Supplemental Material for "Multi-time correlators in continuous measurement of qubit observables"

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Section A: Proof of the "collapse recipe"

In this section we prove the "collapse recipe", which states that in the absence of phase backaction, the multi-time correlator

$$K_{\ell_1\ell_2...\ell_N}(t_1,t_2,...t_N) \equiv \langle I_{\ell_N}(t_N)...\, I_{\ell_2}(t_2)\, I_{\ell_1}(t_1)\rangle \quad (1)$$

can be calculated by replacing the actual continuous measurement with projective measurement of operators σ_{ℓ_k} at time moments t_k ($k=1,2,...\,N$, $t_1<t_2<...<t_N$), while the qubit evolution at $t\neq t_k$ is replaced with the ensemble-averaged evolution. In the proof we show that the value of the correlator K obtained in this way coincides with the value obtained from the quantum Bayesian formalism, in which the qubit evolution is described by the stochastic equation (in Itô interpretation)

$$\dot{\boldsymbol{r}} = \Lambda_{\rm ens}(\boldsymbol{r}-\boldsymbol{r}_{\rm st}) + \sum_{\ell=1}^{N_{\rm d}} \frac{\boldsymbol{n}_\ell - (\boldsymbol{n}_\ell \boldsymbol{r})\,\boldsymbol{r}}{\sqrt{\tau_\ell}}\,\xi_\ell(t), \quad (2)$$

where $\boldsymbol{r}$ is the vector of Bloch-sphere components, $\rho = \mathbb{1}/2+\boldsymbol{r\sigma}/2$, the output signal of the ℓth detector continuously measuring the qubit operator $\sigma_\ell = \boldsymbol{n}_\ell\boldsymbol{\sigma}$ is

$$I_\ell(t) = \boldsymbol{n}_\ell \boldsymbol{r}(t) + \sqrt{\tau_\ell}\,\xi_\ell(t), \quad (3)$$

τ_ℓ is the corresponding measurement (collapse) time (the quantum efficiency η_ℓ is not important for correlators), and $\xi_\ell(t)$ are the uncorrelated white noises with $\langle \xi_\ell(t)\,\xi_{\ell'}(t')\rangle = \delta_{\ell\ell'}\delta(t-t')$. The Markovian ensemble-averaged evolution of the qubit state is given by Eq. (2) without the noise term,

$$\dot{\boldsymbol{r}}_{\rm ens} = \Lambda_{\rm ens}(\boldsymbol{r}_{\rm ens}-\boldsymbol{r}_{\rm st}). \quad (4)$$

The evolution is assumed to start with some initial state $\boldsymbol{r}_{\rm in}$ at time $t_{\rm in}\leq t_1$. All parameters of the measurement and evolution ($\Lambda_{\rm ens}$, $\boldsymbol{r}_{\rm st}$, $\boldsymbol{n}_\ell$, τ_ℓ, η_ℓ) can be time-dependent.

We will first prove the collapse recipe in a simple way and then will prove it in another, more formal way.

1. *Simple proof*

The simple proof of the collapse recipe closely follows the proof for two-time correlators in Refs. [1] and [2]. Understanding of this proof is easier after understanding of proofs in Refs. [1] and [2].

The proof uses linearity of quantum mechanics. In particular, from the linearity, the correlator (1) can depend on the initial state $\boldsymbol{r}_{\rm in}$ only linearly, $K=\boldsymbol{v}\boldsymbol{r}_{\rm in}+C$, where the vector $\boldsymbol{v}$ and the number C can depend on all parameters for the correlator, but do not depend on $\boldsymbol{r}_{\rm in}$. The linearity is better seen by introducing 4-vectors for unnormalized density matrices, $\tilde{\boldsymbol{r}}=(u,x,y,z)$ for $\rho=(u\mathbb{1}+x\sigma_x+y\sigma_y+z\sigma_z)/2$; then $K=\tilde{\boldsymbol{v}}\tilde{\boldsymbol{r}}_{\rm in}$ with some 4-vector $\tilde{\boldsymbol{v}}$, which does not depend on $\tilde{\boldsymbol{r}}_{\rm in}$. Note that quantum evolution is linear for 4-vectors $\tilde{\boldsymbol{r}}$, but is not necessarily linear for 3-vectors $\boldsymbol{r}$. The evolution, which is linear for 3-vectors $\boldsymbol{r}$, is called unital.

The correlator (1) is the average over the ensemble of quantum trajectories, starting with initial state $\boldsymbol{r}_{\rm in}$ at time $t_{\rm in}$. Let us discretize time into small but still non-zero timesteps Δt, so that the noises $\xi_\ell(t)$ are not infinitely large ($|\xi_\ell|\sim 1/\sqrt{\Delta t}$). Since the values of the output signals $I_\ell(t)$ at $t\neq t_k$ do not affect the correlator K, we can pretend that during these timesteps the signals $I_\ell(t)$ are not available to any observer, and therefore the qubit evolution is equivalent [3] to ensemble-averaged evolution given by Eq. (4). Thus, we need to take into account the full Bayesian evolution (2) only during timesteps t_k. Moreover, since at time t_k only the output from ℓ_kth detector affects the correlator, in Eq. (2) we can neglect all the terms in the sum except for $\ell=\ell_k$. Integrating Eq. (2) over the timestep Δt around t_k, we obtain the "Bayesian kick"

$$\Delta \boldsymbol{r}_k \equiv \Delta\boldsymbol{r}(t_k) = \frac{\boldsymbol{n}_{\ell_k} - (\boldsymbol{n}_{\ell_k}\boldsymbol{r}_k)\,\boldsymbol{r}_k}{\sqrt{\tau_{\ell_k}}}\,\xi_{\ell_k}(t_k)\,\Delta t, \quad (5)$$

where $\boldsymbol{r}_k\equiv\boldsymbol{r}(t_k)$ and $|\xi_{\ell_k}|\sim 1/\sqrt{\Delta t}$. With $\Delta t\to 0$, this information-induced kick becomes infinitesimally small, so its effect on further evolution is infinitesimally small. However, its contribution to the correlator (1) is significant, since the signal $I_{\ell_k}(t_k)$ in the correlator contains the term $\sqrt{\tau_{\ell_k}}\,\xi_{\ell_k}(t_k)$ [see Eq. (3)], so the effect of the Bayesian kick (5) is proportional to the product

$$\sqrt{\tau_{\ell_k}}\,\xi_{\ell_k}(t_k)\Delta\boldsymbol{r}_k = [\boldsymbol{n}_{\ell_k} - (\boldsymbol{n}_{\ell_k}\boldsymbol{r}_k)\,\boldsymbol{r}_k]\,\xi^2_{\ell_k}(t_k)\,\Delta t, \quad (6)$$

which is non-zero since on average $\xi^2_{\ell_k}(t_k)\,\Delta t=1$.

Let us prove that we can apply the collapse recipe to the measurement at time t_k in the correlator (1). This means that the value of the correlator would not change if we replace the actual signal $I_{\ell_k}(t_k)$ in Eq. (1) by $I_{\ell_k}(t_k) = \pm 1$ with probabilities

$$p_k^{\pm} = \frac{1 \pm \boldsymbol{n}_{\ell_k} \boldsymbol{r}_k}{2}, \tag{7}$$

and correspondingly greatly increase the Bayesian kick by starting the further evolution with the state $\boldsymbol{r}(t_k+0) = \pm \boldsymbol{n}_{\ell_k}$ (i.e., the corresponding eigenstate of the measured operator σ_{ℓ_k}). In the proof we assume fixed (though arbitrary) values for all previous measurements $I_{\ell_{k'}}(t_{k'<k})$, so that $\boldsymbol{r}_k$ is fixed. Then the N-time correlator (1) reduces to a product of $I_{\ell_1}(t_1)\, I_{\ell_2}(t_2) \ldots I_{\ell_{k-1}}(t_{k-1})$ and the remaining $(N+1-k)$-time correlator. Therefore, this correlator depends linearly on $\boldsymbol{r}_k$ (better to say, on 4-vector $\tilde{\boldsymbol{r}}_k$ – see discussion above).

Let us separate the correlator (1) [with fixed $I_{\ell_{k'}}(t_{k'<k})$] into two terms, $K = K_k^{(1)} + K_k^{(2)}$, which correspond to the two terms in Eq. (3) at time t_k, i.e., $K_k^{(1)} = \langle I_{\ell_N}(t_N) \ldots I_{\ell_{k+1}}(t_{k+1})\rangle\, \boldsymbol{n}_{\ell_k} \boldsymbol{r}_k\, I_{\ell_{k-1}} \ldots I_{\ell_1}(t_1)$ and $K_k^{(2)} = \langle I_{\ell_N}(t_N) \ldots I_{\ell_{k+1}}(t_{k+1}) \sqrt{\tau_{\ell_k}}\, \xi_{\ell_k}(t_k)\rangle\, I_{\ell_{k-1}} \ldots I_{\ell_1}(t_1)$. Because of the quantum linearity, the value of $K_k^{(1)}$ will not change if we replace $\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k$ with $I_{\ell_k} = +1$ and start the further evolution with the unnormalized density matrix $\rho_1^+(t_k+0) = (\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)\, \rho(t_k) = (\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)\, \mathbb{1}/2 + (\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)(\boldsymbol{r}_k \boldsymbol{\sigma}/2)$. Note that we need to multiply all elements of ρ by $\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k$; this is why the normalization changes, $\mathrm{Tr}(\rho_1^+) = \boldsymbol{n}_{\ell_k} \boldsymbol{r}_k$. This is necessary because for non-zero stationary state $\boldsymbol{r}_{\rm st}$, the evolution (4) of the Bloch-sphere components is non-linear, even though the evolution of the density matrix ρ is linear.

The same linearity-based idea for $K_k^{(2)}$ needs to take into account the Bayesian kick (5). It is easy to see that $K_k^{(2)}$ will not change if we replace $\sqrt{\tau_{\ell_k}}\, \xi_{\ell_k}(t_k)$ with $I_{\ell_k} = +1$ and start the further evolution with $\rho_2^+(t_k+0) = [\boldsymbol{n}_{\ell_k} - (\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)\, \boldsymbol{r}_k]\, \boldsymbol{\sigma}/2$ – see Eq. (6). Note the zero trace of ρ_2^+; this is because the Bayesian kick does not change the trace.

Adding the contributions from $K_k^{(1)}$ and $K_k^{(2)}$ and using the linearity, we see that the correlator K will not change if we replace I_{ℓ_k} with $I_{\ell_k} = +1$ and start the further evolution with $\rho^+(t_k+0) = \rho_1^+(t_k+0) + \rho_2^+(t_k+0) = (\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)\, \mathbb{1}/2 + \boldsymbol{n}_{\ell_k} \boldsymbol{\sigma}/2$. Using the linearity again, we see that K will also not change if we replace I_{ℓ_k} with $I_{\ell_k} = -1$ and start the further evolution with $\rho^-(t_k+0) = -\rho^+(t_k+0) = -(\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)\, \mathbb{1}/2 - \boldsymbol{n}_{\ell_k} \boldsymbol{\sigma}/2$. The value of K will also not change if we use one of these two replacements probabilistically. Note that $\rho^{\pm}(t_k+0)$ differ from the normalized eigenstates $\mathbb{1}/2 \pm \boldsymbol{n}_{\ell_k} \boldsymbol{\sigma}/2$ of the measured operator σ_{ℓ_k} only by $(\pm \boldsymbol{n}_{\ell_k} \boldsymbol{r}_k - 1)\, \mathbb{1}/2$. If we choose the replacements $I_{\ell_k} = \pm 1$ with probabilities given by Eq. (7), then the effect of this difference will be cancelled on average since $\sum_{\pm} \pm(\pm \boldsymbol{n}_{\ell_k} \boldsymbol{r}_k - 1)(1 \pm \boldsymbol{n}_{\ell_k} \boldsymbol{r}_k) = 0$. Therefore, the value of K does not change if we start the further evolution with the states $\mathbb{1}/2 \pm \boldsymbol{n}_{\ell_k} \boldsymbol{\sigma}/2$, as if after the standard projective measurement of σ_{ℓ_k}.

Thus, we have proven that we can apply the collapse recipe to the measurement at time t_k, assuming fixed measurement results for the previous measurements. Since the values of the previous measurement results are arbitrary, the assumption of fixed results is not needed. Finally, since the collapse recipe can be applied separately to measurement at any time moment t_k in the correlator (1), it can be applied to all of them. This completes the proof of the collapse recipe for multi-time correlators (1).

Note that instead of using the collapse recipe and working with normalized states, we can also calculate the correlator using the described above procedure based on unnormalized states. In this case at each moment t_k, we replace $I_{\ell_k}(t_k)$ with $I_{\ell_k} = +1$ and start the further evolution with $\rho^+(t_k+0) = (\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)\, u_k\, \mathbb{1}/2 + \boldsymbol{n}_{\ell_k} \boldsymbol{\sigma}/2$, where $u_k = \mathrm{Tr}[\rho(t_k)]$ accounts for possibly unnormalized state ρ before t_k. Since this procedure can be applied for all N moments t_k and then the product of all I_{ℓ_k} is 1, the value of the correlator is simply the norm of the state after the last time moment t_N. Therefore this new "one-path recipe" for calculating the N-time correlator (1) is the following. Start with the initial (normalized) state $\tilde{\mathbf{r}}_{\rm in}$ at the initial time $t_{\rm in}$ and propagate it using the ensemble-averaged evolution (4) (which does not change the norm), also adding the "state jumps" (which change the norm) at time moments t_k as $\rho(t_k+0) = (\boldsymbol{n}_{\ell_k} \boldsymbol{r}_k)\, u_k\, \mathbb{1}/2 + \boldsymbol{n}_{\ell_k} \boldsymbol{\sigma}/2$. Then the norm of the resulting state $\rho(t_N+0)$ is the value of the correlator.

This one-path recipe can be easily generalized to arbitrary measurement operators in an arbitrary system. For a continuous measurement of an arbitrary Hermitian observable A, the quantum Bayesian evolution due to informational backaction (in the absence of a unitary backaction) is (the derivation is simple, the result is the same as in the Quantum Trajectory theory [4])

$$\dot{\rho} = \frac{A\rho A - (A^2\rho + \rho A^2)/2}{2\eta S} + \frac{A\rho + \rho A - 2\rho \mathrm{Tr}(A\rho)}{\sqrt{2S}}\, \xi(t), \tag{8}$$

where $\xi(t)$ is the normalized white noise, $\langle \xi(t)\, \xi(t')\rangle = \delta(t-t')$, extracted from the normalized detector signal, $I(t) = \mathrm{Tr}(A\rho) + \sqrt{S/2}\, \xi(t)$, S is the single-sided spectral density of the detector signal noise, and η is the quantum efficiency (so that the fraction η of the noise S is "quantum-limited"). Since the first term in Eq. (8) is obviously the ensemble-averaged (Lindblad) evolution, the evolution due to measurement of several (generally noncommuting) observables A_ℓ in the presence of additional unitary evolution and decoherence (but still without unitary backaction from measurement) is

$$\dot{\rho} = \mathcal{L}[\rho] + \sum\nolimits_{\ell} \frac{A_\ell \rho + \rho A_\ell - 2\rho \mathrm{Tr}(A_\ell \rho)}{\sqrt{2S_\ell}}\, \xi_\ell(t), \tag{9}$$

where $I_\ell(t) = \mathrm{Tr}(A_\ell \rho) + \sqrt{S_\ell/2}\, \xi_\ell(t)$, $\langle \xi_\ell(t)\, \xi_{\ell'}(t')\rangle = \delta_{\ell\ell'}\, \delta(t-t')$, and $\mathcal{L}[\rho]$ is the ensemble-averaged Lindbdad evolution, with the contribution from measurement $\mathcal{L}_{\rm m}[\rho] = \sum_\ell [A_\ell \rho A_\ell - (A_\ell^2\rho + \rho A_\ell^2)/2]/(2\eta_\ell S_\ell)$. Following the same idea as described above, for each time moment t_k in the correlator (1), we can remove $I_{\ell_k}(t_k)$ from the correlator K, separating it into two parts, $K = K_k^{(1)} + K_k^{(2)}$, so that for $K_k^{(1)}$ the (unnormalized) state jumps from $\rho_k = \rho(t_k)$ to $\rho_1(t_k{+}0) = {\rm Tr}(A_{\ell_k}\rho_k)\,\rho_k$, while for $K_k^{(2)}$ the "Bayesian kick" changes ρ_k into $\rho_2(t_k+0) = (A_{\ell_k}\rho_k + \rho_k A_{\ell_k})/2 - {\rm Tr}(A_{\ell_k}\rho_k)\,\rho_k$ – see Eq. (9). Therefore, we can remove $I_{\ell_k}(t_k)$ from the correlator (1), replacing it with the state jump

$$\rho(t_k+0) = (A_{\ell_k}\rho_k + \rho_k A_{\ell_k})/2. \tag{10}$$

Thus, the one-path recipe for the N-time correlator (1) in the general case is

$$K_{\ell_1\ell_2...\ell_N}(t_1,t_2,...t_N) = {\rm Tr}\big[\mathcal{M}_{t_N}\,\mathcal{E}(t_N|t_{N-1})\,\mathcal{M}_{t_{N-1}}... \\ \mathcal{M}_{t_2}\,\mathcal{E}(t_2|t_1)\,\mathcal{M}_{t_1}\,\mathcal{E}(t_1|t_{\rm in})\,\rho_{\rm in}\big], \tag{11}$$

where $\mathcal{E}(t|t')$ is the trace-preserving ensemble-averaged evolution (operation) from time t' to t due to Lindblad term $\dot\rho = \mathcal{L}[\rho]$, while $\mathcal{M}_{t_k}\rho = (A_{\ell_k}\rho + \rho A_{\ell_k})/2$ is the trace-changing operation, related to measurement (without unitary backaction) of the operator A_{ℓ_k} at time t_k. If a unitary backaction of the form $\sum_\ell -i[B_\ell,\rho]\,\xi_\ell(t)/\sqrt{2S_\ell}$ is added into Eq. (9) (B_ℓ are Hermitian), with the contribution to the ensemble-averaged evolution absorbed by $\mathcal{L}[p]$, then the additional Bayesian kick produces an extra term in Eq. (10): $\mathcal{M}_{t_k}\rho = (A_{\ell_k}\rho + \rho A_{\ell_k})/2 - i[B_{\ell_k},\rho]/2$. The one-path recipe is similar to the result of a recent paper [5] by Tilloy. Note the similarity of Eq. (11) to the quantum regression formula.

The one-path recipe (11) based on unnormalized states can be reduced to the physically transparent collapse recipe (based on physical states) only when $B_\ell = 0$ and A_ℓ^2 are positive numbers, i.e., scaled unity operators. (In the general case, it is still possible to generalize the collapse recipe to work with normalized, but unphysical states; however, then the physical meaning becomes obscure.) In particular, the collapse recipe is fully applicable for continuous measurement of multi-qubit Pauli operators in an arbitrary system of qubits, because then $A_\ell^2 = \mathbb{1}$. One can see this by noticing that Eq. (10) in this case can be written as $\rho(t_k+0) = \sum_\pm \pm{\rm Tr}[\rho_k\Pi^\pm_{\ell_k}]\left(\dfrac{\Pi^\pm_{\ell_k}\rho_k\Pi^\pm_{\ell_k}}{{\rm Tr}[\rho_k\Pi^\pm_{\ell_k}]}\right)$, where $\Pi^\pm_{\ell_k}$ is the projection operator corresponding to the eigenvalue ±1 of A_{ℓ_k}. This form corresponds to the result ±1 of the projective measurement of A_{ℓ_k}, with probability ${\rm Tr}[\rho_k\Pi^\pm_{\ell_k}]$ and with the density matrix inside the parenthesis being the normalized state after the projective multi-qubit collapse.

Completing the brief digression into the general case, we remind that the main purpose of this section is the proof of the collapse recipe for the case of a single qubit, considered in this paper.

2. Alternative proof

Now let us prove the collapse recipe for the single-qubit case in a different, more formal way. In this derivation we will also obtain the correlator factorization result for unital evolution, Eq. (11) of the main text.

In addition to the correlator K given by Eq. (1), let us introduce the vector-valued correlator

$$\boldsymbol{K}_{\ell_1...\ell_N}(t_1,...t_N) \equiv \big\langle \boldsymbol{r}(t_N)\,I_{\ell_{N-1}}(t_{N-1})\cdots I_{\ell_1}(t_1)\big\rangle. \tag{12}$$

Note that in this notation for $\boldsymbol{K}$, the last subscript ℓ_N is not needed, but we keep it to remind us that $\boldsymbol{K}$ is an average product of N terms. We will usually assume $t_1 < t_2 < ... < t_N$ (as for the correlator K), but at some point in the derivation we will need the time moment t_N to cross t_{N-1}. The correlator K can be easily obtained from $\boldsymbol{K}$ as

$$K_{\ell_1...\ell_N}(t_1,...t_N) = \boldsymbol{n}_{\ell_N}\boldsymbol{K}_{\ell_1...\ell_N}(t_1,...t_N), \tag{13}$$

since the noise contribution $\sqrt{\tau_{\ell_N}}\,\xi_{\ell_N}(t_N)$ to the output signal $I_{\ell_N}(t_N)$ [see Eq. (3)] is not correlated with past qubit states.

Let us separate $\boldsymbol{K}$ into two terms, $\boldsymbol{K} = \boldsymbol{K}^{(1)} + \boldsymbol{K}^{(2)}$, which correspond to the two terms in Eq. (3) for the signal $I_{\ell_{N-1}}(t_{N-1})$,

$$\boldsymbol{K}^{(1)}_{\ell_1...\ell_N}(t_1,...t_N) \equiv \big\langle \boldsymbol{r}(t_N)\big[\boldsymbol{n}_{\ell_{N-1}}\boldsymbol{r}(t_{N-1})\big] \\ \times I_{\ell_{N-2}}(t_{N-2})\cdots I_{\ell_1}(t_1)\big\rangle, \tag{14}$$

$$\boldsymbol{K}^{(2)}_{\ell_1...\ell_N}(t_1,...t_N) \equiv \big\langle \boldsymbol{r}(t_N)\big[\sqrt{\tau_{\ell_{N-1}}}\,\xi_{\ell_{N-1}}(t_{N-1})\big] \\ \times I_{\ell_{N-2}}(t_{N-2})\cdots I_{\ell_1}(t_1)\big\rangle. \tag{15}$$

The derivative of $\boldsymbol{K}^{(1)}$ over the last time moment t_N can be obtained from Eq. (2),

$$\partial_{t_N}\boldsymbol{K}^{(1)}_{\ell_1...\ell_N}(t_1,...t_N) = \Lambda_{\rm ens}(t_N)\Big(\boldsymbol{K}^{(1)}_{\ell_1...\ell_N}(t_1,...t_N) \\ - \boldsymbol{r}_{\rm st}(t_N)\,K_{\ell_1...\ell_{N-1}}(t_1,...t_{N-1})\Big), \tag{16}$$

where we included possible dependence of $\Lambda_{\rm ens}$ and $\boldsymbol{r}_{\rm st}$ on time. The initial condition at $t_N = t_{N-1}+0$ is

$$\boldsymbol{K}^{(1)}_{\ell_1...\ell_N}(t_1,...t_{N-1},t_{N-1}+0) = \big\langle \boldsymbol{r}(t_{N-1}) \\ \times \big(\boldsymbol{n}_{\ell_{N-1}}\boldsymbol{r}(t_{N-1})\big)\,I_{\ell_{N-2}}(t_{N-2})\cdots I_{\ell_1}(t_1)\big\rangle. \tag{17}$$

The time derivative of $\boldsymbol{K}^{(2)}$ over t_N can also be obtained from Eq. (2), which gives

$$\partial_{t_N}\boldsymbol{K}^{(2)}_{\ell_1...\ell_N}(t_1,...t_N) = \Lambda_{\rm ens}(t_N)\,\boldsymbol{K}^{(2)}_{\ell_1...\ell_N}(t_1,...t_N) \\ + \Big\langle \big[\boldsymbol{n}_{\ell_{N-1}} - \boldsymbol{r}(t_{N-1})\big(\boldsymbol{n}_{\ell_{N-1}}\boldsymbol{r}(t_{N-1})\big)\big] \\ \times I_{\ell_{N-2}}(t_{N-2})\cdots I_{\ell_1}(t_1)\Big\rangle\,\delta(t_N - t_{N-1}). \tag{18}$$

Note that $\boldsymbol{K}^{(2)} = 0$ for $t_N < t_{N-1}$ because of causality, so the second term in Eq. (18) sets the initial condition at $t_N = t_{N-1} + 0$, caused by the Bayesian kick. Also note that at $t_N > t_{N-1}$, the evolution of $\boldsymbol{K}^{(2)}$ is linear (due to $\Lambda_{\rm ens}$); it does not have the inhomogeneous term containing $\boldsymbol{r}_{\rm st}$ as for the evolution of $\boldsymbol{K}^{(1)}$ in Eq. (16).

Solving Eqs. (16)–(18), we find $\boldsymbol{K}^{(1)}$ and $\boldsymbol{K}^{(2)}$ for $t_N > t_{N-1}$, starting with the value (17) of $\boldsymbol{K}^{(1)}$ at $t_N = t_{N-1} + 0$,

$$\boldsymbol{K}^{(1)}_{\ell_1\ldots\ell_N}(t_1,\ldots t_N) = \mathcal{P}(t_N|t_{N-1})\,\boldsymbol{K}^{(1)}_{\ell_1\ldots\ell_N}(t_1,\ldots t_{N-1}+0) + \boldsymbol{\mathcal{P}}_{\rm st}(t_N|t_{N-1})\,K_{\ell_1\ldots\ell_{N-1}}(t_1,\ldots t_{N-1}), \tag{19}$$

$$\boldsymbol{K}^{(2)}_{\ell_1\ldots\ell_N}(t_1,\ldots t_N) = -\mathcal{P}(t_N|t_{N-1})\boldsymbol{K}^{(1)}_{\ell_1\ldots\ell_N}(t_1,\ldots t_{N-1}+0) + \mathcal{P}(t_N|t_{N-1})\,\boldsymbol{n}_{\ell_{N-1}}K_{\ell_1\ldots\ell_{N-2}}(t_1,\ldots t_{N-2}), \tag{20}$$

where $\mathcal{P}(t|t')$ is the 3×3 propagator matrix for the homogeneous part of the ensemble-averaged evolution (4), so that $\partial_t\mathcal{P}(t|t') = \Lambda_{\rm ens}(t)\,\mathcal{P}(t|t')$ for $t > t'$ and $\mathcal{P}(t|t) = \mathbb{1}$, while $\boldsymbol{\mathcal{P}}_{\rm st}(t|t')$ is the contribution from the inhomogeneous part,

$$\boldsymbol{\mathcal{P}}_{\rm st}(t|t') = -\int_{t'}^{t}\mathcal{P}(t|t'')\,\Lambda_{\rm ens}(t'')\,\boldsymbol{r}_{\rm st}(t'')\,dt''. \tag{21}$$

From Eqs. (19)–(21) and (13)–(15) we find the recursive formula, which relates the N-time correlator $K_{\ell_1\ldots\ell_N}(t_1,\ldots t_N)$ with $(N-1)$-time correlator and $(N-2)$-time correlator (for $N > 2$)

$$K_{\ell_1\ldots\ell_N}(t_1,\ldots t_N) = \boldsymbol{n}_{\ell_N}\,\mathcal{P}(t_N|t_{N-1})\,\boldsymbol{n}_{\ell_{N-1}} \times K_{\ell_1\ldots\ell_{N-2}}(t_1,\ldots t_{N-2}) + \boldsymbol{n}_{\ell_N}\boldsymbol{\mathcal{P}}_{\rm st}(t_N|t_{N-1})\,K_{\ell_1\ldots\ell_{N-1}}(t_1,\ldots t_{N-1}). \tag{22}$$

Note that for $N = 2$, the only difference in the derivation is that the product $I_{\ell_{N-2}}(t_{N-2})\cdots I_{\ell_1}(t_1)$ in Eq. (18) should be replaced with 1. As a consequence, the $(N-2)$-time correlator in Eqs. (20) and (22) should be replaced with 1. Therefore, Eq. (22) is also valid for $N = 2$ if we define the 0-time correlator as being equal to 1.

Thus, the recursive relation (22) is sufficient to derive explicit formulas for N-time correlators, if we complement it with the correlator for $N = 1$, which is simple,

$$K_{\ell_1}(t_1) = \boldsymbol{n}_{\ell_1}\boldsymbol{r}(t_1). \tag{23}$$

Now let us show that the N-time correlators obtained via Eqs. (22) and (23) coincide with the correlators obtained using the collapse recipe. Since for $N = 1$ the collapse recipe obviously gives Eq. (23), we only need to prove that the recursive relation (22) also follows from the collapse recipe (with the correlator for $N = 0$ defined as 1). Note that applicability of the collapse recipe to the two-time correlator was proven in Ref. [2].

Let us rewrite Eq. (7) of the main text (following from the collapse recipe, as indicated by the superscript below) in the form

$$K^{\rm coll}_{\ell_1\ldots\ell_N}(t_1,\ldots t_N) = \sum\nolimits_{\{I_{\ell_k}=\pm1\}}^{2^N} I_{\ell_N} \times \frac{1 + I_{\ell_N}\boldsymbol{n}_{\ell_N}\boldsymbol{r}_{\rm ens}\big(t_N\big|I_{\ell_{N-1}}\boldsymbol{n}_{\ell_{N-1}},t_{N-1}\big)}{2} \times \prod\nolimits_{k=2}^{N-1}\big[I_{\ell_k}p\big(I_{\ell_k},t_k\big|I_{\ell_{k-1}},t_{k-1}\big)\big] \times I_{\ell_1}p(I_{\ell_1},t_1), \tag{24}$$

where

$$\boldsymbol{r}_{\rm ens}\big(t_N\big|I_{\ell_{N-1}}\boldsymbol{n}_{\ell_{N-1}},t_{N-1}\big) = I_{\ell_{N-1}}\mathcal{P}(t_N|t_{N-1})\,\boldsymbol{n}_{\ell_{N-1}} + \boldsymbol{\mathcal{P}}_{\rm st}(t_N|t_{N-1}), \tag{25}$$

is the solution of the ensemble-averaged evolution (4) with the initial condition $I_{\ell_{N-1}}\boldsymbol{n}_{\ell_{N-1}}$ at time t_{N-1}. Note that the last line of Eq. (24) summed over all combinations of $I_{\ell_k} = \pm1$ except summation over I_{ℓ_N}, is the $(N-1)$-time correlator $K^{\rm coll}_{\ell_1\ldots\ell_{N-1}}(t_1,\ldots t_{N-1})$.

The term 1 in the second line of Eq. (24) can be removed because of summation over $I_{\ell_N} = \pm1$. After removing 1, we see that I_{ℓ_N} in the first and second lines cancel each other since $I^2_{\ell_N} = 1$. Therefore, Eq. (24) can be rewritten as

$$K^{\rm coll}_{\ell_1\ldots\ell_N}(t_1,\ldots t_N) = \sum\nolimits_{\{I_{\ell_k}=\pm1\}}^{2^{N-1}}\boldsymbol{n}_{\ell_N} \times \big[I_{\ell_{N-1}}\mathcal{P}(t_N|t_{N-1})\,\boldsymbol{n}_{\ell_{N-1}} + \boldsymbol{\mathcal{P}}_{\rm st}(t_N|t_{N-1})\big] \times \prod\nolimits_{k=2}^{N-1}\big[I_{\ell_k}p\big(I_{\ell_k},t_k\big|I_{\ell_{k-1}},t_{k-1}\big)\big] \times I_{\ell_1}p(I_{\ell_1},t_1), \tag{26}$$

where there is already no summation over the last output I_{ℓ_N}, and we used Eq. (25) for $\boldsymbol{r}_{\rm ens}$. Let us separate $K^{\rm coll}$ into two terms, corresponding to contributions from the two terms in the second line of Eq. (26). The second term (containing $\boldsymbol{\mathcal{P}}_{\rm st}$) is $\boldsymbol{n}_{\ell_N}\boldsymbol{\mathcal{P}}_{\rm st}(t_N|t_{N-1})K^{\rm coll}_{\ell_1\ldots\ell_{N-1}}(t_1,\ldots t_{N-1})$, thus coinciding with the third line of Eq. (22). In the remaining first term, let us substitute the product $\prod_{k=2}^{N-1}$ with product $\prod_{k=2}^{N-2}$ multiplied by the corresponding factor for $k = N-1$, and then use relations $I^2_{\ell_{N-1}} = 1$ and $\sum_{I_{\ell_{N-1}}=\pm1}p(I_{\ell_{N-1}},t_{N-1}|I_{\ell_{N-2}},t_{N-2}) = 1$. This gives us $\boldsymbol{n}_{\ell_N}\mathcal{P}(t_N|t_{N-1})\,\boldsymbol{n}_{\ell_{N-1}}K^{\rm coll}_{\ell_1\ldots\ell_{N-2}}(t_1,\ldots t_{N-2})$, which is the first term in Eq. (22). Thus, we have obtained the same recursive relation (22) for $K^{\rm coll}$. Therefore, we have proven that the collapse recipe gives the same result for N-time correlators as the calculation based on the stochastic evolution equation (2).

Note that the recursive relation (22) can be used directly to derive the main result of the paper: factorization of the N-time correlator in the case of unital evolution, Eq. (11) of the main text. Since $\boldsymbol{r}_{\rm st} = 0$ for unital evolution, from Eq. (21) we obtain $\boldsymbol{\mathcal{P}}_{\rm st} = 0$, so the recursive formula (22) relates the N-time correlator only with the $(N-2)$-time correlator. It is easy to see that the coefficient $\boldsymbol{n}_{\ell_N}\mathcal{P}(t_N|t_{N-1})\,\boldsymbol{n}_{\ell_{N-1}}$ is the two-time correlator $K_{\ell_{N-1}\ell_N}(t_{N-1},t_N)$, as also follows from Eq. (22) for $N = 2$, since $K = 1$ for $N = 0$. Thus, for unital evolution, the N-time correlator is a product of two-time correlator for the two latest time moments and the remaining $(N-2)$-time correlator. This gives Eq. (11) of the main text.

Section B: Experimental multi-time correlators

Our theoretical results for the correlators have been checked against experimental data from the experiment, in which a physical qubit (transmon), embedded into a 3D Al cavity, is subject to relatively fast Rabi oscillations with frequency $\Omega_{\rm R}/2\pi = 40$ MHz. The physical qubit is dispersively coupled to the two lowest cavity modes; each of them is off-resonantly driven with two sideband tones at frequencies $\omega_{{\rm r},i} \pm \Omega_{\rm rf}$ (where $\Omega_{\rm rf} \approx \Omega_{\rm R}$), with a relative phase δ_i. Here $i = z, \varphi$ labels the cavity mode that performs continuous measurement of the observable σ_i of the effective (rotating frame) qubit, and $\omega_{{\rm r},i}$ is the frequency of the corresponding cavity mode. Details of the measurement technique are discussed in Ref. [6] (see also Ref. [2]).

The measured effective qubit is defined in the frame, which rotates with frequency $\Omega_{\rm rf}$ with respect to the laboratory frame of the physical qubit. The measurement axes on the Bloch sphere of the effective qubit are determined by the relative phases δ_i of the sideband tones, with position of the effective z axis defined arbitrarily within the xz plane of the physical qubit rotations. We choose one of the measurements to be exactly the σ_z measurement; the other measurement direction is shifted by an angle φ, thus corresponding to the observable $\sigma_\varphi \equiv \sigma_z \cos\varphi + \sigma_x \sin\varphi$. In the experiment $\varphi = n\pi/10$ with integer $n = 0, 1, 2, ... 10$. The effective qubit is initialized at $t = 0$ in the middle between the measurement axes, i.e., at the states $\boldsymbol{r}_0^\pm = \pm\{\sin(\varphi/2), 0, \cos(\varphi/2)\}$. Approximately 200,000 readout trajectories are recorded for each angle φ, with approximately 100,000 trajectories for each initial state (we use only trajectories, selected by heralding the ground state at the start of a run and checking that transmon is still within the two-level subspace after the run [2]).

The ensemble-averaged evolution for the effective qubit is [2]

$$\dot{x} = -\Gamma_z x - \Gamma_\varphi \cos\varphi\,(x\cos\varphi - z\sin\varphi) + \tilde{\Omega} z - \gamma x, \tag{27}$$

$$\dot{y} = -(\Gamma_z + \Gamma_\varphi + T_2^{-1})\, y, \tag{28}$$

$$\dot{z} = \Gamma_\varphi \sin\varphi\,(x\cos\varphi - z\sin\varphi) - \tilde{\Omega} x - \gamma z, \tag{29}$$

where Γ_z and Γ_φ are the measurement-induced dephasing rates in the corresponding bases of the two measurement channels, $\tilde{\Omega}_{\rm R} = \Omega_{\rm R} - \Omega_{\rm rf}$ is a small residual Rabi oscillation frequency, $\gamma = (T_1^{-1} + T_2^{-1})/2$, and T_1 and T_2 are the intrinsic energy relaxation and dephasing times for the physical qubit. In the experiment $\Gamma_z \approx \Gamma_\varphi \approx (1.3\,\mu{\rm s})^{-1}$ (denoted Γ in the main text), $T_1 = 60\,\mu$s, $T_2 = 30\,\mu$s, and $\tilde{\Omega}_{\rm R}/2\pi \simeq 12$ kHz.

Experimental three-time correlators are calculated using the experimental (unnormalized and slightly shifted) output signals $\tilde{I}_z(t)$ and $\tilde{I}_\varphi(t)$ from the two measurement channels as

$$K^{\pm}_{\varphi z\varphi}(\Delta t_{21}, \Delta t_{32}) = \int_{t_a}^{t_a+T} \Big\langle \frac{\tilde{I}_\varphi(t+\Delta t_{21}+\Delta t_{32}) - \tilde{I}_\varphi^{\rm off}}{\Delta\tilde{I}_\varphi/2} \times \frac{\tilde{I}_z(t+\Delta t_{21}) - \tilde{I}_z^{\rm off}}{\Delta\tilde{I}_z/2}\, \frac{\tilde{I}_\varphi(t) - \tilde{I}_\varphi^{\rm off})}{\Delta\tilde{I}_\varphi/2} \Big\rangle \frac{dt}{T}, \tag{30}$$

where the time integration over duration $T = 0.2\,\mu$s is needed to reduce correlator fluctuations, the small constant offsets $\tilde{I}^{\rm off}_{z,\varphi}$ are less than 0.2 in magnitude (see [2] for details), and experimental responses are $\Delta\tilde{I}_z = 4.2$ (in Ref. [2] we used 4.0) and $\Delta\tilde{I}_\varphi = 4.4$. To avoid initial transients in the data, we use $t_a = 1\,\mu$s. The superscipts in the correlators $K^{\pm}_{\varphi z\varphi}$ correspond to the initial states $\boldsymbol{r}_0^\pm$, the ensemble averaging is over the corresponding $\sim$ 100,000 trajectories. Since theoretically $K^{-}_{\varphi z\varphi} = -K^{+}_{\varphi z\varphi}$, in Fig. 1 of the main text we plot the difference,

$$K_{z\varphi z}(\Delta t_{21}, \Delta t_{32}) = \big[K^{+}_{z\varphi z}(\Delta t_{21}, \Delta t_{32}) - K^{-}_{z\varphi z}(\Delta t_{21}, \Delta t_{32})\big]/2. \tag{31}$$

The experimental four-time correlators plotted in Fig. 2 of the main text are calculated as

$$K_{z\varphi z\varphi}(\Delta t_{21}, \Delta t_{32}, \Delta t_{43}) = \int_{t_a}^{t_a+T} \Big\langle \frac{\tilde{I}_\varphi(t+\Delta t_{41}) - \tilde{I}_\varphi^{\rm off}}{\Delta\tilde{I}_\varphi/2} \times \frac{\tilde{I}_z(t+\Delta t_{31}) - \tilde{I}_z^{\rm off}}{\Delta\tilde{I}_z/2}\, \frac{\tilde{I}_\varphi(t+\Delta t_{21}) - \tilde{I}_\varphi^{\rm off}}{\Delta\tilde{I}_\varphi/2} \times \frac{\tilde{I}_z(t) - \tilde{I}_z^{\rm off}}{\Delta\tilde{I}_z/2} \Big\rangle \frac{dt}{T}, \tag{32}$$

where $\Delta t_{31} = \Delta t_{32} + \Delta t_{21}$, $\Delta t_{41} = \Delta t_{43} + \Delta t_{31}$, the averaging is now over all trajectories (starting from both $\boldsymbol{r}_0^+$ and $\boldsymbol{r}_0^-$), and we use $T = 500\,\mu$s and $t_a = 1\,\mu$s. We need a larger averaging window T since the four-time correlators are noisier than the three-time correlators.

The theoretical lines in Figs. 1 and 2 of the main text are calculated using the two-time correlator $K_{z\varphi}(\tau) = K_{z\varphi}(t_1, t_1+\tau)$ derived in Ref. [2] (see below); for the three-time correlator we also need the average $\langle I_\varphi(t)\rangle = z(t)\cos\varphi + x(t)\sin\varphi$, which is calculated using Eqs. (27)–(29) with the initial condition $\boldsymbol{r}(0) = \boldsymbol{r}_0^\pm$. The two-time correlator $K_{z\varphi}(\tau)$ is calculated analytically as [2]

$$K_{z\varphi}(\tau) = \frac{(\Gamma_z + \Gamma_\varphi)\cos\varphi + 2\tilde{\Omega}_{\rm R}\sin\varphi}{2(\Gamma_+ - \Gamma_-)} \left(e^{-\Gamma_-\tau} - e^{-\Gamma_+\tau}\right) + \frac{\cos\varphi}{2}\left(e^{-\Gamma_-\tau} + e^{-\Gamma_+\tau}\right), \tag{33}$$

where

$$\Gamma_\pm = \frac{\Gamma_z + \Gamma_\varphi \pm \left[\Gamma_z^2 + \Gamma_\varphi^2 + 2\Gamma_z\Gamma_\varphi\cos(2\varphi) - 4\tilde{\Omega}_{\rm R}^2\right]^{1/2}}{2}. \tag{34}$$

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