Bell-inequality violation versus entanglement in the presence of local decoherence

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We analyze the effect of local decoherence of two qubits on their entanglement and the Bell-inequality violation. Decoherence is described by Kraus operators, which take into account dephasing and energy relaxation at an arbitrary temperature. We show that in the experiments with superconducting phase qubits the survival time for entanglement should be much longer than for the Bell-inequality violation.

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Entanglement of separated systems is a genuine quantum effect and an essential resource in quantum information processing [1]. Experimentally, a convincing evidence of a two-qubit entanglement is a violation of the Bell inequality [2] in its Clauser-Horne-Shimony-Holt [3] (CHSH) form. However, only for pure states the entanglement always [4] results in a violation of the CHSH inequality. In contrast, some mixed entangled two-qubit states (as we will see, most of them) do not violate the CHSH inequality [5], though they may still exhibit nonlocality in other ways [6]. Distinction between entanglement and Bell-CHSH-inequality violation, in its relevance to experiments with superconducting phase qubits [7], is the subject of our paper.

The two-qubit entanglement is usually characterized by the concurrence [8] C or by the entanglement of formation [9], which is a monotonous function [8] of C. Nonentangled states have $C=0$, while $C=1$ corresponds to maximally entangled states. There is a straightforward way [8] to calculate C for any two-qubit density matrix ρ. The Bell inequality in the CHSH form [3] is $|S|\leq 2$, where $S=E(\vec{a},\vec{b})-E(\vec{a},\vec{b}')+E(\vec{a}',\vec{b})+E(\vec{a}',\vec{b}')$ and $E(\vec{a},\vec{b})$ is the correlator of results (± 1) for measurement of two qubits (pseudospins) along directions $\vec{a}$ and $\vec{b}$. This inequality should be satisfied by any local hidden-variable theory, while in quantum mechanics it is violated up to $|S|=2\sqrt{2}$ for maximally entangled (e.g., spin-zero) states. Mixed states produce smaller violation (if any), and there is a straightforward way [10] to calculate the maximum value S_+ of $|S|$ for any two-qubit density matrix.

For states with a given concurrence C, there is an exact bound [11] for $S_+ : 2\sqrt{2}C \leq S_+ \leq 2\sqrt{1+C^2}$: (we consider only $S_+>2$), so that the CHSH inequality violation is guaranteed if $C>1/\sqrt{2}$. For any pure state the upper bound is reached: $S_+=2\sqrt{1+C^2}$, so that nonzero entanglement always leads to $S_+>2$. The distinction between entanglement and CHSH inequality violation has been well studied for so-called Werner states [5] which have the form $\rho=f\rho_s+(1-f)\rho_{\rm mix}$, where ρ_s denotes the maximally entangled (singlet) state, and $\rho_{\rm mix}=1/4$ is the density matrix of the completely mixed state. The Werner state is entangled for [5] $f>1/3$, while it violates the CHSH inequality only when [10] $f>1/\sqrt{2}$.

The Werner states, however, are not relevant to most of experiments (including those with superconducting phase qubits [7]), in which an initially pure state becomes mixed due to decoherence (Werner states are produced due to so-called depolarizing channel [1]). Recently a number of authors have analyzed effects of qubit decoherence on the Bell-CHSH-inequality violation [12–16] and entanglement [17–23]. Best-studied models of decoherence in this context are pure dephasing [12,13,15,19,21,23] and zero-temperature energy relaxation [14,16,18,22], while there are also papers considering a combination of these mechanisms [17,20], high-temperature energy relaxation [14], and nonlocal decoherence [14,15,23]. In particular, for the case of pure dephasing it has been shown [19,20] that the concurrence C decays as a product of decoherence factors for the two qubits, and therefore a state remains entangled for arbitrarily long time; moreover, the calculation of S_+ shows [12,13] that the CHSH inequality is always violated also. For the case of zero-temperature energy relaxation it has been shown that entanglement can still last forever [16,18,22] (depending on the initial state), while a finite survival time has been obtained [16] for the CHSH inequality violation.

In this paper we consider a two-qubit state decoherence due to general (Markovian) local decoherence of each qubit (including dephasing and energy relaxation at a finite temperature) and assume absence of any other evolution. For this model we compare for how long an initial state remains entangled ($C>0$), and for how long it can violate the Bell-CHSH inequality ($S_+>2$). In particular, we show that for typical (best) present-day parameters for phase qubits [7] these durations differ by ~8 times.

Before analyzing this problem let us discuss which fraction of the entangled two-qubit states violate the Bell-CHSH inequality. This question is well-posed only if we introduce a particular metric (distance) and corresponding measure (volume) in the 15-dimensional space of density matrices. Various metrics are possible; let us choose the Hilbert-Schmidt metric [1,24], for which the geometry in the space of states is Euclidean. Then random states ρ with the uniform probability distribution can be generated as [24] $\rho=A^\dagger A/{\rm tr}(A^\dagger A)$, where A is a 4×4 matrix, all elements of which are independent Gaussian complex variables with the same variance and zero mean. Using this method, we performed Monte Carlo simulation, generating 10^9 random states and checking if they are entangled [25–27] and if they violate the CHSH inequality [10]. In this way we confirmed that 75.76% of all states are entangled [28] and found that only 0.822% of all states violate the Bell-CHSH inequality. Therefore, only a small fraction, 1.085% of entangled states violate the Bell-CHSH inequality.

Now let us discuss the effect of decoherence. For one qubit it can be described by the Bloch equations [29] (we use the basis of the ground state $|0\rangle$ and excited state $|1\rangle$) and characterized by the energy relaxation time T_1, dephasing time T_2 ($T_2 \le 2T_1$), and the Boltzmann factor $h = \exp(-\Delta/\theta)$, where Δ is the energy separation of the states and θ is the temperature. The usual solution of the Bloch equations can be translated into the language of time-dependent superoperator $\mathcal{L}$ for the one-qubit density matrix ρ, so that $\rho(t) = \mathcal{L}[\rho(0)] = \Sigma_{i=1}^{4} K_i \rho(0) K_i^\dagger$, where four Kraus operators K_i can be chosen as

$$K_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{g} & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \sqrt{1-g} & 0 \\ 0 & \lambda/\sqrt{1-g} \end{pmatrix}, \tag{1}$$

$$K_3 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - hg - \dfrac{\lambda^2}{1-g}} \end{pmatrix}, \quad K_4 = \begin{pmatrix} 0 & \sqrt{hg} \\ 0 & 0 \end{pmatrix},$$

where $g = [1 - \exp(-t/T_1)]/(1+h)$, $\lambda = \exp(-t/T_2)$, and in our notation $|1\rangle = (1,0)^T$, $|0\rangle = (0,1)^T$. It is easy to check that the term under the square root in K_3 is always non-negative, and equals 0 (for $t>0$) only if $T_2 = 2T_1$ and $\theta = 0$. Notice that choice of the Kraus operators K_i is not unique.

In general, decoherence of two qubits is described by many parameters (out of 240 parameters describing a general quantum operation only 15 parameters describe unitary evolution). We choose a relatively simple but physically relevant model when the decoherence is dominated by local decoherence of each qubit. (Nonlocal decoherence would be physically impossible in the case of large distance between the qubits.) The model now involves six parameters: $T_1^{a,b}$, $T_2^{a,b}$, and $h_{a,b} = \exp(-\Delta_{a,b}/\theta_{a,b})$, where subscripts (or superscripts) a and b denote qubits, and the evolution is described by the tensor-product superoperator $\mathcal{L} = \mathcal{L}_a \otimes \mathcal{L}_b$ (which is completely positive because of complete positivity of $\mathcal{L}_{a,b}$). This superoperator contains 16 terms: $\rho(t) = \mathcal{L}[\rho(0)] = \Sigma_{i,j=1}^{4} K_{ij} \rho(0) K_{ij}^\dagger$, $K_{ij} = K_i^a \otimes K_j^b$, where operators $K_i^{a,b}$ are given by Eq. (1) for each qubit.

As an initial state we consider an "odd" pure state

$$|\Psi\rangle = \cos\beta|10\rangle + e^{i\alpha}\sin\beta|01\rangle \tag{2}$$

($0 < \beta < \pi/2$), which is relevant for experiments with the phase qubits [7]. Since the parameter α corresponds to z rotation of one of the qubits, while decoherence as well as values of C and S_+ are insensitive to such rotation, all our results have either trivial or no dependence on α. The evolution of the state (2) due to local decoherence $\mathcal{L}$ can be calculated analytically, and at time t the nonvanishing elements of the two-qubit density matrix ρ are

$$\rho_{11}(t) = (1-g_a)h_b g_b \cos^2\beta + h_a g_a (1-g_b)\sin^2\beta, \tag{3}$$

$$\rho_{22}(t) = (1-g_a)(1-h_b g_b)\cos^2\beta + h_a g_a g_b \sin^2\beta,$$

$$\rho_{33}(t) = g_a h_b g_b \cos^2\beta + (1-h_a g_a)(1-g_b)\sin^2\beta,$$

$$\rho_{44}(t) = g_a(1-h_b g_b)\cos^2\beta + (1-h_a g_a)g_b \sin^2\beta,$$

$$\rho_{32}(t) = \rho_{23}^*(t) = \exp(-t/T_2^a - t/T_2^b)e^{i\alpha}(\sin 2\beta)/2,$$

where $g_{a,b}$ are defined below Eq. (1) and ρ_{ij} subscripts $i,j = 1,2,3,4$ correspond to the basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$. These equations become very simple at zero temperature because then $h_a = h_b = 0$. Notice that the dephasing times $T_2^{a,b}$ enter Eqs. (3) only through the combination $1/T_2^a + 1/T_2^b$ (this is not so for a general initial state) so that the two-qubit dephasing can be characterized by one parameter $T_2 \equiv 2/(1/T_2^a + 1/T_2^b)$.

For the state (3) the concurrence is [14,20]

$$C = 2\max\{0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\}, \tag{4}$$

and the CHSH inequality parameter S_+ is [10,16]

$$S_+ = 2\max\{2\sqrt{2}|\rho_{23}|, \sqrt{4|\rho_{23}|^2 + (1 - 2\rho_{11} - 2\rho_{44})^2}\}, \tag{5}$$

while for the initial state $C = \sin 2\beta > 0$ and $S_+ = 2\sqrt{1+C^2} > 2$. Notice that the first and second terms in Eq. (5) correspond to the "horizontal" and "vertical" measurement configurations, using the terminology of Ref. [30]. Equations (3)–(5) are all we need to analyze entanglement and CHSH inequality violation.

Notice that for a pure dephasing ($T_1^a = T_1^b = \infty$) we have $\rho_{11} = \rho_{44} = 0$, and therefore

$$C = \exp(-2t/T_2)\sin 2\beta, \quad S_+ = 2\sqrt{1 + C^2}. \tag{6}$$

In this case at any t the state remains entangled [19,20] and violates the CHSH inequality [12,13]. (It also remains within the class of states producing maximal CHSH inequality violation for a given concurrence [11].) In the case when both dephasing and energy relaxation are present but temperature is zero, $\theta_a = \theta_b = 0$, the concurrence C is still given by Eq. (6) and lasts forever [16,22]; however S_+ does not satisfy Eq. (6) and the CHSH inequality is no longer violated after a finite time [16]. Finally, in presence of energy relaxation at nonzero temperature (at least for one qubit) the entanglement also vanishes after a finite time, as seen from Eq. (4), in which $\lim_{t\to\infty} \rho_{11}\rho_{44} \neq 0$.

Let us consider in more detail the case when both dephasing and energy relaxation are present, but temperature is zero and $T_1^a = T_1^b \equiv T_1$. Then Eq. (5) for S_+ becomes very simple since $\rho_{11} = 0$ and $\rho_{44} = 1 - \exp(-t/T_1)$. The time dependence $S_+(t)$ consists of three regions: at small t it is always determined by the second term [31] in Eq. (5), then after some time t_1 the first term becomes dominating, while after a later time t_2 the second term becomes dominating again. Notice that in the second region $S_+ = 4\sqrt{2}|\rho_{23}| = 2\sqrt{2}C$, so such state provides minimal S_+ for a given concurrence C [11,32]. The time τ_B after which the Bell-CHSH inequality is no longer violated falls either into the first or second region, because $S_+(t_2) < 2$. The time τ_B can be easily calculated if $S_+(t_1) > 2$, so that τ_B falls into the second region and therefore

$$\tau_B = (T_2/2)\ln(\sqrt{2}\sin 2\beta). \tag{7}$$

This case is realized when pure dephasing is relatively weak: $T_1/T_2 \le \ln(\sqrt{2}\sin 2\beta)/[2\ln(4 - 2\sqrt{2})]$; since $T_1/T_2 \ge 1/2$, it also requires $\sin 2\beta \ge 2\sqrt{2} - 2$. (For $T_1/T_2 = 1/2$ Eq. (7) has been obtained in Ref. [16].) Notice that τ_B in Eq. (7) corre-

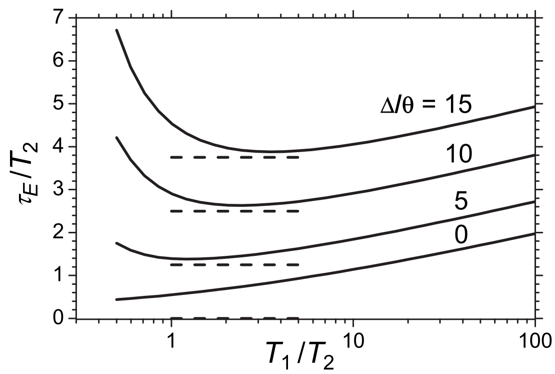

FIG. 1. The two-qubit entanglement duration τ_E in units of the dephasing time T_2 for the maximally entangled initial state ($\beta=\pi/4$) and several values of the temperature θ. Dashed lines correspond to Eq. (8).

sponds to the condition $C=1/\sqrt{2}$, while in general τ_B corresponds to $C\leq 1/\sqrt{2}$.

Now let us focus on calculating the duration τ_E of the entanglement survival, duration τ_B of the Bell-CHSH-inequality violation, and their ratio τ_E/τ_B at nonzero temperature. For simplicity we limit ourselves to the case of maximally entangled initial state ($\beta=\pi/4$), and we also assume equal energy relaxation, splitting and temperature for both qubits: $T_1^a=T_1^b\equiv T_1$, $\Delta_a=\Delta_b\equiv\Delta$, and $\theta_a=\theta_b\equiv\theta$. As follows from Eq. (4), the entanglement duration τ_E can be calculated numerically using the equation $|\rho_{23}|=\sqrt{\rho_{11}\rho_{44}}$. Figure 1 shows τ_E (normalized by T_2) as a function of the ratio T_1/T_2 for several values of the normalized inverse temperature Δ/θ. As we see, in a typical experimental regime [7] when $\Delta/\theta\sim 10$, the ratio τ_E/T_2 does not depend much on T_1/T_2 when T_1 is larger but comparable to T_2 (which is also typical experimentally). In other words, τ_E is approximately proportional to T_2, and in this regime τ_E also has crudely inverse dependence on temperature [see Eq. (8) below].

Analytical formulas for τ_E can be easily obtained in the limiting cases. In absence of pure dephasing ($T_1/T_2=1/2$) and at low temperature ($\theta\ll\Delta$) we find $\tau_E/T_2\approx\Delta/2\theta-\ln(2\sqrt{2}+2)/2\approx\Delta/2\theta-0.79$, while at high temperature ($\theta\gg\Delta$) we have $\tau_E/T_2\approx\ln(\sqrt{2}+1)/2\approx 0.44$. In the case of strong dephasing ($T_1/T_2\gg 1$) we find (neglecting some corrections) $\tau_E/T_2\simeq\Delta/(4\theta)+\ln(T_1/T_2)/2$.

However, these asymptotic formulas are not very relevant to a typical experimental situation with phase qubits [7], in which $T_1\gtrsim T_2$. As another way to approximate τ_E we have chosen the value at the minimum of the curves in Fig. 1; this minimum occurs at the ratios T_1/T_2 somewhat close to the experimental values, and the result is naturally not much sensitive to T_1/T_2 in a significantly broad range. For sufficiently small temperatures ($\Delta/\theta>2$) we have obtained approximation $(\tau_E/T_2)_{\min}\approx\Delta/4\theta+\ln(3^{3/4}/2)\approx\Delta/4\theta+0.13$ and found that the minimum occurs at $T_1/T_2\approx(\tau_E/T_2)_{\min}/\ln 3$. So, as the crudest approximation in the experimentally relevant regime ($\theta/\Delta\sim 10^{-1}$, $T_1/T_2\gtrsim 1$), the two-qubit entanglement lasts for (see dashed lines in Fig. 1)

$$\tau_E\simeq T_2\Delta/4\theta. \tag{8}$$

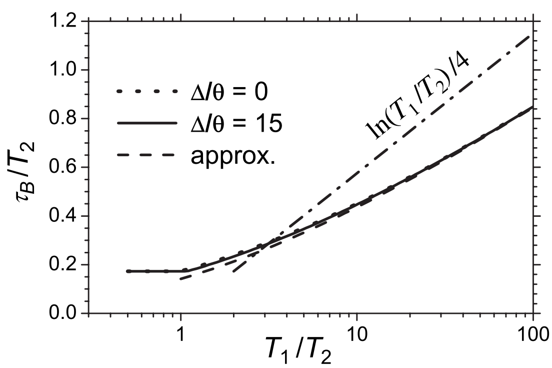

FIG. 2. The duration τ_B of the Bell-CHSH-inequality violation (assuming $\beta=\pi/4$) for $\Delta/\theta=15$ (solid line) and $\Delta/\theta=0$ (dotted line). The dashed line: $\tau_B/T_2=\ln[T_1/(4\tau_B)]/4$.

The duration τ_B of the Bell-CHSH-inequality violation is calculated using Eq. (5) as $S_+(\tau_B)=2$. Solid and dotted lines in Fig. 2 show numerical results for τ_B (in units of T_2) as a function of the ratio T_1/T_2 for low and high temperatures: $\Delta/\theta=15$ and 0. The curves are almost indistinguishable, that means that τ_B is practically independent of the temperature for fixed T_1 and T_2. Notice that each curve consists of a constant (horizontal) part and an increasing part, which correspond to two terms in Eq. (5). It can be shown that at zero temperature the horizontal part is realized at $T_1/T_2\leq\ln 2/[4\ln(4-2\sqrt{2})]\approx 1.1$, while at high temperature ($\theta\gg\Delta$) it is realized at $T_1/T_2\leq 1$. The horizontal part corresponds to the first term in Eq. (5) dominating at τ_B: $S_+=2\sqrt{2}\exp(-2t/T_2)$, so at sufficiently weak pure dephasing we have $\tau_B/T_2=\ln 2/4\approx 0.17$ [see also Eq. (7)]. In the opposite case of strong pure dephasing ($T_1/T_2\gg 1$) the duration τ_B is the solution of the equation $\tau_B/T_2=\ln[T_1/(4\tau_B)]/4$ (dashed line in Fig. 2), so roughly $\tau_B/T_2\simeq\ln(T_1/T_2)/4$ (dot-dashed line in Fig. 2). Combining these results, we get a crude estimate

$$\tau_B\simeq T_2\max\{0.17,0.25\ln(T_1/T_2)\}. \tag{9}$$

Figure 3 shows the ratio τ_E/τ_B of the survival durations of entanglement and the Bell-CHSH-inequality violation. We see that the ratio τ_E/τ_B increases with the decrease of temperature and decrease of the pure dephasing contribution,

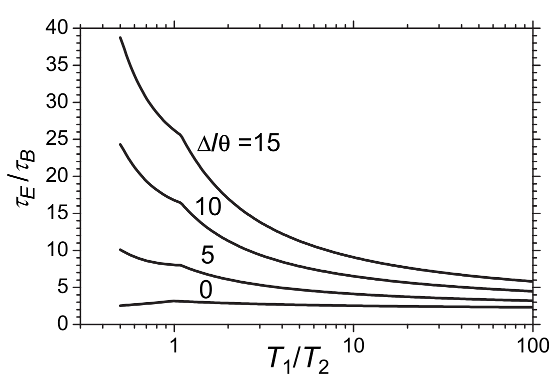

FIG. 3. The ratio τ_E/τ_B for the maximally entangled initial state and several values of the temperature θ.

which are both the desired experimental regimes. (This rule does not work in the experimentally irrelevant regime $\theta \gg \Delta$ and $T_1 < T_2$.) Notice that the kinks on the curves correspond to the change of the dominating term in Eq. (5). In absence of pure dephasing ($T_1/T_2=1/2$) the low-temperature result ($\theta \ll \Delta$) is $\tau_E/\tau_B \approx (2/\ln 2)[\Delta/\theta - \ln(2\sqrt{2}+2)]$, while at $\theta \gg \Delta$ the ratio is $\tau_E/\tau_B \approx 2\ln(\sqrt{2}+1)/\ln 2 \approx 2.5$. In the limit of strong pure dephasing ($T_1/T_2 \gg 1$) the asymptotic result is $\tau_E/\tau_B \approx 2+(\Delta/\theta)/\ln(T_1/T_2)$ (as we see, $\tau_E > 2\tau_B$ for any parameters). In the experimentally relevant regime when $\theta/\Delta \sim 10^{-1}$ and $T_1/T_2 \gtrsim 1$, the ratio can be obtained from Eqs. (8) and (9), giving a crude estimate $\tau_E/\tau_B \simeq (\Delta/\theta)\min\{1.5, 1/\ln(T_1/T_2)\}$.

For an experimental estimate let us choose parameters typical for best present-day experiments with superconducting phase qubits [7]: $\Delta/2\pi\hbar \simeq 6$ GHz, $\theta \simeq 50$ mK, $T_1 \simeq 450$ ns, $T_2 \simeq 300$ ns. Then $\Delta/\theta \simeq 6$, $T_1/T_2 \simeq 1.5$, and we obtain $\tau_E \simeq 470$ ns, $\tau_B \simeq 60$ ns, and $\tau_E/\tau_B \simeq 7.7$.

In conclusion, we have found that in the Hilbert-Schmidt metric only 1.085% of entangled states violate the Bell-CHSH inequality, thus explaining why entanglement can last for a significantly longer time (τ_E) than the Bell-CHSH-inequality violation (τ_B). Using the technique of Kraus operators, we have considered local decoherence due to dephasing and energy relaxation at finite temperature, and for this model calculated τ_E, τ_B, and their ratio τ_E/τ_B.

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[31] Notice that when $T_1^a \neq T_1^b$, the second term in Eq. (5) is maximized for a nonmaximally entangled state $\beta \neq \pi/4$ though the benefit is not significant if we need $S_+ \gtrsim 2.2$.
[32] The statement in Ref. [11] that any mixed state with $S_+ = 2\sqrt{2}C > 2$ is maximally entangled, is incorrect (here maximum entanglement means that C cannot be increased by any two-qubit unitary transformation). As a counterexample, consider the states $\rho = f|\Psi\rangle\langle\Psi| + (1-f)|00\rangle\langle 00|$, produced from the initial state (2) due to zero-temperature energy relaxation ($T_2 = 2T_1$, $\theta=0$, $f=e^{-t/T_1}$). Any two such states with the same f but different initial parameter β can obviously be connected by a unitary transformation (involving only the subspace spanned by $|01\rangle$ and $|10\rangle$), while they have different concurrence C given by Eq. (6). Finally, as follows from our analysis, there is a finite range of parameters f and β, in which $S_+ = 2\sqrt{2}C$; in this range the concurrence can still be varied by unitary transformations varying β, contradicting the statement of Ref. [11].