pair-box qubit.

In spite of significant experimental rf-SET activity, we are aware of only few theoretical papers on the rf-SETs. The basic theory of the shot-noise-limited sensitivity of the rf-SET has been developed in Ref. 13. A similar theory has been applied to the analysis of the sensitivity of the rf-SET-based micromechanical displacement detector.[14,15] Some theoretical analysis of the transmission-type rf-SET can be found in Ref. 10.

In this letter, we extend the theory of Ref. 13 to the case of arbitrary large $Q$-factor of the tank circuit, removing the assumption (violated in the present-day experiments) of $Q$ being much smaller than the impedance-matching value. We calculate the response and sensitivity of the normal-metal rf-SET and find the optimal values numerically. Besides the usual case of the carrier wave in resonance with the tank circuit, we also consider the regime of a resonant overtone and find a comparable rf-SET performance in this case.

We consider a SET (Fig. 1) consisting of two tunnel junctions with capacitances $C_{1j}$ and $C_{2j}$ and resistances $R_1$ and $R_2$. The measured charge source $q_S$ has the capacitance $C_S = C_{S1} + C_{S2}$ and is coupled to the SET via capacitance $C_g$. Assuming constant $q_S$ (neglecting back-action), the SET can be reduced to the effective double-junction SET with capacitances $C_1 = C_{1j} + C_g C_{S1}/(C_g + C_S)$, $C_2 = C_{2j} + C_g C_{S2}/(C_g + C_S)$ and background charge $q_0 = q_{00} + q_S C_g/(C_g + C_S)$, where $q_{00}$ is the initial contribution. We will calculate the rf-SET response and sensitivity in respect to $q_0$, while the corresponding quantities in respect to the measured charge $q_S$ differ by the factor $C_g/(C_g + C_S)$.

The current $I(t)$ through the SET affects the quality factor of the tank circuit consisting of the capacitance $C_T$ and inductance $L_T$. In the linear approximation, the SET can be replaced by an effective resistance $R_d$, and the total ("loaded") quality factor $Q_L = (1/Q + 1/Q_{\text{SET}})^{-1}$ has contributions from the "unloaded" $Q$-factor $Q = \sqrt{L_T/C_T}/R_0$ and damping by the SET: $Q_{\text{SET}} = R_d/\sqrt{L_T/C_T}$. For the incoming voltage wave $\hat{V}_{\text{in}} \exp(i\omega t)$, the reflected wave $\alpha \hat{V}_{\text{in}} \exp(i\omega t)$ depends on the reflection coefficient $\alpha = (Z - R_0)/(Z + R_0)$, where $Z = i\omega L_T + (i\omega C_T + 1/R_d)^{-1}$; close to the resonance, $\omega \approx \omega_0 = (L_T C_T)^{-1/2}$, it can be approximated as $Z \approx L_T/C_T R_d + 2i(L_T/C_T)^{1/2}\Delta\omega/\omega_0$, $\Delta\omega \equiv \omega - \omega_0$. Since an increment of the measured charge $q_S$ leads to an increment
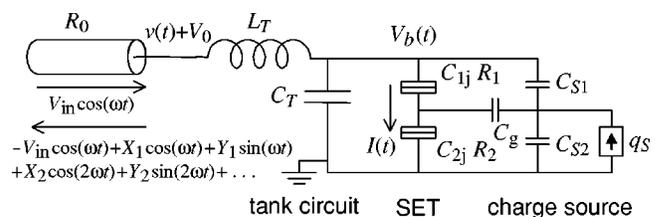


FIG. 1. Schematic of the rf-SET.

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of $R_d$, the rf-SET response is proportional to $d\alpha/dR_d$. However, the amplitude $\hat{V}_b = \hat{V}_{in}(Z+R_0)(i\omega C_T + 1/R_d)/2$ of the SET bias voltage oscillations should be determined by the Coulomb blockade threshold; so a more representative quantity is

$$\frac{d\alpha}{dR_d}\frac{\hat{V}_{in}}{\hat{V}_b} \approx \frac{-iR_0}{R_d^2}\frac{Q}{1+Q^2R_0/R_d}\frac{1}{1+2iQ_L\Delta\omega/\omega_0}. \tag{1}$$

This equation shows that the rf-SET response reaches the maximum at $Q = (R_d/R_0)^{1/2}$, which is the case of matched impedances at resonance, $Z \approx R_0$, and corresponds to the condition $Q = Q_{SET} = 2Q_L$.

The linear analysis can only be used as an estimate because of the significant nonlinearity of the SET $I$–$V$ dependence. For the full analysis, we use the differential equation[13] $\ddot{v}/\omega_0^2 + \dot{v}/Q\omega_0 + v = 2(1-\omega^2/\omega_0^2)V_{in}\cos\omega t - R_0[I(t)-\langle I\rangle]$, where $v(t) = V_{in}\cos\omega t + V_{out}(t)$ is the voltage at the end of the cable with subtracted dc component $V_0$ (see Fig. 1; do not use complex representation any more). The SET current $I(t)$ and its dc value $\langle I\rangle$ are found self-consistently from the SET bias voltage $V_b(t) = V_0 + v(t) + [2V_{in}\omega\sin\omega t + \dot{v}(t)]Q/\omega_0$ using the "orthodox" model[1] and assuming continuous SET current ($\omega \ll I/e$).

In the steady state, the reflected wave can be represented as $V_{out}(t) = -V_{in}\cos\omega t + \Sigma_{n=1}^{\infty}[X_n\cos n\omega t + Y_n\sin n\omega t]$ with the following coefficients $X_n$ and $Y_n$:

$$X_n = \{R_0Q[n\tilde{\omega}a_n - Q(1-n^2\tilde{\omega}^2)b_n] + 2Q^2(1-\tilde{\omega}^2)^2$$
$$\times V_{in}\delta_{1n}\}/[n^2\tilde{\omega}^2 + Q^2(1-n^2\tilde{\omega}^2)^2], \tag{2}$$

$$Y_n = \{-R_0Q[n\tilde{\omega}b_n + Q(1-n^2\tilde{\omega}^2)a_n] + 2Q\tilde{\omega}(1-\tilde{\omega}^2)$$
$$\times V_{in}\delta_{1n}\}/[n^2\tilde{\omega}^2 + Q^2(1-n^2\tilde{\omega}^2)^2], \tag{3}$$

$$a_n = 2\langle I(t)\sin n\omega t\rangle, \quad b_n = 2\langle I(t)\cos n\omega t\rangle, \tag{4}$$

where $\tilde{\omega} \equiv \omega/\omega_0$, $\delta_{1n}$ is the Kronecker symbol, and averaging is over the oscillation period, while $I(t)$ is determined by the SET voltage $V_b(t) = V_0 + 2Q\tilde{\omega}V_{in}\sin\omega t + \Sigma_{n=1}^{\infty}[(X_n + Qn\tilde{\omega}Y_n)\cos n\omega t + (Y_n - Qn\tilde{\omega}X_n)\sin n\omega t]$. Notice that the linear approximation corresponds to neglecting the contribution of overtones ($n \geq 2$); then $R_d = \pi A/[\int_0^{2\pi}I(V_0 + A\sin x)\sin x\,dx]$, where $A$ is the amplitude of $V_b$ oscillations, $V_b(t) = V_0 + A\sin(\omega t + \phi)$, while there is no effective reactance contribution. We used the self-consistent linear approximation as a starting point for the iterative solution of Eqs. (2)–(4).

The rf-SET response in respect to monitoring the quadrature component $X_n$ can be defined as a derivative $dX_n/dq_0$ (similarly, $dY_n/dq_0$ for $Y_n$ monitoring). Other experimentally relevant definitions are for monitoring the optimized phase-shifted combination $X_n\cos\varphi + Y_n\sin\varphi$ and/or the reflected wave amplitude; however, in the cases considered subsequently there is only one leading quadrature, so that different definitions practically coincide.

The corresponding noise-limited sensitivity (minimal detectable charge for the measurement bandwidth $\Delta f$) is defined as $\delta q_0/\sqrt{\Delta f} = \sqrt{S_{Xn}}/|dX_n/dq_0|$ (similarly, $\sqrt{S_{Yn}}/|dY_n/dq_0|$), where the low-frequency spectral densities $S_{Xn}$ and $S_{Yn}$ of quadrature fluctuations are

$$S_{Xn} = c_n^2\langle S_I(t)\sin^2 n\omega t\rangle + d_n^2\langle S_I(t)\cos^2 n\omega t\rangle$$
$$-c_nd_n\langle S_I(t)\sin 2n\omega t\rangle, \tag{5}$$
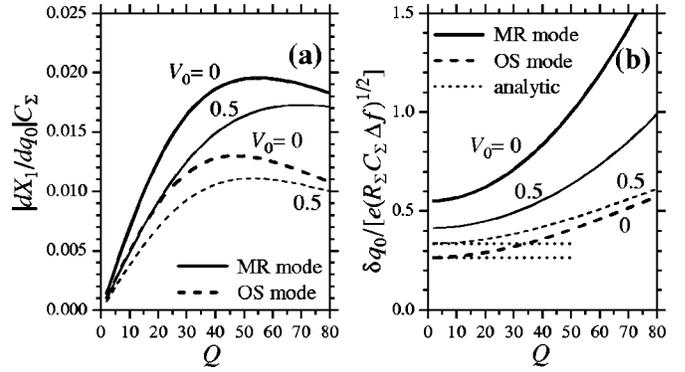


FIG. 2. (a) Rf-SET response and (b) sensitivity as functions of the $Q$-factor in the maximum response (MR) and optimal sensitivity (OS) modes. $T = 0.01\,e^2/C_\Sigma$, $R_\Sigma/R_0 = 2000$, $\omega = \omega_0$.

$$S_{Yn} = d_n^2\langle S_I(t)\sin^2 n\omega t\rangle + c_n^2\langle S_I(t)\cos^2 n\omega t\rangle$$
$$+ c_nd_n\langle S_I(t)\sin 2n\omega t\rangle, \tag{6}$$

where $c_n = (2R_0Qn\tilde{\omega})/[n^2\tilde{\omega}^2 + Q^2(1-n^2\tilde{\omega}^2)^2]$, $d_n = c_nQ(1-n^2\tilde{\omega}^2)/n\tilde{\omega}$, $S_I(t)$ is the low-frequency spectral density of the SET shot noise[12] with the time dependence due to oscillating bias voltage, and the averaging is over the period $2\pi/\omega$.

Figure 2 shows the numerically calculated rf-SET response and sensitivity as functions of the "unloaded" $Q$-factor for a symmetric SET,[16] $C_1 = C_2 = C_\Sigma/2$, $R_1 = R_2 = R_\Sigma/2$, with $R_\Sigma = 100$ k$\Omega$ at temperature $T = 0.01e^2/C_\Sigma$ for the case of resonant carrier frequency, $\omega = \omega_0$. Both the response and sensitivity are shown with respect to the quadrature $X_1$ since all other components are small. The rf-SET performance is optimized over the wave amplitude $V_{in}$ and the charge $q_0$ to provide either maximum response (MR mode; solid lines) or optimized sensitivity (OS mode; dashed lines).[17] We show the results for two values of the dc bias voltage $V_0$. The case $V_0 = 0$ provides the best MR response and the best OS sensitivity, and corresponds to the symmetric SET operation with respect to positive and negative bias voltages. The other value shown, $V_0 = 0.5e/C_\Sigma$, represents a typical case when only one branch of the SET $I$–$V$ curve is used, and corresponds to the plateau-like region[13] of the response and sensitivity dependences on $V_0$.

As seen from Fig. 2(a), the maximum rf-SET response is achieved at $Q$-factors (somewhat different in different regimes) comparable to the rough estimate $(R_\Sigma/R_0)^{1/2} \simeq 45$ (although this maximum does not actually correspond to the minimum of reflection). In contrast to the response behavior, the rf-SET sensitivity [Fig. 2(b)] worsens monotonically with $Q$. Qualitatively, this happens because the noise $S_{X1}$ in Eq. (5) is proportional to $Q^2$, while the response does not grow as fast as $Q$. At low $Q$, the OS sensitivity is fitted well by the analytical result[13] $\delta q_0 \simeq 2.65e(R_\Sigma C_\Sigma\Delta f)^{1/2}(TC_\Sigma/e^2)^{1/2}$ for $V_0 = 0$, and $\delta q_0 \simeq 3.34C_\Sigma(R_\Sigma T\Delta f)^{1/2}$ for the asymmetric operation (shown by dotted lines). However, at realistic $Q$-factors, $\delta q_0$ is significantly larger (by about 50% at $Q = 50$ for data in Fig. 2). Another interesting observation from Fig. 2 is that the response in the MR mode is only moderately ($\sim 30\%$) better than in the OS mode. These findings show that in order to improve experimental rf-SET sensitivity, it is preferable to use smaller $Q$-factor (at least not exceeding the
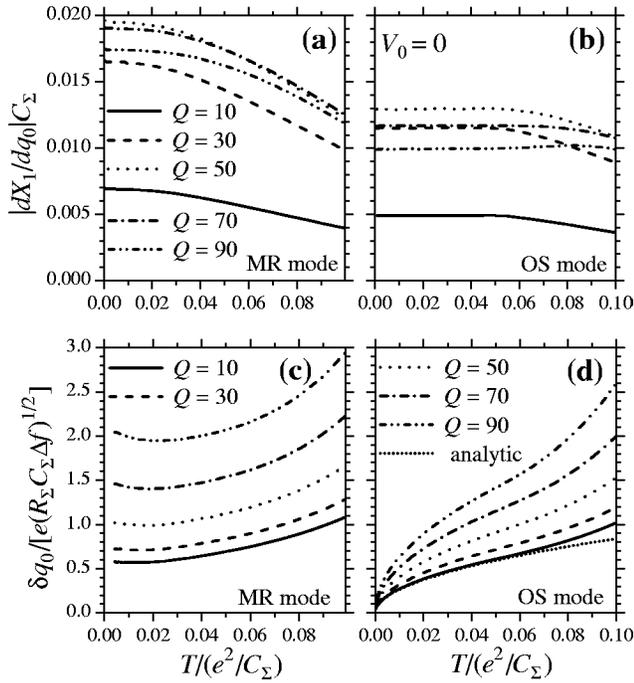
FIG. 3. Dependence of (a) and (b) rf-SET response and (c) and (d) sensitivity on temperature $T$ in the MR and OS modes for several $Q$-factors. $V_0=0$, $R_\Sigma/R_0=2000$, and $\omega=\omega_0$.



FIG. 4. (a) Rf-SET response and (b) sensitivity in the regimes when the second or third overtone of the incident rf wave is in resonance with the tank circuit. $T=0.01\,e^2/C_\Sigma$, $R_\Sigma/R_0=2000$.

matching value) and smaller rf wave amplitude (to be closer to the OS mode than to MR mode).

Figure 3 shows the temperature dependence of the rf-SET response and sensitivity in the MR and OS modes. Even though the low-$T$ analytical formula for the OS sensitivity (mentioned previously) works well only for small $Q$, the $T^{1/2}$ dependence at $T\lesssim0.05e^2/C_\Sigma$ remains valid for large $Q$-factors (at very small $T$, the OS sensitivity is limited by the neglected here contribution from cotunneling processes[11,18]). The OS response practically does not depend on temperature at $T\lesssim0.05e^2/C_\Sigma$. The performance in the MR mode saturates below $T\simeq0.03e^2/C_\Sigma$.

So far, we have been considering the usual case $\omega=\omega_0$. In spite of significant SET $I$–$V$ nonlinearity (the SET nonlinearity has been recently used[19] for rf mixing), the contribution of overtones in this case is small because they are off resonance. Even though the formally calculated sensitivities with respect to overtones are comparable to the $X_1$ sensitivity (worse by less than two times for $n=2$ and 3), the responses are much smaller and therefore monitoring of overtones is impractical. However, the contribution of $n$th overtone becomes significant if $\omega\simeq\omega_0/n$. Figure 4 shows the rf-SET response and sensitivity for $\omega=\omega_0/2$ and $\omega=\omega_0/3$, in respect to monitoring $Y_2$ and $Y_3$, correspondingly (the $X$-quadratures are small). We use $V_0=0$ in the case $\omega=\omega_0/3$ and $V_0=0.5e/C_\Sigma$ in the case $\omega=\omega_0/2$ (for $V_0=0$ there is no second overtone because of the $I$–$V$ curve symmetry).

Comparing Figs. 2 and 4 (the parameters are the same), we see that the rf-SET performance in the regime of a resonant overtone is comparable to the performance in the conventional regime $\omega=\omega_0$ (the MR response and OS sensitivity are worse by about 1.5 times).[20] On the other hand, the frequency separation between the incident wave and monitored reflected wave may be an important advantage for some applications. In addition, it may be advantageous to have the absence of the monitored wave when the SET is off (no current), while for the conventional mode, this case corresponds to the largest reflected power. The disadvantage is a larger incident wave amplitude $V_{in}$ than for a conventional rf-SET regime, which may lead to heating problems. Nevertheless, we hope that the proposed mode of the resonant overtone will be practically useful.
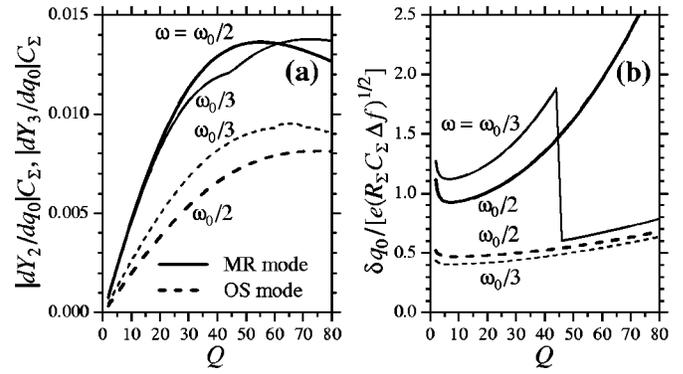
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