

### Intrinsic noise of the single-electron transistor

A. N. Korotkov

*Institute of Nuclear Physics, Moscow State University, Moscow 119899 GSP, Russia  
and Department of Physics, State University of New York, Stony Brook, New York 11794*

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The paper is devoted to calculation of the “classical” (thermal and/or shot) intrinsic noise of the single-electron transistor (SET) caused by the stochastic character of electron tunneling. Exact solution of the master equation describing the dynamics of the SET is obtained in the frequency representation. The low-frequency limit for the spectral calculations is considered in detail.

#### I. INTRODUCTION

The progress of modern nanotechnology allows the observation of several effects associated with charging of small tunnel structures by single electrons.<sup>1-3</sup> These single-electron effects are of considerable interest not only because of their physics, but also because of the possibility to create various analog and digital “single-electron” devices.<sup>1,4,5</sup>

The most simple, but very promising, device is the so-called single-electron transistor (SET). Its basic circuit consists of two small-area tunnel junctions connected in series and biased by a dc voltage source [Fig. 1(a)]. If the temperature  $T$  and the junction capacitances  $C_j$  ( $j = 1, 2$ ) are small enough,

$$T \ll e^2/C_j, \tag{1}$$

while the tunnel resistances  $R_j$  are well above the quantum unit  $R_Q$ ,

$$R_j \gg R_Q = \pi\hbar/2e^2, \tag{2}$$

then the tunneling events become correlated. These correlations result in several remarkable features of the dc  $I$ - $V$  curve.<sup>6-13</sup> In particular, it is sensitive to fractional variations (in units of the electron charge  $e$ ) of the background charge  $Q_0$  of the central electrode (Fig. 2).

One can control the current through the SET by variation of the charge  $Q_0$ . Figure 1(b) shows an equivalent circuit of the SET coupled via the capacitance  $C_0$  to a signal source (with finite internal capacitance  $C_s$ ) generating the charge  $Q_s$ . In the case of large internal internal capacitance,  $C_s \gg C_0$ , it is possible to describe the

signal source solely in terms of its voltage  $U_s = Q_s/C_s$ .

Another way to control the SET is to couple it resistively to the signal source. However, it is much more difficult to realize this idea experimentally.

The most important feature of the SET is its very high sensitivity to variation of the charge  $Q_0$ . The best achieved experimental charge sensitivity of a SET, limited by the noise of the system as a whole, is on the order of  $10^{-4} e/\text{Hz}^{1/2}$ .<sup>14-16</sup> The ultimate sensitivity is limited only by the intrinsic noise of the SET, caused by the stochastic character of the tunneling process.

This paper is devoted to the calculation of the “classical” part of the intrinsic noise, which can be described by the “orthodox” theory of the SET.<sup>7-13</sup> The classical noise includes the thermal and shot noises; however, these two components cannot generally be separated from one another. We calculate the spectral density of the current through the transistor, the spectral density of the central-electrode potential, and their mutual spectral density. The expressions obtained are simplified using the exact solution of the master equation for the SET in the frequency representation. The low-frequency limit ( $\omega RC \ll 1$ ) is the most interesting case in practice; it is

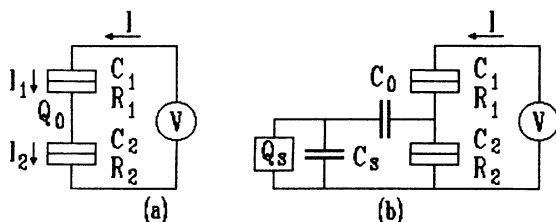


FIG. 1. (a) Basic SET transistor: two tunnel junctions in series. (b) Equivalent circuit of the capacitively coupled SET.

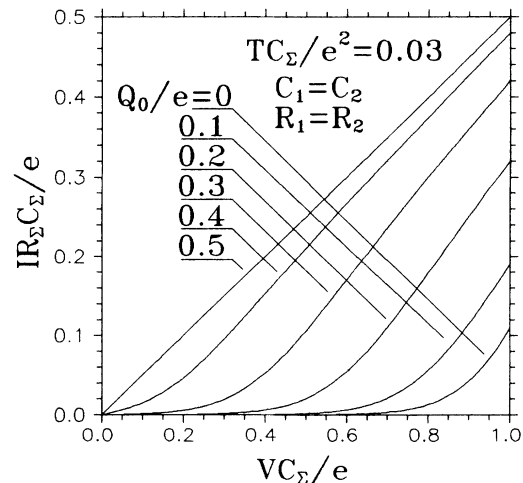


FIG. 2.  $I$ - $V$  curve of the SET for different subelectron charges  $Q_0$  on the central electrode.  $C_\Sigma = C_1 + C_2, R_\Sigma = R_1 + R_2$ .

considered in more detail. All the final expressions are convenient for numerical calculations.

Extensive numerical calculations of the classical noise were performed in Ref. 17 for optimization of the sensitivity of the SET. The present paper contains a detailed explanation of the calculation method used there and the derivation of the basic equations.

Besides the classical noise, some other noise processes can be important. For small temperatures, at voltages below or near the Coulomb-blockade threshold, cotunneling processes become essential.<sup>18,19</sup> Using the estimation of the corresponding quantum noise,<sup>17</sup> one can find the validity range of the classical noise theory. The sensitivity of the SET in present-day experiments is limited by  $1/f$  noise.<sup>16,20</sup> The most probable source of this noise is the fluctuations of the impurity charge configurations.<sup>15,16,20</sup> One can hope that progress in fabrication technology could reduce the excess noise to values less than or of the order of the intrinsic noise of the SET.

The paper is organized as follows. The next section contains a brief review of the orthodox theory of the SET. The general formalism for the classical noise in the SET is presented in the third section [Eqs. (12), (13), (18)–(22)]. A simple matrix expression for the exact solution of the master equation [Sec. IV, Eq. (26)] allows us to obtain much more convenient equations for the classical noise [Sec. V, Eqs. (27)–(31)]. The low-frequency limit for these calculations is considered in Sec. VI. In particular, in this limit Eqs. (27)–(31) should be replaced with Eqs. (34)–(36). An alternative method of low-frequency noise calculation is presented in Sec. VII. This method is based on the idea of division of the random process into independent elementary segments (this idea can be applied also for calculations of some other single-electron processes). In the last section, several results for the ultimate sensitivity of SET are presented.

## II. ORTHODOX THEORY OF THE SINGLE-ELECTRON TRANSISTOR

Let us consider two small-area metal tunnel junctions connected in series and biased by a voltage source  $V$  [Fig. 1(a)]. We follow the orthodox theory of this device<sup>6–13</sup> and consider the electron transport as a sequence of instantaneous tunneling events. The rates  $\Gamma_{1,2}^{\pm}(n)$  of electron tunneling to (+) or from (–) the central electrode through the first or second junction depend on the number  $n$  of excess electrons in the central electrode (charge state  $n$ )

$$\Gamma_j^{\pm}(n) = [W_j^{\pm}(n)/(e^2 R_j)]/[1 - \exp(-W_j^{\pm}(n)/T)], \quad (3)$$

$$W_j^{\pm}(n) = \frac{e^2}{C_{\Sigma}} \left[ \mp (-1)^j \frac{VC_1 C_2}{eC_j} - \frac{1}{2} \mp n \mp \frac{Q_0}{e} \right]. \quad (4)$$

Here  $C_{1,2}, R_{1,2}$  are the capacitance and resistances of the junctions,  $C_{\Sigma} = C_1 + C_2$ ,  $Q_0$  is the background charge, and  $T$  is the temperature. The energy gain  $W_j^{\pm}(n)$  is the decrease of the Gibbs energy of the system

$$U(n_1, n_2) = \frac{Q^2}{2C_{\Sigma}} - \frac{eV}{C_{\Sigma}}(C_1 n_2 + C_2 n_1) + \text{const}, \quad (5)$$

$$Q = en + Q_0, \quad n = n_1 - n_2,$$

sed to the particular tunneling event. In this last expression,  $n_1, n_2$  are the number of electrons passed through the first and second junctions (the positive direction is chosen from the first to second junction). The electron charge is considered as positive for simplicity (then the directions of the current and electron motion coincide); this choice does not change any final result. Note that for semiconductor-based SET's, all general expressions for the noise derived in this paper remain valid, except that tunneling rates  $\Gamma_j^{\pm}(n)$  are given by expressions more complicated than Eq. (3).<sup>21,22</sup>

Because the tunneling events are considered as instantaneous, the current  $I(t)$  consists of  $\delta$ -like peaks, and the potential of the central electrode  $\varphi(t)$  is steplike:

$$\varphi(t) = VC_1/C_{\Sigma} + [en(t) + Q_0]/C_{\Sigma}, \quad (6)$$

$$I(t) = I_1(t)C_2/C_{\Sigma} + I_2(t)C_1/C_{\Sigma}, \quad (7)$$

$$I_1(t) = e\dot{n}_1(t), \quad I_2(t) = e\dot{n}_2(t).$$

Here  $I_1(t)$  and  $I_2(t)$  are the currents through the junctions and  $I(t)$  is the total current [see Fig. 1(a)].

The dynamics of electron tunneling in single-electron circuits can be described in orthodox theory by two different approaches. In the Langevin approach we consider random jumps between the different charge states of the system. This method is useful for long arrays of small tunnel junctions<sup>23</sup> and other complicated systems. We will use this approach below for derivation of Eqs. (18), (19), and in Sec. VII. In the second, somewhat simpler approach based on a Fokker-Planck-type equation, one considers the deterministic dynamics of the values averaged over the statistical ensemble.

In the latter approach the dynamics of tunneling in the SET is described by the master equation for the probability  $\sigma(n)$  of the charge state  $n$  of the double junction system:

$$\frac{d\sigma(n,t)}{dt} = \sum_{j=1,2} \{ \Gamma_j^+(n-1)\sigma(n-1,t) + \Gamma_j^-(n+1)\sigma(n+1,t) - [\Gamma_j^+(n) + \Gamma_j^-(n)]\sigma(n,t) \}. \quad (8)$$

The stationary probability distribution  $\sigma_{\text{st}}(n)$  satisfies the equations

$$\sigma_{\text{st}}(n)[\Gamma_1^+(n) + \Gamma_2^+(n)] = \sigma_{\text{st}}(n+1)[\Gamma_1^-(n+1) + \Gamma_2^-(n+1)], \quad (9)$$

$$\sum_n \sigma_{\text{st}}(n) = 1.$$

In stationary state the average currents through the junctions are equal to each other:

$$\langle I_1(t) \rangle = \langle I_2(t) \rangle = \langle I(t) \rangle = I.$$

$$I = e \sum_{n=-\infty}^{+\infty} [\Gamma_1^+(n) - \Gamma_1^-(n)] \sigma_{st}(n) \quad (10)$$

$$= e \sum_n [\Gamma_2^-(n) - \Gamma_2^+(n)] \sigma_{st}(n),$$

and the average potential  $\langle \varphi(t) \rangle$  is given by the expression

$$\langle \varphi(t) \rangle = \sum_n \varphi(n) \sigma_{st}(n). \quad (11)$$

### III. GENERAL FORMALISM FOR THE CLASSICAL NOISE

In this section we calculate the spectral density of the current  $S_{II}(\omega)$ , the spectral density of the central-electrode potential  $S_{\varphi\varphi}(\omega)$ , and their mutual spectral density  $S_{I\varphi}(\omega)$ . It is well known that the mutual spectral density  $S_{XY}(\omega)$  for two arbitrary (possibly identical) stationary random real processes  $X(t)$  and  $Y(t)$  can be calculated as the Fourier transform of the correlation function  $K_{XY}(\tau)$ :

$$S_{XY}(\omega) = 2 \int_{-\infty}^{+\infty} K_{XY}(\tau) \cos(\omega\tau) d\tau - 2i \int_{-\infty}^{+\infty} K_{XY}(\tau) \sin(\omega\tau) d\tau, \quad (12)$$

$$K_{XY}(\tau) = \langle X(t+\tau)Y(t) \rangle - \langle X \rangle \langle Y \rangle.$$

Note that for  $X(t) = Y(t)$  the imaginary part of  $S_{XY}(\omega)$  is absent.

The autocorrelation function  $K_{\varphi\varphi}(\tau)$  for the central-electrode potential can be easily calculated using Eq. (8):

$$K_{\varphi\varphi}(\tau) = -\langle \varphi \rangle^2 + \sum_{n,n'} \varphi(n') \sigma(n', \tau | n) \varphi(n) \sigma_{st}(n), \quad \tau \geq 0 \quad (13)$$

$$K_{\varphi\varphi}(-\tau) = K_{\varphi\varphi}(\tau).$$

Here  $\sigma(n', \tau | n)$  means the probability of the charge state

$$K_{I_j I_j}(\tau) = -I^2 + \sum_{n,n'} e \{ [\Gamma_j^+(n') - \Gamma_j^-(n')] [\sigma(n', \tau | n+1) e \Gamma_j^+(n) - \sigma(n', \tau | n-1) e \Gamma_j^-(n)] \sigma_{st}(n) + A_j \delta(\tau), \quad \tau \geq 0, \quad (18)$$

$$K_{I_j I_j}(-\tau) = K_{I_j I_j}(\tau).$$

The last singular term in Eq. (18) is the consequence of the consideration of tunneling events as instantaneous. At  $\tau=0$  the derivation above fails, and the constants  $A_j$  should be calculated independently (see below).

Let us emphasize that in Eq. (18) the initial state for the evolution (during time  $\tau$ ) is not  $n$  but  $n \pm 1$ . One can say that the current at  $t=0$  behaves as an operator which changes the charge state by  $\pm 1$ .

In order to calculate the autocorrelation function  $K_{II}(\tau)$  for the total current one should take into account tunneling events in both junctions. The expression below can be derived in a similar way to Eq. (18), and this result is quite clear if we consider the current as an operator:

$$K_{II}(\tau) = -I^2 + \sum_{n,n'} e^2 \{ [\Gamma_1^+(n') - \Gamma_1^-(n')] C_2 / C_\Sigma + [\Gamma_2^-(n') - \Gamma_2^+(n')] C_1 / C_\Sigma \} \\ \times \{ \sigma(n', \tau | n+1) [\Gamma_1^+(n) C_2 / C_\Sigma - \Gamma_2^+(n) C_1 / C_\Sigma] \\ + \sigma(n', \tau | n-1) [\Gamma_2^-(n) C_1 / C_\Sigma - \Gamma_1^-(n) C_2 / C_\Sigma] \} \sigma_{st}(n) + A \delta(\tau), \quad \tau \geq 0, \quad (19)$$

$$K_{II}(-\tau) = K_{II}(\tau).$$

$n'$  at the moment  $t=\tau$  ( $\tau \geq 0$ ) if the charge state  $n$  was realized at  $t=0$ . It is the solution of Eq. (8) satisfying the initial condition  $\sigma(n', 0 | n) = \delta_{n,n'}$ .

It is important that for the autocorrelation function of the current  $K_{II}(\tau)$ , an expression similar to Eq. (13) is not valid. The reason for this difference is that the current  $I(t)$  is not a function of  $n$ , but depends on the derivatives of  $n_{1,2}(t)$  [see Eq. (7)]. In order to derive the correct expression for  $K_{II}(\tau)$  we should use the Langevin approach.

The only nonvanishing contributions to the tunneling currents  $I_j$  ( $j=1,2$ ) in orthodox theory come from moments  $t_j$  corresponding to the tunneling through the  $j$ th function:

$$I_j(t) = (-1)^{j+1} e \sum_i s_i \delta(t - t_i), \quad (14)$$

where  $s_i = 1$  for the process increasing the internal charge state  $n$  by unity and  $s_i = -1$  when  $n$  is decreased. Thus, the product  $dQ_j(t_1) dQ_j(t_2) \equiv I_j(t_1) dt_1 I_j(t_2) dt_2$  is nonvanishing only if we have tunneling events inside each of the infinitesimal time intervals  $dt_1, dt_2$ . In this case,

$$dQ_j(t_1) dQ_j(t_2) = e^2 s_1 s_2. \quad (15)$$

The probability of such a double event is

$$P_{s_1 s_2}(t_1, t_2) = \Gamma_j^\pm(n_2) dt_2 \sigma(n_2, t_2 - t_1 | n_1 + s_1) \\ \times \Gamma_j^\pm(n_1) dt_1 \sigma_{st}(n_1), \quad (16)$$

where  $n(t_1) = n_1$ ,  $n(t_1 + dt_1) = n_1 + s_1$ ,  $n(t_2) = n_2$ ,  $n(t_2 + dt_2) = n_2 + s_2$ , and the choice of the upper indices  $\pm$  corresponds to the signs of  $s_1, s_2$ . Now, summing all the contributions to the product

$$\langle I_j(t_1) I_j(t_2) \rangle = \frac{\langle dQ_j(t_1) dQ_j(t_2) \rangle}{dt_1 dt_2}, \quad (17)$$

i.e., summing over the different  $n_1, n_2, s_1, s_2$  with the corresponding probabilities (16), one obtains the following expression for the autocorrelation function for the current  $I_j$ :

Constants  $A, A_j$  in Eqs. (18), (19) can be calculated from the high-frequency limit for spectral densities. Taking into account that for  $\omega RC \gg 1$ , four noise processes (two junctions, two directions) can be considered as uncorrelated, let us use the shot-noise formula:

$$\begin{aligned} \lim_{\omega \rightarrow \infty} S_{II}(\omega) &= 2A, \quad \lim_{\omega \rightarrow \infty} S_{I_j I_j}(\omega) = 2A_j, \\ A &= (C_2/C_\Sigma)^2 A_1 + (C_1/C_\Sigma)^2 A_2, \\ A_j &= e(I_j^+ + I_j^-), \quad I_j^\pm = e \sum_n \sigma_{st}(n) \Gamma_j^\pm(n). \end{aligned} \quad (20)$$

Here  $A_j/e$  is the sum of the components of the current flowing through the  $j$ th junction in opposite directions.

For completeness, let us consider the correlation between currents  $I_1(t)$  and  $I_2(t)$ . Using the same method of considering the current as an operator, one can write the following equation ( $\tau \geq 0$ ):

$$K_{I_1 I_2}(\tau) = -I^2 + \sum_{n, n'} e[\Gamma_1^+(n') - \Gamma_1^-(n')] [\sigma(n', \tau|n-1)e\Gamma_2^-(n) - \sigma(n', \tau|n+1)e\Gamma_2^+(n)] \sigma_{st}(n). \quad (21)$$

The expression for  $K_{I_2 I_1}(\tau)$  can be readily obtained from Eq. (21) by exchanging indices  $1 \leftrightarrow 2$ . The case  $\tau \leq 0$  is covered by the equation  $K_{I_1 I_2}(\tau) = K_{I_2 I_1}(-\tau)$ . Note that  $K_{I_1 I_2}(\tau)$  shows a jump at  $\tau=0$  but has no singularity because the probability of simultaneous tunneling events is equal to zero in orthodox theory.

Deriving the expression for the mutual correlation function  $K_{I_\varphi}(\tau)$  [which is the linear combination of functions  $K_{I_j \varphi}(\tau)$ ], one should consider two cases. If the current is taken at the moment after the potential ( $\tau > 0$ ), then the evolution during time interval  $\tau$  starts from the same charge state  $n$  as just before the earlier moment and the expression is similar to Eq. (13). However, if  $\tau < 0$ , the "operator of current" changes the charge state before the evolution, which now starts from the state  $n \pm 1$  and lasts during time  $-\tau$ . In this case the expression is similar to Eq. (18). Taking into account both cases, one obtains

$$\begin{aligned} K_{I_\varphi}(\tau) &= (C_2/C_\Sigma) K_{I_1 \varphi}(\tau) + (C_1/C_\Sigma) K_{I_2 \varphi}(\tau), \\ K_{I_j \varphi}(\tau) &= -I \langle \varphi \rangle + \sum_{n, n'} e(-1)^{j+1} [\Gamma_j^+(n') - \Gamma_j^-(n')] \sigma(n', \tau|n) \varphi(n) \sigma_{st}(n) \quad \text{for } \tau > 0, \\ K_{I_j \varphi} &= -I \langle \varphi \rangle + e(-1)^{j+1} \sum_{n, n'} \varphi(n') [\sigma(n', -\tau|n+1) \Gamma_j^+(n) - \sigma(n', -\tau|n-1) \Gamma_j^-(n)] \sigma_{st}(n) \quad \text{for } \tau < 0. \end{aligned} \quad (22)$$

This correlation function is not symmetrical and shows a jump at  $\tau=0$  [the singularity is absent because  $\varphi(t)$  is finite]. As a consequence, the mutual spectral density  $S_{I_\varphi}(\omega)$  is complex.

Equations (12), (13), (18)–(22) give the complete method for calculation of the classical noise described by the orthodox theory of the SET. However, it requires solution of the master equation (8), which is a set of linear differential equations. In the next two sections we describe a simple matrix method to solve this problem. A similar matrix method for solution of the master equation can be useful for calculations of some other single-electron circuits, for example, for calculation of the spectrum of the current in an array of junctions.<sup>24</sup>

#### IV. MATRIX METHOD FOR THE MASTER EQUATION

Let us solve the master equation (8) in the frequency representation, following Ref. 25. Equation (8) has the following matrix form:

$$\begin{aligned} \frac{d\sigma(n)}{dt} &= \sum_m \Gamma_{nm} \sigma(m), \\ \Gamma_{nm} &= \delta_{m, n-1} [\Gamma_1^+(m) + \Gamma_2^+(m)] \\ &\quad + \delta_{m, n+1} [\Gamma_1^-(m) + \Gamma_2^-(m)] \\ &\quad - \delta_{m, n} [\Gamma_1^+(m) + \Gamma_1^-(m) + \Gamma_2^+(m) + \Gamma_2^-(m)]. \end{aligned} \quad (23)$$

Let us introduce a new function, which is a kind of retarded Green's function for the master equation:

$$\bar{\sigma}_{nm}(\tau) = \begin{cases} \sigma(n, \tau|m) - \sigma_{st}(n), & \tau > 0, \\ 0, & \tau \leq 0. \end{cases} \quad (24)$$

Note that  $\sum_n \bar{\sigma}_{nm}(\tau) = 0$ , and  $\bar{\sigma}_{nm} \rightarrow 0$  at  $\tau \rightarrow \infty$ .

For this function one can write the following differential equation

$$\frac{d\bar{\sigma}_{nm}(\tau)}{d\tau} = \sum_k \Gamma_{nk} \bar{\sigma}_{km}(\tau) + [\delta_{nm} - \sigma_{st}(n)] \delta(\tau). \quad (25)$$

The Fourier transforms  $\bar{\sigma}_{nm}(\omega)$  of the functions  $\bar{\sigma}_{nm}(\tau)$  satisfy a linear algebraic equation which has the solution

$$\begin{aligned} \bar{\sigma}_{nm}(\omega) &= \sum_k \left[ \frac{1}{i\omega \hat{\Gamma} - \hat{\Gamma}} \right]_{nk} [\delta_{km} - \sigma_{st}(k)] \\ &= \left[ \frac{1}{i\omega \hat{\Gamma} - \hat{\Gamma}} \right]_{nm} - \frac{\sigma_{st}(n)}{i\omega}, \end{aligned} \quad (26)$$

where  $\hat{\Gamma}$  is the unit matrix, and  $\hat{\Gamma}$  has matrix elements  $\Gamma_{ij}$  (23). This equation is the exact solution of the master equation (8) in the frequency representation.

It is possible to use another definition of  $\bar{\sigma}_{nm}$  where the term  $-\sigma_{st}(n)$  is removed from Eqs. (24)–(26). However,

we use the definition above because it provides Fourier transforms without singularity at zero frequency.

### V. MATRIX FORM FOR THE SPECTRAL DENSITIES

Now we can considerably simplify the method of calculation of  $S_{II}(\omega)$ ,  $S_{\varphi\varphi}(\omega)$ , and  $S_{I\varphi}(\omega)$  described in Sec. III. Let us substitute  $\tilde{\sigma}_{n',n}(\tau) + \sigma_{st}(n')$  for  $\sigma(n',\tau|n)$  in Eqs. (13), (18), (19), (21), and (22). The terms containing  $\sigma_{st}(n')$  exactly cancel the terms  $-I^2$  and  $-I\langle\varphi\rangle$ . Using Eq. (12) it is possible to express the spectral densities in terms of  $\tilde{\sigma}_{n',n}(\omega)$ .

For the potential  $\varphi$ , one obtains the following spectral density:

$$\begin{aligned} S_{\varphi\varphi}(\omega) &= 4 \sum_{n,n'} \varphi(n') [\text{Re} \tilde{\sigma}_{n',n}(\omega)] \varphi(n) \sigma_{st}(n) \\ &= 4 \sum_{n,n'} \varphi(n') B_{n',n} \varphi(n) \sigma_{st}(n), \end{aligned} \quad (27)$$

where  $\hat{B} = \text{Re}(1/(i\omega\hat{1} - \hat{\Gamma}))$ .

In the same manner using Eqs. (12), (18), (19), (21), (26) for the spectral densities of the currents, one obtains

$$S_{I_1 I_1}(\omega) = 2A_j + 4e^2 \sum_{n,n'} [\Gamma_j^+(n') - \Gamma_j^-(n')] B_{n',n} [\Gamma_j^+(n-1)\sigma_{st}(n-1) - \Gamma_j^-(n+1)\sigma_{st}(n+1)], \quad (28)$$

$$\begin{aligned} S_{I_1 I_2}(\omega) &= 2e^2 \sum_{n,n'} [\Gamma_1^+(n') - \Gamma_1^-(n')] \left[ \frac{1}{i\omega\hat{1} - \hat{\Gamma}} \right]_{n',n} [\Gamma_2^-(n+1)\sigma_{st}(n+1) - \Gamma_2^+(n-1)\sigma_{st}(n-1)] \\ &\quad + 2e^2 \sum_{n,n'} [\Gamma_2^-(n') - \Gamma_2^+(n')] \left[ \frac{1}{-i\omega\hat{1} - \hat{\Gamma}} \right]_{n',n} [\Gamma_1^+(n-1)\sigma_{st}(n-1) - \Gamma_1^-(n+1)\sigma_{st}(n+1)], \end{aligned} \quad (29)$$

$$\begin{aligned} S_{II}(\omega) &= 2A + 4(e/C_\Sigma)^2 \sum_{n,n'} \{ [\Gamma_1^+(n') - \Gamma_1^-(n')] C_2 + [\Gamma_2^-(n') - \Gamma_2^+(n')] C_1 \} \\ &\quad \times B_{n',n} \{ [\Gamma_1^+(n-1)C_2 - \Gamma_2^+(n-1)C_1] \sigma_{st}(n-1) \\ &\quad + [\Gamma_2^-(n+1)C_1 - \Gamma_1^-(n+1)C_2] \sigma_{st}(n+1) \}. \end{aligned} \quad (30)$$

Here the shift of the variable  $n \rightarrow n \pm 1$  was made for convenience.

Finally we get the following expressions for the mutual spectral density:

$$S_{I\varphi}(\omega) = C_2/C_\Sigma S_{I_1\varphi}(\omega) + C_1/C_\Sigma S_{I_2\varphi}(\omega),$$

$$\begin{aligned} S_{I_1\varphi}(\omega) &= 2e(-1)^{j+1} \sum_{n,n'} [\Gamma_j^+(n') - \Gamma_j^-(n')] \left[ \frac{1}{i\omega\hat{1} - \hat{\Gamma}} \right]_{n',n} \varphi(n) \sigma_{st}(n) \\ &\quad + 2e(-1)^{j+1} \sum_{n,n'} \varphi(n') \left[ \frac{1}{-i\omega\hat{1} - \hat{\Gamma}} \right]_{n',n} [\Gamma_j^+(n-1)\sigma_{st}(n-1) - \Gamma_j^-(n+1)\sigma_{st}(n+1)]. \end{aligned} \quad (31)$$

Note that in Eqs. (27)–(31) the contribution of the term  $-\sigma_{st}(n)/i\omega$  from Eq. (26) is equal to zero. It is the obvious consequence of the possibility of defining  $\tilde{\sigma}_{nm}$  in different ways. Expressions (27)–(31) are convenient for numerical calculations, because the inverse to the three-diagonal matrix  $i\omega\hat{1} - \hat{\Gamma}$  can be calculated very rapidly.

The numerical results for  $S_{II}(\omega)$ ,  $S_{I_1 I_1}(\omega)$ , and  $S_{I_2 I_2}(\omega)$  in one particular case are shown in Fig. 3. It is interesting that  $S_{I_1 I_1}(\omega)$  grow with frequency, because the corresponding correlation functions are negative (such an anticorrelation is the consequence of the Coulomb repulsion of the electrons, which reduces the probability of a definite tunneling event after tunneling through the same junction in the same direction). At high frequency the curves satisfy Eq. (20). They do not tend to zero at  $\omega \rightarrow \infty$  because of the assumption of instantaneous tunneling events accepted in the orthodox theory. One can see that the spectral densities of the currents do not have peaks, in contrast to the case of a single tunnel junction biased by the current, where single-electron oscillations take place.<sup>1,2</sup>

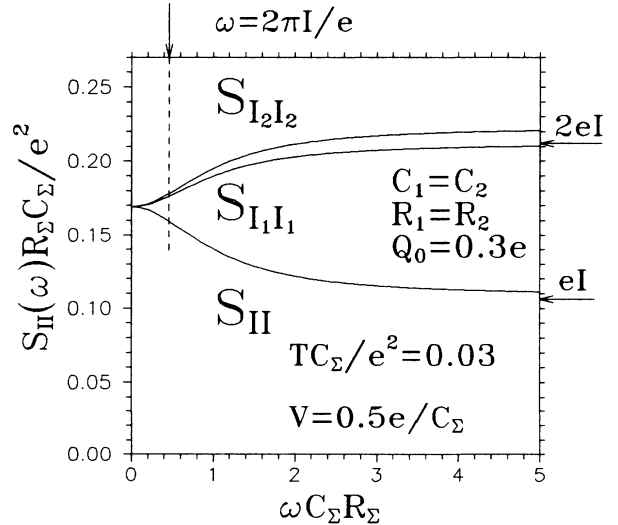


FIG. 3. Spectral densities of the currents  $I_1, I_2$  through tunnel junctions and spectral density of the total current  $I$  as functions of the frequency  $\omega$ .

To conclude this section, let us discuss the possibility of calculating spectral densities of fluctuations by Monte Carlo simulation of the random tunneling process.<sup>12,23</sup> The spectral density of the current through a SET was calculated by this method in Ref. 12. However, the results obtained there were not very accurate (the standard deviation of the calculations was about 20%). The reason for this inaccuracy is that the spectral density was calculated in Ref. 12 (as well as in Ref. 23 where fluctuations in an array of junctions were considered) using Eq. (12), and this is not the best choice for numerical calculations. Let us discuss briefly a different method (it was developed for arrays of junctions, but it is valid for any single-electron circuit, and, as a simple example, for a SET).

From Eq. (12) it is easy to derive the following equation:

$$\int_{-\infty}^{+\infty} S_{XY}(\omega + \Omega) \frac{1 - \cos(\Omega T)}{\pi T \Omega^2} d\Omega = 2 \left\langle \left[ \int_t^{t+T} X(\tau_1) \exp(i\omega\tau_1) d\tau_1 \right] \times \int_t^{t+T} Y(\tau_2) \exp(-i\omega\tau_2) d\tau_2 \right\rangle. \quad (32)$$

One can see that in the limit  $T \rightarrow \infty$  the left side of this equation transforms to the spectral density  $S_{XY}(\omega)$ . Even for finite  $T$  this is a good approximation of  $S_{XY}(\omega)$ . In the numerical calculations one should divide the simulation process into segments with time duration  $T$  ( $t = t_k = kT, k = 1, 2, \dots$ ) and then average the right side of Eq. (32) over different segments. It is convenient to calculate spectral densities simultaneously at many equidistant frequency points  $\omega_i = i\omega_0, i = 1, 2, \dots$ . To avoid inaccurate results at low frequencies [when the point  $\omega = 0$  can give an essential contribution to the left side of Eq. (32)], one should take  $\omega_0/2\pi$  equal to an integer multiple (the best choice is 1) of  $T^{-1}$ .

This method decreases the computer time needed for achievement of the desired accuracy by a factor of about  $10^2$ . As a consequence, it becomes possible to obtain more accurate results.

We have compared the numerical results of exact calculations [Eqs. (27)–(30)] and Monte Carlo simulations for the spectral density of the current and the central-electrode potential of the SET. The results coincide, but the latter method required much more computer time to

achieve a comparable accuracy. Thus, Monte Carlo simulations is not the best choice for calculation of the fluctuations in a SET; however, it is the only practical method for more complicated single-electron circuits.

## VI. LOW-FREQUENCY LIMIT

In most applications one needs only the low-frequency limit ( $\omega RC \ll 1$ ) of the noise intensity, because the characteristic frequency in the master equation,  $(RC)^{-1} \sim 10^9 - 10^{11}$  Hz, is much higher than the typical frequency of the signal which controls the SET. Let us consider this limit in more detail.

At  $\omega \rightarrow 0$  the spectral densities for the currents  $I(t)$ ,  $I_1(t)$ , and  $I_2(t)$  coincide (see Fig. 3). This is a consequence of the equation

$$I_1(t) - I_2(t) = e\dot{n}(t) \quad (33)$$

and the finite dispersion of  $n$ . The mutual spectral density  $S_{I\varphi}(0)$  is real, in accordance with Eq. (12).

It is important that Eqs. (27)–(31) cannot be applied at  $\omega = 0$  directly because the matrix  $\hat{\Gamma}$  is singular ( $\sum_n \Gamma_{nm} = 0$ ) and cannot be inverted. However, modification of these equations and the correct use of the matrix  $\hat{\Gamma}^{-1}$  can solve the problem.

If some column  $X$  has a zero sum of elements, then the expression  $\hat{\Gamma}^{-1}X$  has mathematical sense, i.e., it is possible to find a column  $Y$  which satisfies the equation  $X = \hat{\Gamma}Y$ . However, the infinite number of columns  $Y$  which have different additional terms  $\text{const} \times \sigma_{st}(n)$  can be regarded as  $\hat{\Gamma}^{-1}X$  because  $\hat{\Gamma}\sigma_{st} = 0$ . So, to avoid the difficulty in Eqs. (27)–(31) at  $\omega = 0$ , one should modify these equations in such a way that  $\hat{\Gamma}^{-1}$  acts on a column which has a zero sum of elements, and the arbitrary term  $\text{const} \times \sigma_{st}(n)$  after the matrix operation  $\hat{\Gamma}^{-1}$  is unimportant.

Note that the arbitrary constant can be added to the terms at the left side of the matrix  $[i\omega\hat{\Gamma} - \hat{\Gamma}]^{-1}$  in Eqs. (27)–(31) because  $\sum_m \Gamma_{mn} = 0$ . Also, any constant multiplied by  $\sigma_{st}(n)$  can be added to the terms at the right side of this matrix because  $\hat{\Gamma}\sigma_{st} = 0$ . It is easy to prove that the expressions below are equivalent to Eqs. (27), (28), (31) at  $\omega \rightarrow 0$ ; on the other hand, they are appropriate for the numerical calculations:

$$S_{\varphi\varphi}(0) = 4 \sum_{n, n'} [\varphi(n') - \langle \varphi \rangle] (-\hat{\Gamma}^{-1})_{n'n} [\varphi(n) - \langle \varphi \rangle] \sigma_{st}(n), \quad (34)$$

$$S_{II}(0) = 2A_1 + 4e^2 \sum_{n, n'} [\Gamma_1^+(n') - \Gamma_1^-(n') - I/e] (-\hat{\Gamma}^{-1})_{n'n} \times [\Gamma_1^+(n-1)\sigma_{st}(n-1) - \Gamma_1^-(n+1)\sigma_{st}(n+1) - (I/e)\sigma_{st}(n)], \quad (35)$$

$$S_{I\varphi}(0) = 2e \sum_{n, n'} \{ [\Gamma_1^+(n') - \Gamma_1^-(n') - I/e] (-\hat{\Gamma}^{-1})_{n'n} [\varphi(n) - \langle \varphi \rangle] \sigma_{st}(n) + [\varphi(n') - \langle \varphi \rangle] (-\hat{\Gamma}^{-1})_{n'n} [\Gamma_1^+(n-1)\sigma_{st}(n-1) - \Gamma_1^-(n+1)\sigma_{st}(n+1) - (I/e)\sigma_{st}(n)] \}. \quad (36)$$

The only consequence of the singularity of matrix  $\hat{\Gamma}$  in these equations is that one should not try to find  $\hat{\Gamma}^{-1}$  numerically. Instead, one has to calculate immediately the result of the summation over  $n$  (the usual numerical procedure for a three-diagonal matrix can be used).

The typical numerical results of calculations using Eqs. (34)–(36) are shown in Fig. 4 for the symmetric SET ( $C_1=C_2, R_1=R_2$ ). Instead of the value  $S_{I\varphi}(0)$ , the correlation factor  $S_{I\varphi}(0)/[S_{II}(0)S_{\varphi\varphi}(0)]^{1/2}$  is shown. One can see that  $S_{II}(0)$  and  $S_{\varphi\varphi}(0)$  increase with voltage. Asymptotic values at  $V \rightarrow \infty$  can be calculated analytically as the shot noise in a system with two noise sources:

$$S_{II}(0) = 2eI(R_1^2 + R_2^2)/(R_1 + R_2)^2, \quad (37)$$

$$S_{\varphi\varphi}(0) = 4eI/(R_1^{-1} + R_2^{-1})^2, \quad (38)$$

$$\begin{aligned} S_{I\varphi}(0)/[S_{II}(0)S_{\varphi\varphi}(0)]^{1/2} \\ = (R_1 - R_2)/[2(R_1^2 + R_2^2)]^{1/2}. \end{aligned} \quad (39)$$

The low-voltage case can be treated analytically when the temperature is very large or very small in comparison with the energy  $e^2/C_\Sigma$ .

If  $T \gg e^2/C_\Sigma$  and  $eV \ll T$ , the single-electron effects are negligible and the noise is described by the Nyquist formula

$$S_{II}(0) = \frac{4T}{R_1 + R_2}, \quad S_{\varphi\varphi}(0) = \frac{4TR_1R_2}{R_1 + R_2}, \quad S_{I\varphi}(0) = 0. \quad (40)$$

If  $T \ll e^2/C_\Sigma$  (the most interesting case), at low voltages the tunneling events are strongly correlated. If the voltage is well below the Coulomb-blockade threshold, then the current is practically zero (in the orthodox theory we ignore the cotunneling processes<sup>18,19</sup>), so  $S_{II}(0)$ ,  $S_{\varphi\varphi}(0)$ , and  $S_{I\varphi}(0)$  vanish. There is not much reason to calculate the correlation factor between  $I(t)$  and  $\varphi(t)$  in this range (the curve in Fig. 4 is cut off), although it is not equal to

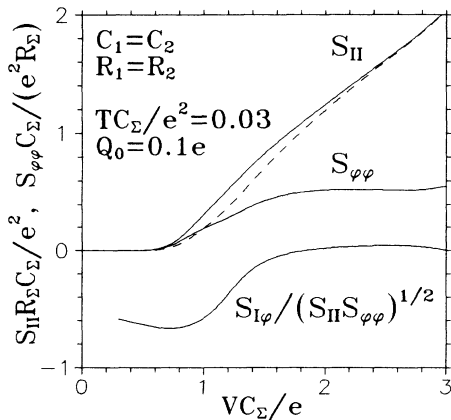


FIG. 4. Low-frequency spectral densities of the current  $I$  and electric potential  $\varphi$  of the central electrode, and their correlation factor as functions of the bias voltage. The dashed line shows the dc  $I$ - $V$  curve of the system (the unit of current is  $e/R_\Sigma C_\Sigma$ ).

zero.

If the voltage is close to the Coulomb-blockade threshold, then in the general case (excepting degenerate cases) only two charge states are essential, and tunneling events in different junctions can occur only in turn. Let the corresponding rates be  $\Gamma_1$  and  $\Gamma_2$ . For this simple particular case, Eqs. (34)–(36) as well as Eqs. (47)–(49), (66)–(77) give the following expressions:

$$S_{II}(0) = 2eI(\Gamma_1^2 + \Gamma_2^2)/(\Gamma_1 + \Gamma_2)^2, \quad I = e\Gamma_1\Gamma_2/(\Gamma_1 + \Gamma_2), \quad (41)$$

$$S_{\varphi\varphi}(0) = (e/C_\Sigma)^2 4\Gamma_1\Gamma_2/(\Gamma_1 + \Gamma_2)^3, \quad (42)$$

$$\begin{aligned} S_{I\varphi}(0)/[S_{II}(0)S_{\varphi\varphi}(0)]^{1/2} \\ = (\Gamma_1 - \Gamma_2)/[2(\Gamma_1^2 + \Gamma_2^2)]^{1/2}. \end{aligned} \quad (43)$$

An expression similar to Eq. (42) was derived for the random process with two time constants in Ref. 26, and Eq. (41) coincides with the results of Ref. 27 [in these papers, spectral densities for arbitrary frequency were derived; the expressions obtained there can be considered as special cases of Eqs. (27), (30)].

The important case for calculation of the ultimate sensitivity of the SET is when the temperature is small and the voltage is very close to the blockade threshold. Then  $\Gamma_1 \gg \Gamma_2$  (or  $\Gamma_1 \ll \Gamma_2$ ) and Eq. (41) transforms to the usual Schottky formula (since pairs of almost simultaneous jumps in different junctions form a random Poisson process):

$$S_{II}(0) = 2eI. \quad (44)$$

The accuracy of this simple expression becomes poor with temperature or voltage increase [see Fig. 4 and Eqs. (37), (40)]; Eq. (44) cannot be used for numerical calculations of the noise intensity.<sup>16,17</sup>

## VII. ALTERNATIVE METHOD FOR THE ZERO-FREQUENCY LIMIT

Along with the method described in Secs. III–VI, a completely different method for calculation of the spectral densities in the zero-frequency limit was also developed. It is based on the idea of the path-integral approach to random walks.<sup>28</sup> The method appeared to be a little faster and was extensively used in Ref. 17 for optimization of the theoretical sensitivity of SET as an electrometer.

Instead of solutions of the master equation (8), one can consider the process (Markov chain) as random sequential jumps between neighboring charge states (the Langevin approach). The first step of the method is division of such random process into elementary segments. The beginning and end of each segment are defined by the moment of the system's transition into a fixed charge state (in the case of the SET this means that  $n$  equals some fixed  $n_0$ ). After derivation of the final expressions, it will be clear that the most convenient  $n_0$  corresponds to the most probable state

$[\sigma_{st}(n_0) \geq \sigma_{st}(n \neq n_0)]$ . However, we assume below  $n_0=0$  for simplicity, while one can modify any expression below for nonzero  $n_0$  by a shift of indices.

Because the process is Markovian, two different segments (for example, neighboring segments) are independent. So we can assemble the whole process of the segments, taking them from the set of all possible segments. In this case each choice should be random and independent of previous choices. Each segment can be characterized by a specific path through charge states (for example,  $0 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 0$ ) and by the specific choice of time intervals spent by the system in each state of this sequence. Note that for the SET any charge-state change can occur in two ways (tunneling through different junctions), so, in addition to the path, information about the sequence of jump directions should also be taken into account. The number of all possible segments is infinite (the number of possible paths is infinite, and the time intervals are continuous variables).

We will need the following characteristics for each possible segment  $\xi$ :

The probability of the segment (more precisely the probability density)  $P(\xi)$ ; it is the frequency of the segment appearance in the sequence of segments;

The duration of the segment as a whole,  $\tau(\xi)$ ;

The number  $k(\xi)$  of electrons which pass through the SET during the segment  $\xi$ ;

The quantity

$$a(\xi) \equiv ek(\xi) - I\tau(\xi) = \int [I(t) - I] dt, \quad (45)$$

where the current  $I(t)$  is defined by Eq. (7), the average current  $I$  is given by Eq. (10), and the integration is carried out through the segment duration;

The variable  $b(\xi) = \int (\varphi(t) - \langle \varphi \rangle) dt$ , where the central-electrode potential  $\varphi(t)$  is defined by Eq. (6), its average value is given by Eq. (11), and the integration is also carried out through the segment duration.

Let us use Eq. (32) at the limit  $T \rightarrow \infty$  as the definition of the mutual spectral density of two random processes  $X(t), Y(t)$ . For the zero-frequency limit, it gives

$$\lim_{\omega \rightarrow 0} S_{XY}(\omega) = \lim_{T \rightarrow \infty} \frac{2}{T} \left\langle \left[ \int_0^T [X(t) - \langle X \rangle] dt \right] \times \left[ \int_0^T [Y(t) - \langle Y \rangle] dt \right] \right\rangle. \quad (46)$$

Because of the independence of the segments composing the process, any such average can be expressed by averaging over the segments:

$$S_{II}(0) = 2\overline{aa} / \overline{\tau} = (2/\overline{\tau})(e^2\overline{k^2} + I^2\overline{\tau^2} - 2eI\overline{k\tau}), \quad (47)$$

$$S_{I\varphi}(0) = 2\overline{ab} / \overline{\tau} = (2/\overline{\tau})(e\overline{k\overline{b}} - I\overline{\tau\overline{b}}), \quad (48)$$

$$S_{\varphi\varphi}(0) = 2\overline{bb} / \overline{\tau}, \quad (49)$$

where the overbar means the average value, with use of the probability  $P(\xi)$  as the weight function, for example  $\overline{ab} = \int a(\xi)b(\xi)P(\xi)d\xi$ .

Note that the average current can be calculated as

$$I = \overline{k} / \overline{\tau} \quad (50)$$

instead of Eq. (10). This fact can be used in numerical calculations as a criterion of their accuracy.

We should be able to calculate all necessary average values using the weight function  $P(\xi)$ . We need both linear expressions  $(\overline{k}, \overline{\tau})$  and quadratic ones  $(\overline{\tau^2}, \overline{k^2}, \overline{k\tau}, \overline{bb}, \overline{k\overline{b}}, \overline{\tau\overline{b}})$ .

The path of any segment can be either "positive," i.e., containing only charge states with  $n > 0$ , or "negative" containing only states  $n < 0$  (remember that the segment starts from  $n = n_0 = 0$ ). Let us consider first the segments with positive paths.

A positive path can be characterized by a sequence of numbers which correspond to the pairs of transitions between neighboring charge states:

$$0 \rightarrow 1 \rightarrow 0, \quad 1 \text{ time}$$

$$1 \rightarrow 2 \rightarrow 1, \quad i_1 \text{ times}$$

$$2 \rightarrow 3 \rightarrow 2, \quad i_2 \text{ times}$$

...

The sequence  $(i_1, i_2, i_3, \dots)$  is finite, i.e.,  $i_n = 0$  above some  $n_{\max}$ .

The averaging procedure can be carried out in two steps. First let us calculate averages for a fixed sequence  $(i_1, i_2, \dots)$  and then take into account the probability of different sequences. The mean time spent in the charge state  $n$  is  $\Gamma(n)^{-1}$ , where

$$\Gamma(n) = \Gamma^+(n) + \Gamma^-(n), \quad \Gamma^\pm(n) = \Gamma_1^\pm(n) + \Gamma_2^\pm(n). \quad (51)$$

The mean square of this time is  $2\Gamma(n)^{-2}$  (as for any Poisson process). For the fixed sequence  $(i_1, i_2, \dots)$  the average segment duration and then average square of this time are the sum of the corresponding values for all steps in the path:

$$\langle \tau \rangle = \tau_0 + i_1\tau_1 + i_2\tau_2 + \dots, \quad (52)$$

$$\langle \tau^2 \rangle = \langle \tau \rangle^2 + \theta_0 + i_1\theta_1 + i_2\theta_2 + \dots,$$

where the bold angular brackets mean the average for the fixed sequence  $\{i_1, i_2, \dots\}$ , and  $\tau_n, \theta_n$  correspond to the pair of jumps between charge states  $n$  and  $n+1$ :

$$\tau_n = \Gamma(n)^{-1} + \Gamma(n+1)^{-1}, \quad \theta_n = \Gamma(n)^{-2} + \Gamma(n+1)^{-2}. \quad (53)$$

For calculation of the average number  $\langle k \rangle$  of electrons passing through and its mean square  $\langle k^2 \rangle$  for the fixed sequence  $\{i_1, i_2, \dots\}$ , one should take into account the possibility of tunneling through different junctions for the same change of charge state. For example, the transition  $n \rightarrow n+1$  corresponds to tunneling through the first junction with probability  $\Gamma_1^+(n)/\Gamma^+(n)$  and it also corresponds to the tunneling through the second junction with probability  $\Gamma_2^+(n)/\Gamma^+(n)$ . Summing over the pairs of jumps  $n \leftrightarrow n+1$ , one obtains

$$\langle k \rangle = k_0 + i_1k_1 + i_2k_2 + \dots, \quad (54)$$

$$\langle k^2 \rangle = \langle k \rangle^2 + l_0 + i_1l_1 + i_2l_2 + \dots,$$



where

$$k_n = \frac{\Gamma_1^+(n) \Gamma_2^-(n+1)}{\Gamma^+(n) \Gamma^-(n+1)} - \frac{\Gamma_2^+(n) \Gamma_1^-(n+1)}{\Gamma^+(n) \Gamma^-(n+1)} \quad (55)$$

$$l_n = \Gamma_1^+(n)\Gamma_2^+(n)/[\Gamma^+(n)]^2 + \Gamma_1^-(n+1)\Gamma_2^-(n+1)/[\Gamma^-(n+1)]^2 .$$

A similar averaging procedure for  $k\tau$  results in the expression

$$\langle k\tau \rangle = (k_0 + i_1 k_1 + i_2 k_2 + \dots)(\tau_0 + i_1 \tau_1 + i_2 \tau_2 + \dots) \quad (56)$$

Now let us write expressions for averages containing  $b(\xi)$  for a fixed sequence  $\{i_1, i_2, \dots\}$  [their derivation is similar to that of Eqs. (52)–(56)]:

$$\langle bb \rangle = (\vartheta_0 + i_1 \vartheta_1 + i_2 \vartheta_2 + \dots)^2 + \psi_0 + i_1 \psi_1 + i_2 \psi_2 + \dots ,$$

$$\langle kb \rangle = (k_0 + i_1 k_1 + i_2 k_2 + \dots)(\vartheta_0 + i_1 \vartheta_1 + i_2 \vartheta_2 + \dots) ,$$

$$\langle \tau b \rangle = (\tau_0 + i_1 \tau_1 + i_2 \tau_2 + \dots)(\vartheta_0 + i_1 \vartheta_1 + i_2 \vartheta_2 + \dots) + \eta_0 + i_1 \eta_1 + i_2 \eta_2 + \dots , \quad (57)$$

where

$$\vartheta_n = [\varphi(n) - \langle \varphi \rangle] / \Gamma(n) + [\varphi(n+1) - \langle \varphi \rangle] / \Gamma(n+1) , \quad (58)$$

$$\psi_n = \{[\varphi(n) - \langle \varphi \rangle] / \Gamma(n)\}^2 + \{[\varphi(n+1) - \langle \varphi \rangle] / \Gamma(n+1)\}^2 , \quad (59)$$

$$\eta_n = [\varphi(n) - \langle \varphi \rangle] / [\Gamma(n)]^2 + [\varphi(n+1) - \langle \varphi \rangle] / [\Gamma(n+1)]^2 . \quad (60)$$

Finally, we have to calculate average values of  $\langle \tau \rangle, \langle \tau^2 \rangle, \langle k \rangle, \langle k^2 \rangle, \langle k\tau \rangle, \langle bb \rangle, \langle kb \rangle, \langle \tau b \rangle$  taking into account the probability of different sequences  $\{i_1, i_2, \dots\}$ . Because the values  $\tau_n, \theta_n, k_n, l_n, \vartheta_n, \psi_n, \eta_n$  are constant, we have to calculate only two types of averages:  $\bar{i}_n$  and  $\overline{i_n i_m}$  (the weight function is the probability of a sequence  $\{i_1, i_2, \dots\}$ ).

An arbitrary segment with positive path satisfies the description  $\{i_1, i_2, \dots\}$  with probability

$$P\{i_1, i_2, \dots\} = \alpha_0 (\alpha_1)^{i_1} (\alpha_2)^{i_2} \dots C_{i_1+i_2-1}^{i_1} C_{i_2+i_3-1}^{i_2} \dots , \quad (61)$$

$$\bar{\tau} = p_0^+ (\tau_0 + i_1 \tau_1 + i_2 \tau_2 + \dots) + p_0^- (\tau_{-1} + i_{-2} \tau_{-2} + i_{-3} \tau_{-3} + \dots) , \quad (68)$$

$$\bar{\tau^2} = p_0^+ [(\tau_0 + i_1 \tau_1 + i_2 \tau_2 + \dots)^2 + \theta_0 + i_1 \theta_1 + i_2 \theta_2 + \dots] + p_0^- [(\tau_{-1} + i_{-2} \tau_{-2} + i_{-3} \tau_{-3} + \dots)^2 + \theta_{-1} + i_{-2} \theta_{-2} + i_{-3} \theta_{-3} + \dots] , \quad (69)$$

$$\bar{k} = p_0^+ (k_0 + i_1 k_1 + i_2 k_2 + \dots) + p_0^- (k_{-1} + i_{-2} k_{-2} + i_{-3} k_{-3} + \dots) , \quad (70)$$

$$\bar{k^2} = p_0^+ [(k_0 + i_1 k_1 + i_2 k_2 + \dots)^2 + l_0 + i_1 l_1 + i_2 l_2 + \dots] + p_0^- [(k_{-1} + i_{-2} k_{-2} + i_{-3} k_{-3} + \dots)^2 + l_{-1} + i_{-2} l_{-2} + i_{-3} l_{-3} + \dots] , \quad (71)$$

where

$$\alpha_n = p_n^+ p_n^-, \quad p_n^+ = \Gamma^+(n) / \Gamma(n) , \quad (62)$$

$$p_n^- = 1 - p_n^+, \quad C_i^j = \frac{i!}{j!(i-j)!} .$$

The first part of Eq. (61) (containing  $\alpha_n$ ) is the probability of one path, and the second part (containing  $C_i^j$ ) is the number of different paths described by sequence  $\{i_1, i_2, \dots\}$ . Note that Eq. (61) consists of a finite number of terms because any sequence  $\{i_1, i_2, \dots\}$  contains a finite number of nonzero elements.

Using Eq. (61) and the equation

$$\sum_{i=1}^{\infty} C_{i+j}^i \alpha^i = (1-\alpha)^{-j-1} - 1 ,$$

one can write the total probability of a positive path (which is obviously equal to  $p_0^+$ ) as

$$Z \equiv \sum_{\{i_1, i_2, \dots\}} P\{i_1, i_2, \dots\} = \frac{\alpha_0}{1 - \frac{\alpha_1}{1 - \frac{\alpha_2}{1 - \dots}}} . \quad (63)$$

The averages  $\bar{i}_n$  and  $\overline{i_n i_m}$  (over only positive paths) are given by the derivatives of  $Z$ :

$$\bar{i}_n = \frac{1}{Z} \sum_{\{i_1, i_2, \dots\}} i_n P\{i_1, i_2, \dots\} = \frac{\alpha_n}{Z} \frac{dZ}{d\alpha_n} , \quad (64)$$

$$\overline{i_n i_m} = \frac{1}{Z} \sum_{\{i_1, i_2, \dots\}} i_n i_m P\{i_1, i_2, \dots\} = \frac{\alpha_n}{Z} \frac{d}{d\alpha_n} \left[ \alpha_m \frac{dZ}{d\alpha_m} \right] . \quad (65)$$

Using Eqs. (63) and (62) one obtains

$$\bar{i}_n = \frac{\Gamma^+(1) \Gamma^+(2) \dots \Gamma^+(n)}{\Gamma^-(1) \Gamma^-(2) \dots \Gamma^-(n)} , \quad (66)$$

$$\overline{i_n i_m} = \overline{i_m i_n} = \bar{i}_n \left[ 1 + 2 \frac{\Gamma^+(m)}{\Gamma^-(m)} \left[ 1 + \frac{\Gamma^+(m-1)}{\Gamma^-(m-1)} \times \left[ 1 + \dots + \frac{\Gamma^+(1)}{\Gamma^-(1)} \right] \right] \dots \right] \quad \text{for } m \leq n . \quad (67)$$

The consideration of negative paths is quite similar to that of positive paths. The final result for zero-frequency spectral densities is given by Eqs. (47)–(49) with

$$\begin{aligned} \overline{k\tau} = & p_0^+ \overline{(k_0 + i_1 k_1 + i_2 k_2 + \dots)(\tau_0 + i_1 \tau_1 + i_2 \tau_2 + \dots)} \\ & + p_0^- \overline{(k_{-1} + i_{-2} k_{-2} + i_{-3} k_{-3} + \dots)(\tau_{-1} + i_{-2} \tau_{-2} + i_{-3} \tau_{-3} + \dots)}, \end{aligned} \quad (72)$$

$$\begin{aligned} \overline{bb} = & p_0^+ [(\vartheta_0 + i_1 \vartheta_1 + i_2 \vartheta_2 + \dots)^2 + \psi_0 + i_1 \psi_1 + i_2 \psi_2 + \dots] \\ & + p_0^- [(\vartheta_{-1} + i_{-2} \vartheta_{-2} + i_{-3} \vartheta_{-3} + \dots)^2 + \psi_{-1} + i_{-2} \psi_{-2} + i_{-3} \psi_{-3} + \dots], \end{aligned} \quad (73)$$

$$\begin{aligned} \overline{kb} = & p_0^+ \overline{(k_0 + i_1 k_1 + i_2 k_2 + \dots)(\vartheta_0 + i_1 \vartheta_1 + i_2 \vartheta_2 + \dots)} \\ & + p_0^- \overline{(k_{-1} + i_{-2} k_{-2} + i_{-3} k_{-3} + \dots)(\vartheta_{-1} + i_{-2} \vartheta_{-2} + i_{-3} \vartheta_{-3} + \dots)}, \end{aligned} \quad (74)$$

$$\begin{aligned} \overline{\tau b} = & p_0^+ [(\tau_0 + i_1 \tau_1 + i_2 \tau_2 + \dots)(\vartheta_0 + i_1 \vartheta_1 + i_2 \vartheta_2 + \dots) + \eta_0 + i_1 \eta_1 + i_2 \eta_2 + \dots] \\ & + p_0^- [(\tau_{-1} + i_{-2} \tau_{-2} + \dots)(\vartheta_{-1} + i_{-2} \vartheta_{-2} + \dots) + \eta_{-1} + i_{-2} \eta_{-2} + \dots]. \end{aligned} \quad (75)$$

In these expressions, the constants  $\tau_n, \theta_n, k_n, l_n, \vartheta_n, \psi_n, \eta_n$  are defined by Eqs. (53), (55), and (58)–(60),  $p_0^\pm$  are given by Eqs. (62) and (51). Averages  $\overline{i_n}$  are given by Eq. (66) for  $n \geq 1$  and by the expression

$$\overline{i_n} = \frac{\Gamma^-( -1) \Gamma^-( -2) \dots \Gamma^-( n+1)}{\Gamma^+( -1) \Gamma^+( -2) \dots \Gamma^+( n+1)} \quad (76)$$

for  $n \leq -2$ . Averages  $\overline{i_n i_m}$  are given by Eq. (67) for positive  $n$  and  $m$ . For negative indices the corresponding expression has the form

$$\begin{aligned} \overline{i_n i_m} = & \overline{i_m i_n} = \overline{i_n} \left[ 1 + 2 \frac{\Gamma^-( m+1)}{\Gamma^+( m+1)} \left[ 1 + \frac{\Gamma^-( m+2)}{\Gamma^+( m+2)} \right. \right. \\ & \left. \left. \times \left[ 1 + \dots + \frac{\Gamma^-( -1)}{\Gamma^+( -1)} \right] \right] \dots \right], \\ & -2 \geq n \geq m. \end{aligned} \quad (77)$$

Note that for zero temperature Eqs. (68)–(75) consist of a finite number of terms because of zero tunneling rates  $\Gamma^+(n)$  and  $\Gamma^-(n)$  outside the range of possible charge states. For nonzero temperature, the number of terms is infinite; however, the sum converges rapidly. For the fastest convergence one should shift the charge-state indices and count them from the most probable charge state  $n_0$ ; then the averages  $\overline{i_n}, \overline{i_n i_m}$  decrease rapidly with  $|n - n_0|$  and  $|m - n_0|$ .

The expressions (47)–(49) and (68)–(77) are convenient for fast numerical calculations of spectral densities at zero frequency. The method of dividing the process into elementary segments (the division points corresponding to definite charge states) can be useful also for analysis of other single-electron devices. For example, it allows us to write in a simple way the expressions for the dc current in several single-electron circuits in particular cases (when the number of possible paths in the charge space is not large) without solution of the master equation. This method can also be used to substantiate a very efficient acceleration method for Monte Carlo simulation of an arbitrary single-electron circuit in the dc case. The idea of this method is that one should not use random time intervals between jumps. Instead, one should take the average values  $\Delta t = (\sum \Gamma)^{-1}$ , which are exactly determined for each charge state. Such a simulation gives the

correct results for average currents and average potentials; however, the convergence of this simulation procedure is much faster (up to ten or even hundred times) than for the usual method.

## VIII. DISCUSSION

A complete explanation of the calculation method for classical intrinsic noise of the SET (described by the orthodox theory) is presented in this paper. Calculation of the intrinsic noise is necessary, for example, for determination of the ultimate charge sensitivity of a capacitively coupled SET [Fig. 1(b)]. Let us emphasize that we do not consider the response to a definite signal alteration, but the minimal signal change which can be detected (for a definite frequency bandwidth). This problem was considered in detail in Ref. 17, so let us discuss here only the final results.

The minimal detectable charge  $\delta Q_s$ , determined by the classical noise (the bias  $V$  and the background charge  $Q_0$  are taken to be optimal), depends on the temperature  $T$ , parameters  $C_1, C_2, R_1, R_2$  of the SET, the capacitance  $C_s$  of the signal source, and the measurement bandwidth  $\Delta f$ . The case of negligible signal-source capacitance,  $C_s \ll C_\Sigma = C_1 + C_2$ , is shown in Fig. 5. This limit de-

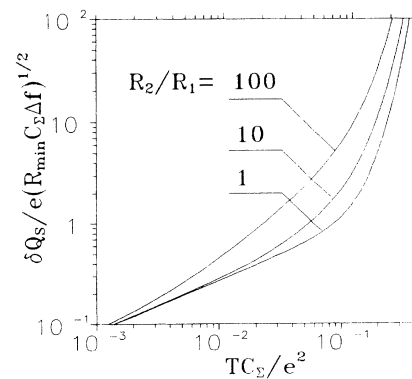


FIG. 5. Ultimate charge sensitive  $\delta Q_s$  (minimal detectable charge) for the capacitively coupled SET in the case of negligible signal-source capacitance (or in the case of small coupling capacitance, then  $\delta Q_s \rightarrow \delta Q_0$ ).  $R_{\min} = \min(R_1, R_2)$ .

scribes also the usual experimental situation of small coupling capacitance  $C_0 \ll C_\Sigma$ , but in this case  $\delta Q_s$  should be substituted by the charge  $\delta Q_0$  injected into the central electrode. One can see that at low temperatures the sensitivity depends mainly on the lesser resistance  $R_{\min} = \min(R_1, R_2)$  and is almost independent of the greater one. It is important that only the sum  $C_\Sigma$  of the junction capacitances (not both values) is necessary for determination of  $\delta Q_s$ .

If the capacitance  $C_s$  is not negligible, one should find the optimal value for the coupling capacitance  $C_0$  (a large  $C_0$  can increase the central-electrode capacitance too much, a small  $C_0$  impedes charge injection). The optimized sensitivity as a function of the signal-source capacitance is shown in Fig. 6 for the symmetric SET ( $C_1 = C_2 = C$ ,  $R_1 = R_2 = R$ ). One can see that for large enough  $C_s$ , the dependence becomes linear (dashed straight lines).<sup>17</sup> In this case it is more convenient to calculate the voltage sensitivity instead of the charge sensitivity. The corresponding value  $\delta U_s = \delta Q_s / C_s$  is shown in Fig. 7 for different resistance ratios.

It is important that the optimal coupling capacitance is rather large even for infinitely large signal-source capacitance (Fig. 8). Though it is not easy to fabricate SET's with  $C_0 > C_{1,2}$ , such structures are achievable now,<sup>29</sup> and they will be necessary for most practical applications of single-electron effects.

In this paper only the classical intrinsic noise is considered. It is obvious that this approach fails when cotunneling (quantum) processes provide a current comparable to the current calculated from the orthodox theory. The corresponding estimate<sup>17,30</sup> shows that the ultimate sensitivity calculation above becomes incorrect if  $R_Q / R_{1,2} > TC_\Sigma / e^2$  (when  $Q_0$  is not close to  $e/2$ ).

There are at least two ways to reduce the quantum noise. The classical tunneling near the Coulomb-blockade threshold  $V_t$  depends mainly on the difference  $V - V_t$ , whereas the cotunneling rates at  $V \approx V_t$  decreases

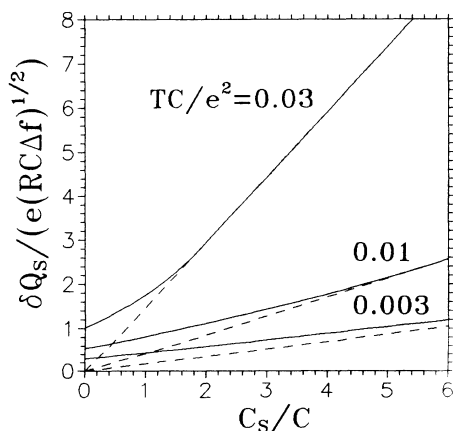


FIG. 6. Ultimate charge sensitivity  $\delta Q_s$  of the capacitively coupled SET at various temperatures as a function of the signal-source capacitance  $C_s$  for  $C_1 = C_2 = C$ ,  $R_1 = R_2 = R$ . The straight dashed lines show the asymptotes of  $\delta Q$  for large  $C_s$ .

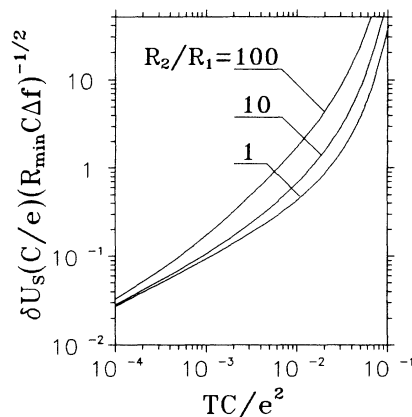


FIG. 7. Ultimate voltage sensitivity  $\delta U_s$  (for large  $C_s$ ) of the capacitively coupled SET as a function of the temperature. The coupling capacitance is assumed to be optimal.  $R_{\min} = \min(R_1, R_2)$ ,  $C_1 = C_2 = C$ .

when  $V_t$  decreases. So, choosing  $Q_0$  close to  $e/2$ , one can reduce the contribution of quantum noise.<sup>17</sup> This method can extend the validity range of the classical theory of the sensitivity approximately<sup>17</sup> up to  $R_Q / R_{1,2} < (TC_\Sigma / e^2)^{1/2}$ .

Another method to decrease the quantum contribution to the noise and to improve the sensitivity is the use of very different resistances  $R_1$  and  $R_2$ . Sensitivity restriction by the classical noise depends mainly on the smaller resistance (see Figs. 5 and 7), so increase of the other resistance is not essential. However, the cotunneling current is proportional to  $1/R_1 R_2$ ,<sup>18</sup> and it decreases when the larger resistance increases.

Note that in Ref. 31 one can find a similar statement that the use of different resistances improves the sensitivity of SET. However, this statement is completely different from the suggestion above, because in contrast to the present paper, in Ref. 31 the word "sensitivity" means  $dI/dQ_0$ . Moreover, it seems that this statement of Ref. 31 is not quite correct (in orthodox theory the optimal value for  $dI/dQ_0$  depends mainly on the smaller

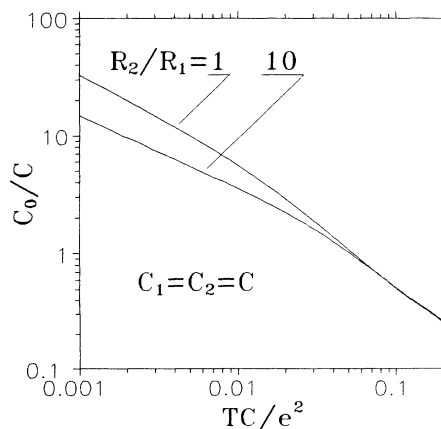


FIG. 8. Optimal coupling capacitance  $C_0$  in the case of large source capacitance  $C_s$ , as a function of the temperature.

resistance but not on the difference between the resistances).

The main restriction on SET sensitivity in contemporary experiments is  $1/f$  noise.<sup>16,20</sup> Usually its intensity exceeds classical noise by one or more orders of magnitude. For example, the sensitivity  $1.5 \times 10^{-4} e/\text{Hz}^{1/2}$  was achieved in the experiment by Geerligs, Anderegg, and Mooij.<sup>15</sup> The theoretical ultimate sensitivity according to the theory above is equal to  $1.2 \times 10^{-5} e/\text{Hz}^{1/2}$  for corresponding experimental parameters ( $C_1 + C_2 = 2.6$  fF,  $R_1 + R_2 = 250$  k $\Omega$ ,  $T(C_1 + C_2)/e^2 = 0.05$ ).

The  $1/f$  noise probably arises from stochastic occupation of the charge traps (impurities).<sup>15,16,20</sup> By improving the fabrication technology, it is possible to reduce  $1/f$

noise. Then one can hope to reach the theoretical limit for sensitivity of the SET, which is of the order of  $10^{-6} e/\text{Hz}^{1/2}$  for experimental parameters available nowadays ( $C_{1,2} \approx 10^{-16}$  F,  $R_{1,2} \approx 10^5 \Omega$ ,  $T \approx 50$  mK).

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