

Analytical calculation of single-electron oscillations in one-dimensional arrays of tunnel junctions

A.N. Korotkov

*Department of Physics, State University of New York, Stony Brook, New York 11794-3800
and Institute of Nuclear Physics, Moscow State University, Moscow 119899, Russia*

(Received 26 July 1994)

We present an analytical expression for the spectral density of the current flowing through a one-dimensional array of identical small-area tunnel junctions. Negligible stray capacitance, zero temperature, and the voltage close to Coulomb blockade threshold were assumed. Single-electron oscillations appear explicitly (i.e., their linewidth becomes less than the average frequency) when the number of junctions is larger than ~ 15 .

Electron transport in systems of small tunnel junctions may show time correlation between tunneling of the single electrons.^{1,2} This correlation leads to a number of effects, in particular, to single-electron-tunneling (SET) oscillations with frequency $f = I/e$, where I is the average current through the single-electron circuit and e is the electron charge. SET oscillations can be observed in a single tunnel junction biased by constant current and also in a voltage-biased array of junctions in series. The existence of SET oscillations in the arrays was confirmed by observation of the peculiarities of the I - V curve when the frequency of oscillations coincides with the frequency of external radiation.^{3,4}

The standard explanation of the origin of SET oscillations in arrays is based on the interaction of single-electron solitons.^{5,6} The soliton (antisoliton) corresponds to polarization of junctions in the vicinity of the additional electron ("hole") on one of the electrodes of the array. Because of the repulsion of solitons (and antisolitons) the charge transport can be described as the movement of a quasiperiodic train of solitons. This periodicity leads to SET oscillations.

This explanation assumes the existence of stray capacitances C_0 of the islands, and requires the length of the array greater than the soliton length, $N \gtrsim 2\sqrt{C/C_0}$, where N is the number of junctions and C is the capacitance of one junction. In the present work we consider the opposite limit of negligible C_0 (when the length of the array is much smaller than the soliton length). It is shown that SET oscillations exist in this case as well, and the linewidth is almost the same (for the same N).

We will use "orthodox" theory¹ of correlated tunneling which describes the transport as instantaneous transitions between different charge states due to single-electron tunneling events. For relatively simple circuits (in particular, for single junctions or double junctions) the corresponding master equation is not complicated, and can be solved almost explicitly. However, for more complex systems (in particular, arrays of junctions) the analytical solution of the master equation is usually not possible, and the most widely used method is Monte Carlo simulation of electron jumps.^{6,7}

In this paper we present analytical calculations of the spectral density of the current through the array of tunnel junctions in a special case. There are five main assumptions: (1) all junctions are identical, (2) there are no stray capacitances in the array, (3) the temperature is negligible in comparison with e^2/C , (4) background charges of islands are negligible, and (5) the voltage across the array is not far from the Coulomb blockade threshold, so that when there is one additional electron (hole) inside the array, the next electron (hole) cannot enter the array before the initial one leaves it (a similar model for high temperatures was used in Ref. 8).

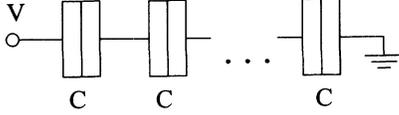
The method described in the present paper was used in Ref. 9 for calculation of SET oscillations in small-area semiconductor superlattices.

Figure 1 shows the array of N identical tunnel junctions. Each junction has the capacitance C and "seed" I - V curve $I_0(v)$ [for metallic junctions $I_0(v) = v/R$; however, in the case of semiconductors the nonlinearity of $I_0(v)$ may be important^{9,10}]. The Coulomb blockade threshold for this system is $V_t = e(N-1)/2C$ [this number is N times the threshold voltage for each junction $v_t = e/2C_{\text{ef}}$, where $C_{\text{ef}} = C + C/(N-1)$ is the effective capacitance of one junction]. For the voltage range

$$V_t < V < V_t + \frac{e}{C} \quad (1)$$

only one additional electron (or hole) in the array is possible, and tunneling events occur in strict order [Fig. 2(a)] discussed below.

Because all junctions are equivalent, we will not follow the position of additional electrons and holes in the array. Instead, we will count the number n of tunnel events starting from the initial state ($n = 0$) when there is no additional charge in the array (hence, one state number corresponds to several different charge configurations). At $n = 0$ the effective voltage across each junction is $v_0 = (V - V_t)/N$, and the total tunneling rate is $\Gamma_0 = NI_0(v_0)/e$.¹ After a tunneling event occurs in some junction ($n = 1$) the voltage drop across this junction decreases by $(e/C)(N-1)/N$ and prevents the tunneling of the next electron (Coulomb blockade). The effective

FIG. 1. The array of N tunnel junctions.

tive voltage drop across any other junction increases by $v_1 - v_0 = e/CN$. After each tunneling event the number of blocked junctions increases by 1, and, at $n = N - 1$, only one junction is still "open." After the next jump the initial state ($n = 0$) is restored. Figure 2(a) is the graphical representation of the master equation

$$\frac{d}{dt} \sigma(n+1) = \sigma(n)\Gamma_n - \sigma(n+1)\Gamma_{n+1}, \quad (2)$$

where $\sigma(n)$ is the probability of the state n (state $n = N$ is equivalent to $n = 0$). The transition rates are given by the expression

$$\Gamma_n = (N-n)e^{-1}I_0 \left(\frac{V - V_t}{N} + \frac{ne}{CN} \right). \quad (3)$$

The simplest way to calculate the average current I through the array is to write the average time of one cycle through all states [Fig. 2(a)] as the sum of average times for each step:

$$I = e \left[\sum_{n=0}^{N-1} (\Gamma_n)^{-1} \right]^{-1}. \quad (4)$$

Note that it is also possible to write an analytical expression for the average current in the case when the voltage corresponds to the next (second) step of the Coulomb staircase,

$$V_t + e/C < V < V_t + 2e/C \quad (5)$$

[the Coulomb staircase is well pronounced for nonlinear $I_0(v)$ — see Ref. 9]. In this case at charge state $n = N - 1$

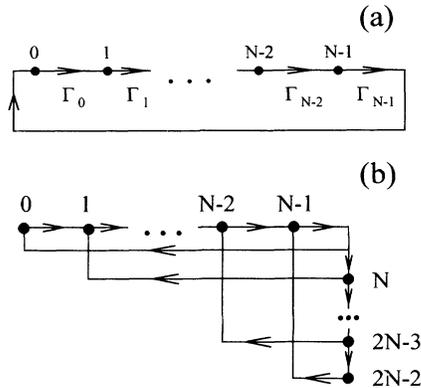


FIG. 2. Configuration space and the diagram of transitions for the voltage ranges given by (a) Eq. (1) (first step of Coulomb staircase) and (b) Eq. (5) (second step of Coulomb staircase).

all junctions are already out of the Coulomb blockade, and a second electron (hole) can enter the array. The diagram which corresponds to the master equation for this case is shown in Fig. 2(b). Using this diagram one can calculate the average current through the array without an explicit solution of the master equation (the method is similar to that explained in Sec. VII of Ref. 11):

$$I = e \left[\sum_{n=0}^{N-2} D_n + E_0 + \sum_{n=0}^{N-2} \left(\prod_{k=0}^n F_k \right) (E_{n+1} - D_n) \right]^{-1}, \quad (6)$$

where

$$D_n^{-1} = \Gamma_n, \quad E_n^{-1} = G_{N+n-1} + (N-n-1)G_{n-1},$$

$$F_n^{-1} = 1 + G_{N+n-1}/(N-n-1)G_{n-1},$$

$$G_n = I_0 \left(\frac{V - V_t + ne/C}{N} \right) / e.$$

Now let us calculate the spectral density $S_I(\omega)$ of the current within the first step of the Coulomb staircase [voltage range (1)] following the method of Refs. 11 and 12. The current in the external circuit

$$I(t) = \frac{e}{N} \sum \delta(t - t_i) \quad (7)$$

contains equal contributions from tunneling through all junctions (t_i are the moments of tunneling events). The autocorrelation function of the current is^{11,12}

$$\begin{aligned} \langle I(\tau)I(0) \rangle &= \langle I(-\tau)I(0) \rangle \\ &= \left(\frac{e}{N} \right)^2 \sum_{n,m} \Gamma_m \sigma(m, \tau | n+1) \Gamma_n \sigma_{st}(n) \\ &\quad + A\sigma(\tau), \end{aligned} \quad (8)$$

where $\sigma(m, \tau | k)$ is the probability of the charge state m at $t = \tau > 0$ given that at $t = 0$ the system had charge number k [it is the solution of Eq. (2) with a definite initial condition], and $\sigma_{st}(n) = \sigma(n, \tau \rightarrow \infty | k)$ is the stationary distribution. The term $A\delta(\tau)$ determines the high-frequency asymptote of $S_I(\omega)$ which satisfies the Schottky formula

$$2A = \lim_{\omega \rightarrow \infty} S_I(\omega) = 2 \frac{e}{N} I. \quad (9)$$

Using the relation $\Gamma_n \sigma_{st}(n) = I/e$ it is possible to simplify Eq. (8):

$$\langle I(\tau)I(0) \rangle = \frac{eI}{N^2} \sum_{n,m} \Gamma_m \sigma(m, \tau | n) + \frac{eI}{N} \delta(\tau). \quad (10)$$

Fourier transformation of the master equation (2) (rewritten in the matrix form) gives the solution for $\sigma(m, \tau | n)$ in the frequency representation:¹¹

$$\sigma_{mn}(\omega) = \left(\frac{1}{i\omega \mathbf{1} - \mathbf{\Gamma}} \right)_{mn} - \frac{\sigma_{st}(m)}{i\omega}, \quad (11)$$

where the elements of the matrix $\mathbf{\Gamma}$ are given by

$$\Gamma_{mk} = -\Gamma_m \delta_{m,k} + \Gamma_{m-1} \delta_{m-1,k}. \quad (12)$$

Note that the second term in Eq. (11) depends on the particular choice of boundary conditions and does not affect the final results. Fourier transformation of Eq. (10) gives the spectral density

$$S_I(\omega) = \frac{2eI}{N} + \frac{4eI}{N^2} \sum_{nm} \Gamma_m \operatorname{Re} \left(\frac{1}{i\omega \mathbf{1} - \Gamma} \right)_{mn}. \quad (13)$$

Because the matrix Γ has the two-diagonal form, it is possible to write $(i\omega \mathbf{1} - \Gamma)^{-1}$ explicitly, which gives the final expression for the spectral density of current,

$$S_I(\omega) = \frac{2e}{N} I + \frac{4eI}{N^2} \operatorname{Re} \left(\left[\prod_{n=0}^{N-1} \left(1 + \frac{i\omega}{\Gamma_n} \right) - 1 \right]^{-1} \times \left\{ N + \sum_{n=1}^{N-1} \sum_{m=1}^N \left[\prod_{k=1}^n \left(1 + \frac{i\omega}{\Gamma_{k+m}} \right) \right] \right\} \right) \Gamma_{N+k} \equiv \Gamma_k. \quad (14)$$

Calculations using this expression are obviously much faster than Monte Carlo simulations.¹³ Typical results [for linear $I_0(v)$] are presented in Fig. 3. The low-frequency limit of spectral density satisfies the equation

$$S_I(0) = 2eI \sum_{n=0}^{N-1} \Gamma_n^{-2} / \left[\sum_{n=0}^{N-1} \Gamma_n^{-1} \right]^2, \quad (15)$$

while the high-frequency limit is given by Eq. (9).

The most interesting feature is the peak near the frequency $\omega/2\pi = I/e$ which represents the SET oscillations. One can see that the quality of oscillations improves with increase of the number of junctions. For the quantitative characterization of oscillations we have calculated the half width $\Delta\omega$ of the peak (at half height) and the ratio $r = S_I(0)/S_I(2\pi I/e)$. Figure 4 shows these quantities as functions of the voltage across the array for different N .

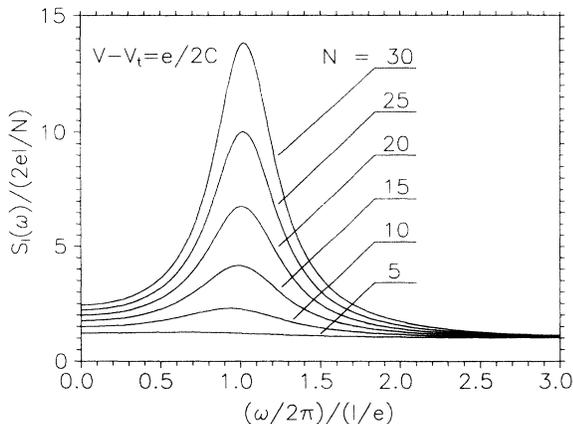


FIG. 3. The normalized spectral density of the current through the array of N junctions for different N . Linear “seed” I - V curve of junctions is assumed. $V = V_t + e/2C$.

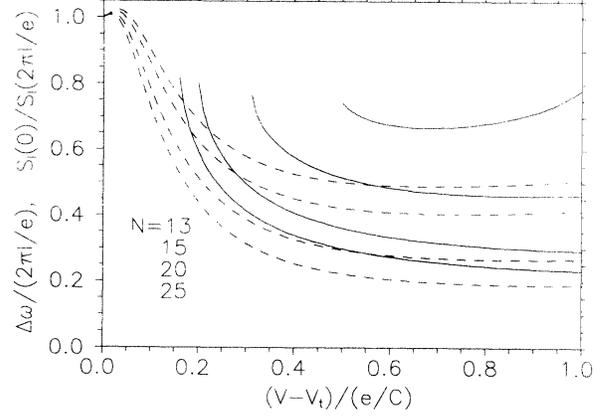


FIG. 4. The relative half linewidth (at half height) of SET oscillations (solid lines) and the ratio $S_I(0)/S_I(2\pi I/e)$ (dashed lines) as functions of the voltage. The number of junctions $N = 13, 15, 20,$ and 25 (from top to bottom).

Oscillations are weak near the Coulomb blockade threshold (the solid curves are cut in the region where it is impossible to determine the linewidth); they improve with the increase of dc voltage bias up to some optimal value (which depends on N), and then become worse again, disappearing at high voltages [our analytical method of calculations can only be applied within the range (1)]. After the optimization over the voltage, we plot (Fig. 5) the dependence of minimal relative half linewidth (squares) and minimal ratio r (triangles) as functions of the number of junctions. [Note that for $N > 20$ the optimal voltage for minimal linewidth is out of the range (1); in this case we used Monte Carlo results represented by solid squares.] Using Fig. 5 one can say that SET oscillations are practically absent for $N < 15$ (full linewidth is larger than the frequency), and they continuously improve with increase of the number of junctions.

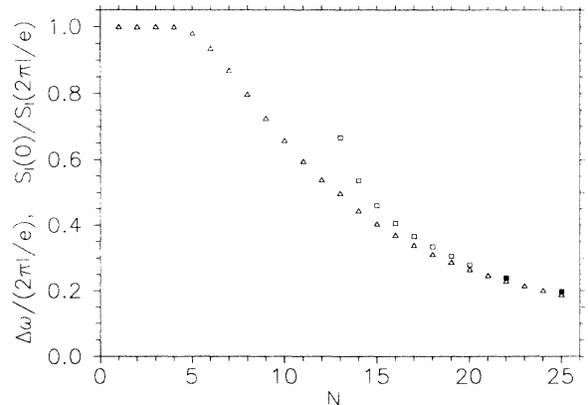


FIG. 5. The minimal (optimized over the voltage) relative half linewidth of oscillations (squares) and ratio $S_I(0)/S_I(2\pi I/e)$ (triangles) as functions of the number of junctions. Solid squares represent results of Monte Carlo simulations.

Finally, let us discuss briefly the applicability range of these results to the case when the assumptions used are not strictly satisfied. Small variations of parameters of different junctions obviously do not affect the results strongly. The method should be valid at small temperatures $T \ll e^2/CN$. Fluctuations of background charge (which can be on the order of the electron charge) may affect the spectral density strongly; however, one can hope that Fig. 5 would not change significantly. The presence of stray capacitances C_0 of islands should be important for $N \gtrsim \sqrt{C/C_0}$. The corresponding curve in Fig. 5 should deviate from our results and eventually saturate

(when $N \gg \sqrt{C/C_0}$). However, it is not clear whether the presence of C_0 can improve oscillations (decrease the linewidth) for particular N . Our preliminary calculations for $N = 21$ and $N = 25$ using the Monte Carlo simulations do not show any improvement over the $C_0 = 0$ case within 5% accuracy.

The author thanks D. V. Averin and K. K. Likharev for fruitful discussions and critical reading of the manuscript. The work was supported in part by AFOSR Grant No. 91-0445 and ONR Grant No. N00014-93-1-0880.

¹D. V. Averin and K. K. Likharev, in *Mesoscopic Phenomena in Solids*, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (Elsevier, Amsterdam, 1991), p. 173.

²P. Delsing, in *Single Charge Tunneling*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1992), p. 249.

³P. Delsing, K. K. Likharev, L. S. Kuzmin, and T. Claeson, *Phys. Rev. Lett.* **63**, 1861 (1989).

⁴P. Delsing, D. B. Haviland, T. Claeson, A. N. Korotkov, and K. K. Likharev, in *Single-Electron Tunneling and Mesoscopic Devices*, edited by H. Koch and H. Lubbig (Springer-Verlag, Berlin, 1992), p. 97.

⁵K. K. Likharev, *IBM J. Res. Dev.* **32**, 144 (1988).

⁶K. K. Likharev, N. S. Bakhvalov, G. S. Kazacha, and S. I. Serdyukova, *IEEE Trans. Magn.* **25**, 1436 (1989).

⁷M. Amman, E. Ben-Jacob, and K. Mullen, *Phys. Lett. A*

142, 431 (1989).

⁸B. Laikhtman, *Phys. Rev. B* **41**, 138 (1990).

⁹A. N. Korotkov, D. V. Averin, and K. K. Likharev, *Phys. Rev. B* **49**, 1915 (1994).

¹⁰A. N. Korotkov and Yu. V. Nazarov, *Physica B* **173**, 217 (1991).

¹¹A. N. Korotkov, *Phys. Rev. B* **49**, 10381 (1994).

¹²A. N. Korotkov, D. V. Averin, K. K. Likharev, and S. A. Vasenko, in *Single-Electron Tunneling and Mesoscopic Devices* (Ref. 4), p. 45.

¹³An effective method for Monte Carlo spectral calculations was discussed in Ref. 11. In particular, this idea was used in computer package MOSES (created by Ruby Chen at SUNY, Stony Brook, NY) which calculates single-electron transport in arbitrary single-electron circuits.