Measurement theory for phase qubits

A. G. Kofman[1], Q. Zhang[1], J. M. Martinis[2], and A. N. Korotkov[1]

[1]Electrical Engineering Department, University of California, Riverside

[2]Department of Physics, University of California, Santa Barbara

Outline

- Two-qubit errors:

  1. Crosstalk after the measurement pulse
  2. Crosstalk during the measurement pulse

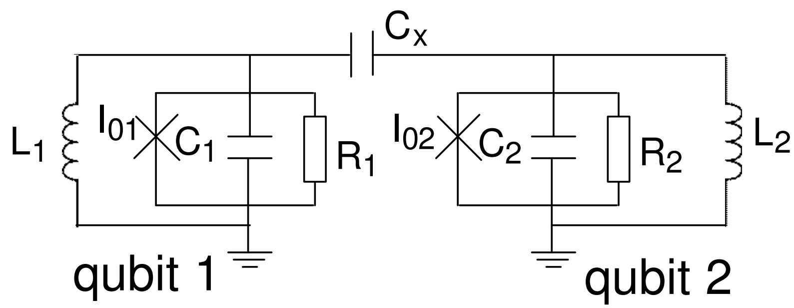
  (related to recent experiments in Martinis' group)

- One-qubit errors during the measurement pulse:

  3. Nonadiabatic effects
  4. Left-well repopulation
  5. Effect of relaxation on qubit switching

  (related to recent experiments in Martinis' group)

Two-qubit errors



A Josephson phase qubit is described by the Hamiltonian

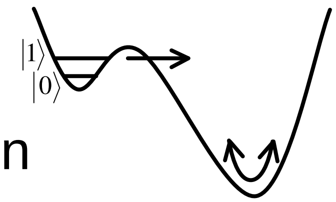
$$H = \frac{p_1^2 + p_2^2 + 2\zeta p_1 p_2}{2(1+\zeta)m} + U_1(\delta_1) + U_2(\delta_2), \;\; \zeta = \frac{C_x}{C + C_x},$$

$$U(\delta_i) = E_J \left[\frac{(\delta_i - \phi_i)^2}{2\beta} - \cos\delta_i \right], \;\; E_J = \frac{\Phi_0 I_0}{2\pi}, \;\; \beta = \frac{2\pi I_0 L}{\Phi_0}, \;\; \phi_i = \frac{2\pi \Phi_i}{\Phi_0},$$

δ_i is the Josephson phase of ith qubit, I_0 is the critical current, $\Phi_0 = h/2e$ is the flux quantum, Φ_i is the external flux, p_i is the canonical momentum.

Dissipation: the resistively shunted junction (RSJ) model, i.e., a friction term in eqs. of motion: $-\dot{\delta}_i/T_1$, where $T_1 = RC$ is the energy relaxation time.

1. Crosstalk after the measurement pulse



Crosstalk for state $|10\rangle$: measurement pulse $\Rightarrow$ tunneling of qubit 1 to the right-hand well (switching) $\Rightarrow$ damped oscillation $\Rightarrow$ microwave voltage $\Rightarrow$ switching of qubit 2 $\Rightarrow$ wrong measurement ($|11\rangle$)

Qubit 1 is treated classically (reasonable: the quantum number $n \gtrsim 150$).

Theoretical approaches for qubit 2:

A. Classical approaches:

- Harmonic oscillator.

- Actual potential, no damping.

- Actual potential, with damping.

B. Quantum approach (actual potential, no damping).

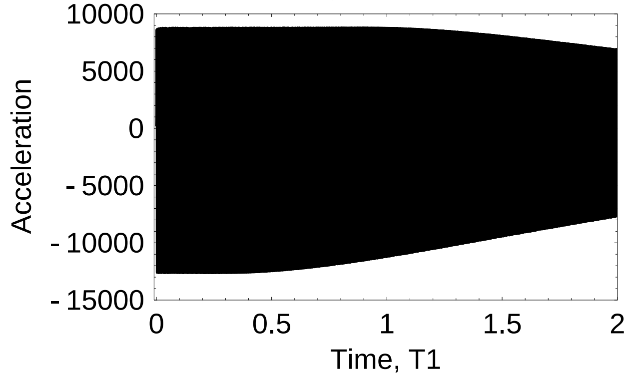
First-qubit dynamics

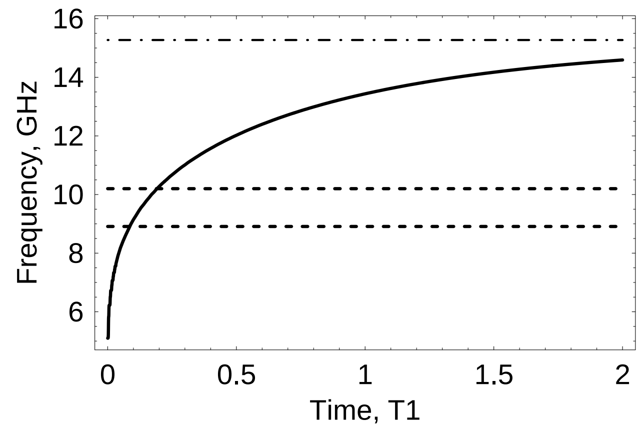
$$\ddot{\delta}_1 + \frac{\dot{\delta}_1}{C''R} + \frac{2\pi I_0}{\Phi_0 C''}\sin\delta_1 + \frac{\delta_1 - \phi_1}{C''L} = 0, \quad C'' = (1+\zeta)C$$

(the charge at the second qubit is neglected) $\Rightarrow$ damped oscillation.

$$C = 700 \text{ fF}, \ J_0 = 1.7\ \mu\text{A}, \ L = 720 \text{ pH}, N_{l1} \equiv \frac{\Delta U_1}{\hbar\omega_{l1}} = 1.36.$$

$\ddot{\delta}_1(t)$, ns^{-2}





$C_x = 6$ fF;

$N_{l2} = 5, \ \omega_{l2} = 8.91$ GHz, $t_c = 0.085 T_1$ ns, $\dot{\omega}_d(t_c) = 110/T_1$ ns^{-2};

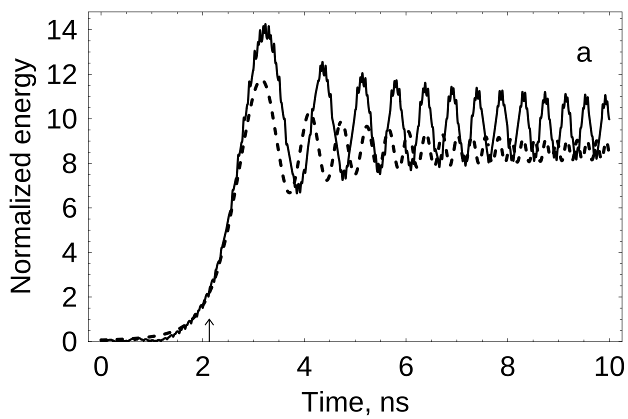
$N_{l2} = 10, \ \omega_{l2} = 10.2$ GHz, $t_c = 0.192 T_1$ ns, $\dot{\omega}_d(t_c) = 57/T_1$ ns^{-2}.

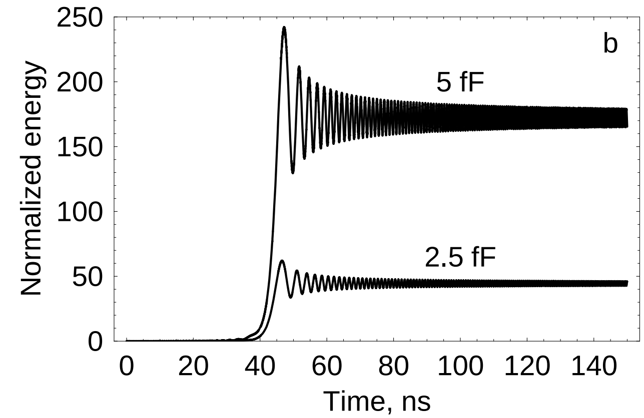
ω_{li} - left-well plasma frequency for qubit i, t_c - resonance time.

2nd qubit: Classical oscillator

$$\ddot{x} + \omega_{l2}^2 x = \zeta \ddot{\delta}_1(t), \;\; x = \delta_2 - \delta_{l2}, \;\; \ddot{\delta}_1(t) = A(t) \exp\left[i \int^t \omega_d(t')\, dt' \right] + \ldots,$$

$\omega_{l2} = 8.91$ GHz ($N_{l2} \equiv \Delta U_2/\hbar\omega_{l2} = 5$).





$T_1 = 25$ ns, $T_1 = 500$ ns

Dashed line - analytical solution with $A = A(t_c)$, $\omega_d(t) = \omega_{l2} + \alpha(t - t_c)$.

$E_{2,\mathrm{max}} = 1.37\pi\zeta^2 m A^2/\alpha$.

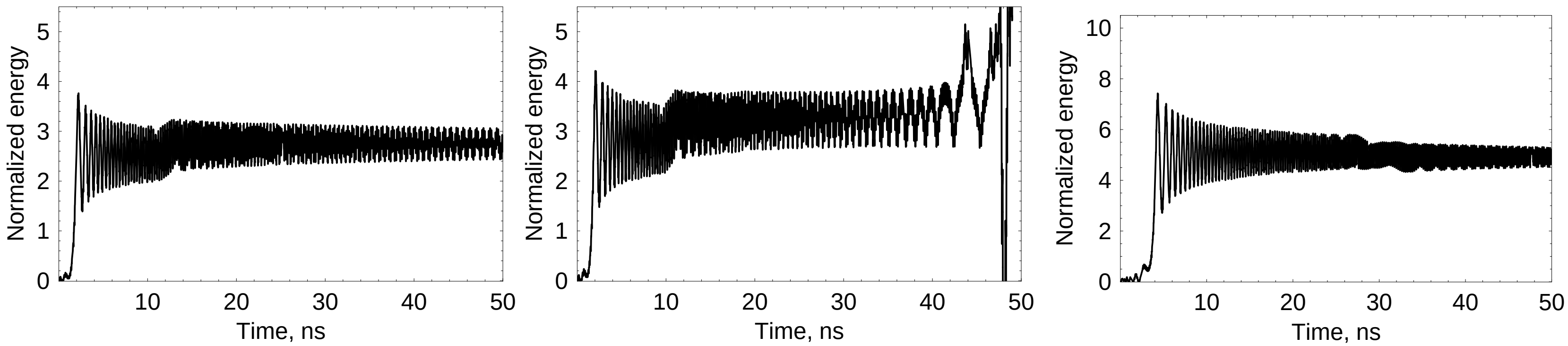
Numerical FT of $\ddot{\delta}_1(t) \Rightarrow A = 4300$ ns^{-2} ($N_{l2} = 5$), $A = 5200$ ns^{-2} ($N_{l2} = 10$).

No escape: C_x[fF] $< B/\sqrt{T_1\text{[ns]}}$. $B = 15$ for $N_{l2} = 5$, $B = 14.3$ for $N_{l2} = 10$ (!)

Classical approach for qubit 2: Actual potential

$$\ddot{\delta}_2 + \frac{\dot{\delta}_2}{(1+\zeta)T_1'} + \frac{2\pi I_0}{\Phi_0 C''}\sin\delta_2 + \frac{\delta_2 - \phi_2}{C''L} = \zeta\ddot{\delta}_1(t), \quad T_1' = \infty \text{ or } T_1.$$

$E_2/\hbar\omega_{l2}$ for $T_1 = 25$ ns, $T_1' = \infty$:



N_{l2}=5, C_x=5 fF N_{l2}=5, C_x=6 fF N_{l2}=10, C_x=6 fF.

Distinctions of nonlinear dynamics from oscillator:

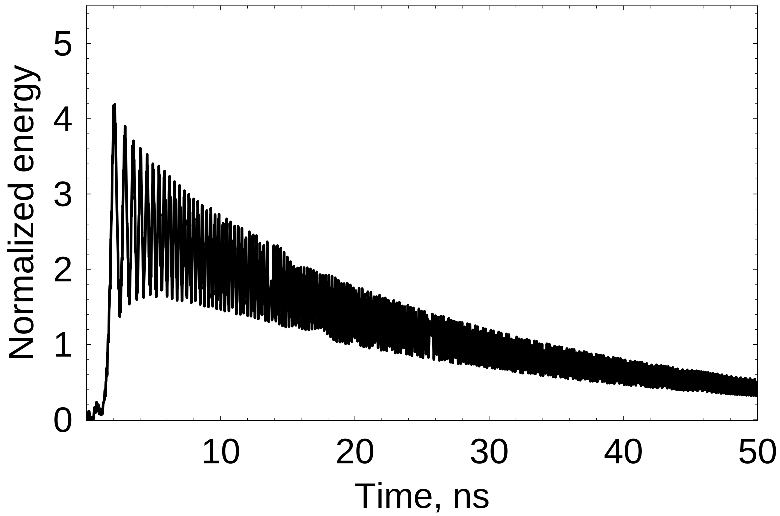
- Earlier excitation.

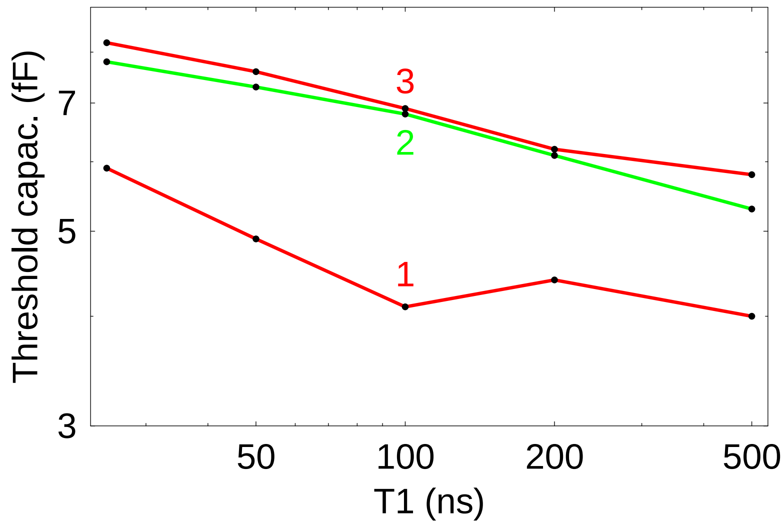
- Less efficient excitation.

- Random changes of oscillation amplitude, sometimes beyond the initial maximum.

- Possibility of escape.

Actual potential, with damping of 2nd qubit





$N_{l2} = 5$, $C_x = 5$ fF, $T_1 = T_1' = 25$ ns.

Threshold capacitance $C_{x,T}$:

$C_x < C_{x,T}$ - no escape, $C_x < C_{x,T}$ - escape is possible;

1 - $N_{l2} = 5,\ T_1' = \infty$;

2 - $N_{l2} = 5,\ T_1' = T_1$;

3 - $N_{l2} = 10,\ T_1' = \infty$;

Quantum approach

Now 2nd qubit is considered quantum-mechanically (still ”classical“ approach for qubit 1).

Hamiltonian for qubit 2:

$$H(t) = \frac{\hat{p}^2 + 2\zeta p_1(t)\hat{p}}{2(1+\zeta)m} + U(\delta), \;\; \hat{p} = -i\hbar\frac{\partial}{\partial\delta}.$$

Canonical (gauge) transformation $\Rightarrow$ more physical form:

$$\Psi(\delta,t) = \Psi'(\delta,t)e^{-i\zeta p_1(t)\delta/\hbar} \Rightarrow H'(t) = H_0 + V(t),$$

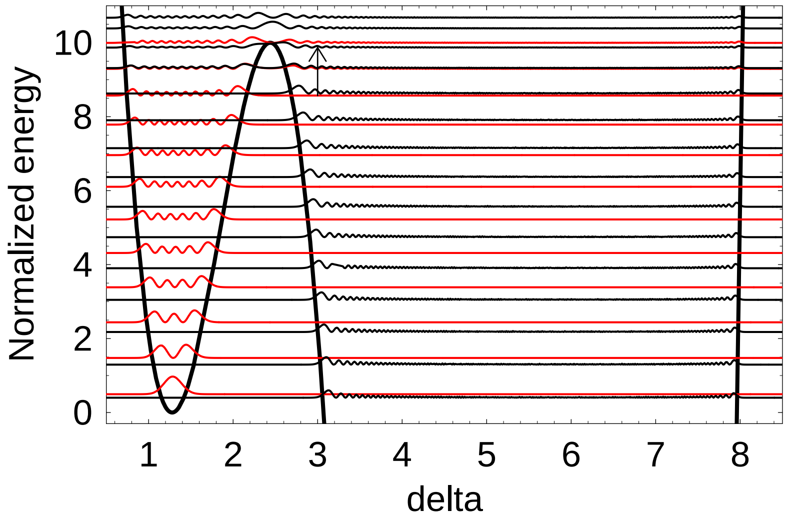
$$H_0 = \frac{\hat{p}^2}{2m''} + U(\delta), \;\; V(t) = -\zeta m''\ddot{\delta}_1(t)\delta, \;\; m'' = (1+\zeta)m.$$

We obtain eigenvalues and eigenfunctions of H_0 by Fourier grid Hamiltonian (or, equivalently, periodic pseudospectral) method.

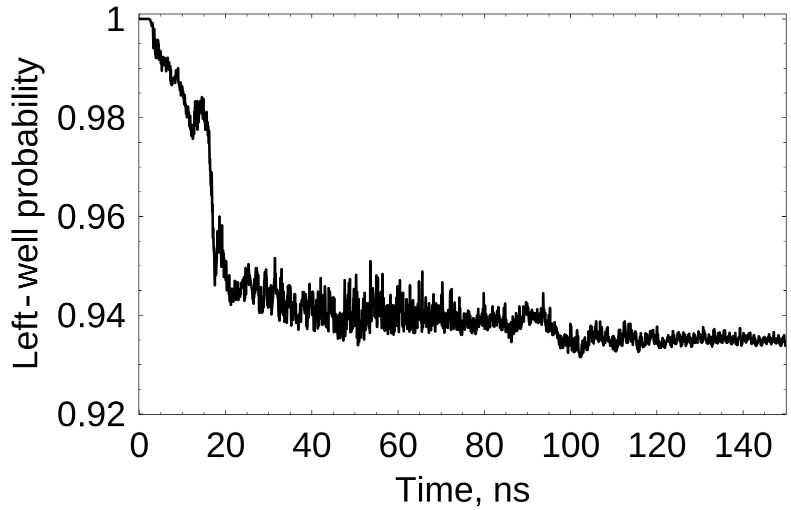
We expand $\Psi(\delta,t)$ over the eigenfunctions of H_0 and obtain from the time-dependent Schroedinger equation a set of **ordinary** differential equations for the expansion coefficients. In our simulations a subset of levels is used.

$$N_l \equiv \Delta U/\hbar\omega_l = 10,\ C_x = 6 \text{ fF},\ T_1 = 25 \text{ ns}$$

$E_n/\hbar\omega_l,\ |\psi_n(\delta)|^2$



No-switching probability



12 states in the left well: $n = 146, 148, \ldots, 166, 169$.

Used in the simulation: $141 \leq n \leq 185$.

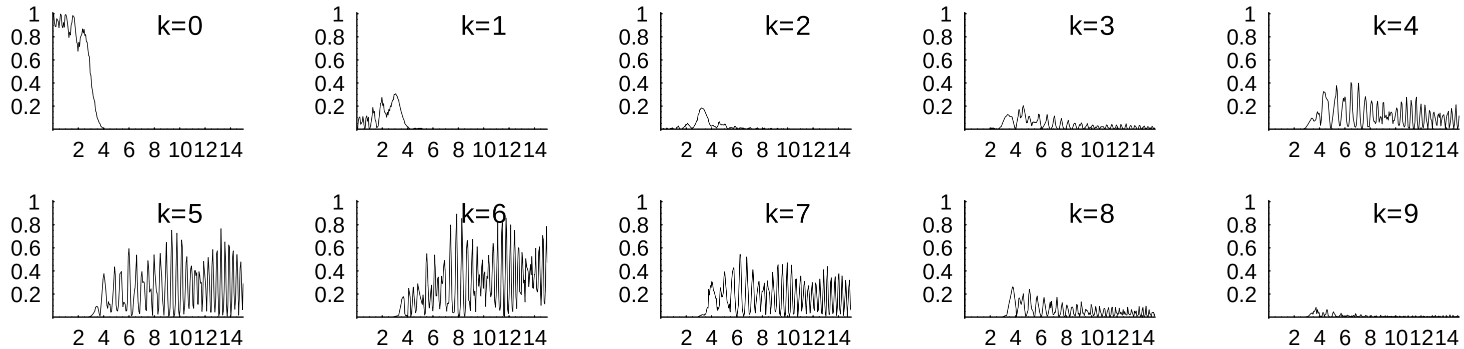
A sharp decrease of the left-well population at $16 < t < 18$ ns is due to the resonance

$$n = 164\ (k = 9) \rightarrow n = 168.$$

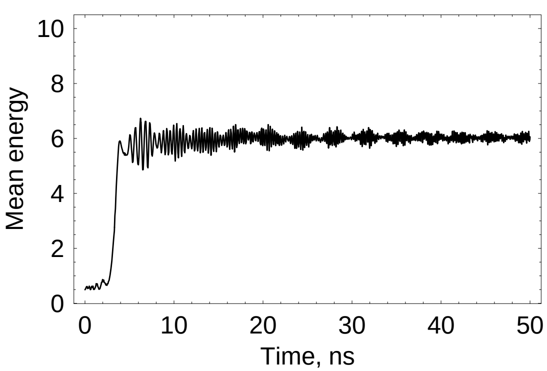
$$\omega_d(16 \text{ ns}) = 12.5 \text{ GHz},\ \omega_{168,164} = 13.2 \text{ GHz},$$

$$\omega_{168,164} - \omega_d(16 \text{ ns}) = 0.7 \text{ GHz} \sim R_{168,164}.$$

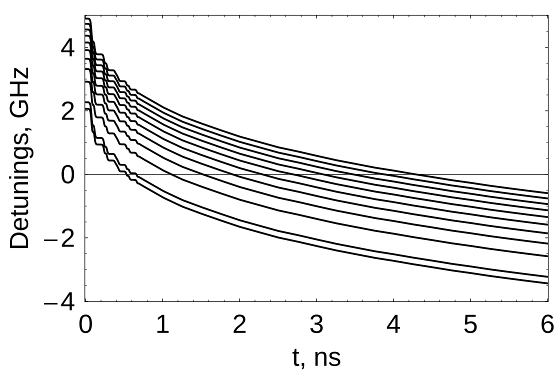
Populations of the bound states, $T_1 = 25$ ns



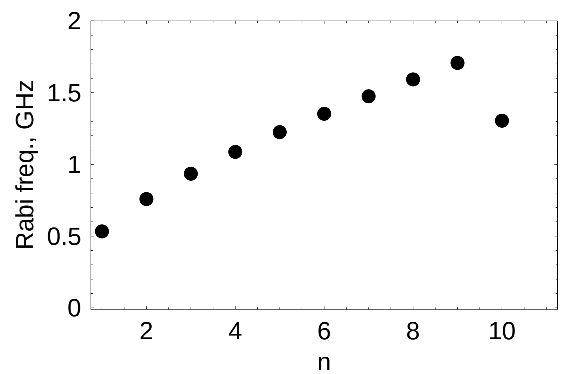
t (ns)



$\omega_{n+1,n} - \omega_d(t)$



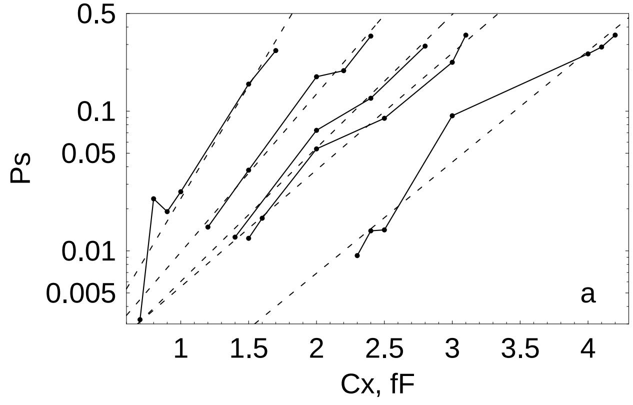
Rabi frequency $R_{n+1,n}$



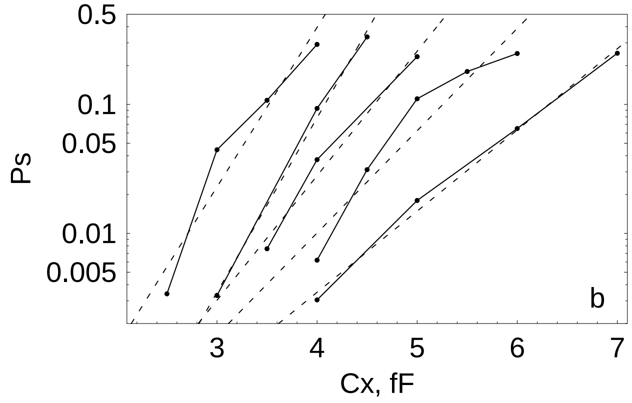
Quantum case: crosstalk = qubit-2 switching probability P_s

Numerical simulations for $T_1 = 25, 50, 100, 200, 500$ ns.

$N_l = 5$



$N_l = 10$

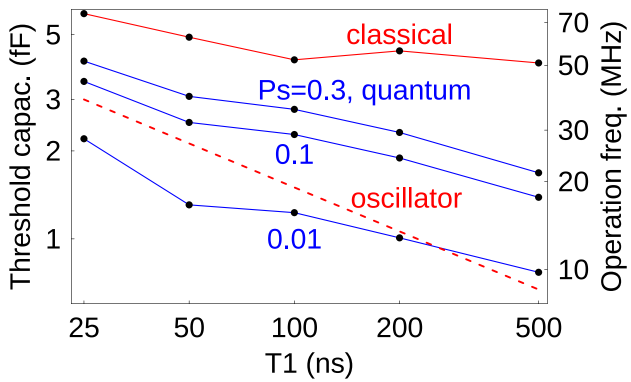


Two-qubit coupling strength (operation frequency) $S = (C_x/C)\omega_{10}$.

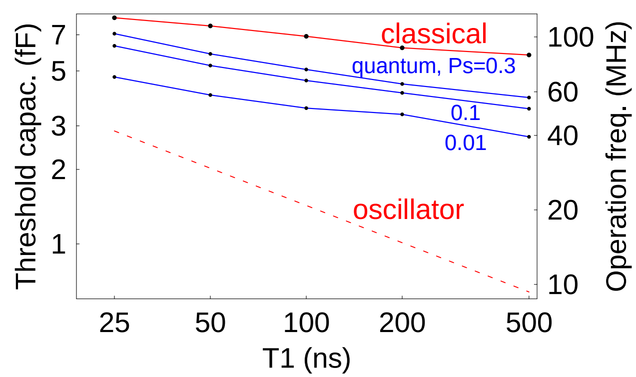
Crosstalk sets limits on C_x and S.

Threshold capacitance $C_{x,T}$ and threshold operation frequency S_T vs. T_1:

$N_l = 5$



$N_l = 10$



$$C_{x,T}(T_1) \approx B T_1^{-\beta}$$

$N_l = \Delta U/\hbar\omega_l$	Oscillator β	Classic β	Quantum β
5	0.5	0.12	0.3
10	0.5	0.12	0.2

Higher barrier $\Rightarrow$ weaker crosstalk $\Rightarrow$ faster operation frequency

2. Crosstalk during the measurement pulse

Dependence on the critical current I_0 (β)

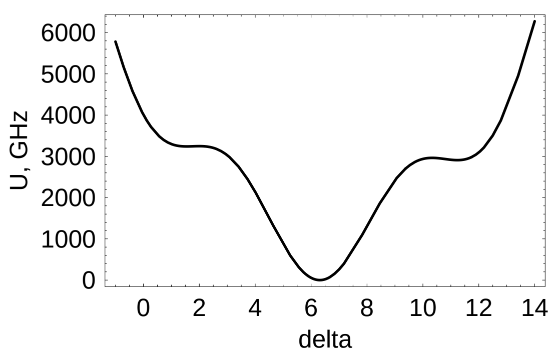
Experiments have shown a significant increase of crosstalk, when critical current increases so that $U(\delta)$ becomes a three-well potential.

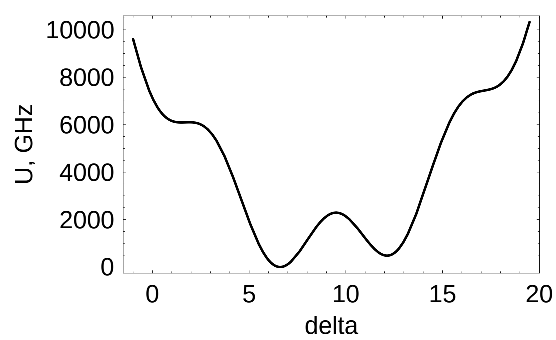
Is three-stable situation really bad?

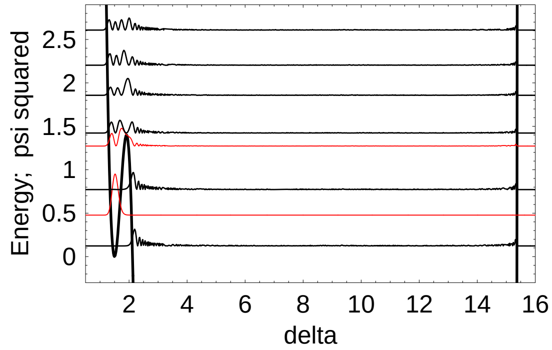
We make simulations in the quantum approach, using parameter values from experiment:

$$C = 1300 \text{ fF}, \quad L = 850 \text{ pH}, \quad C_x = 3 \text{ fF}, N_l = 1.4, \quad T_1 = 120 \text{ ns},$$

$I_0 = 1.9\ \mu$A, $\beta = 4.91$ $\qquad$ $I_0 = 2.9\ \mu$A, $\beta = 7.49$



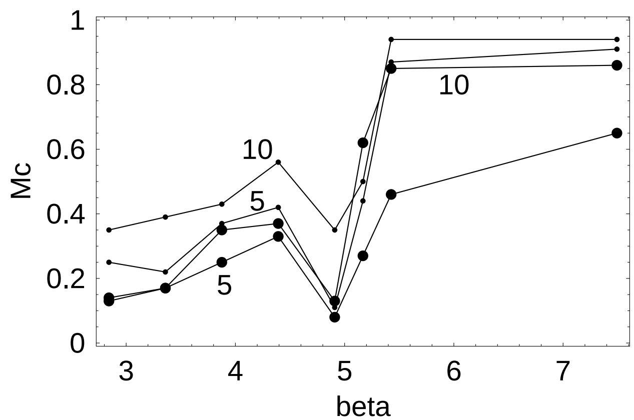


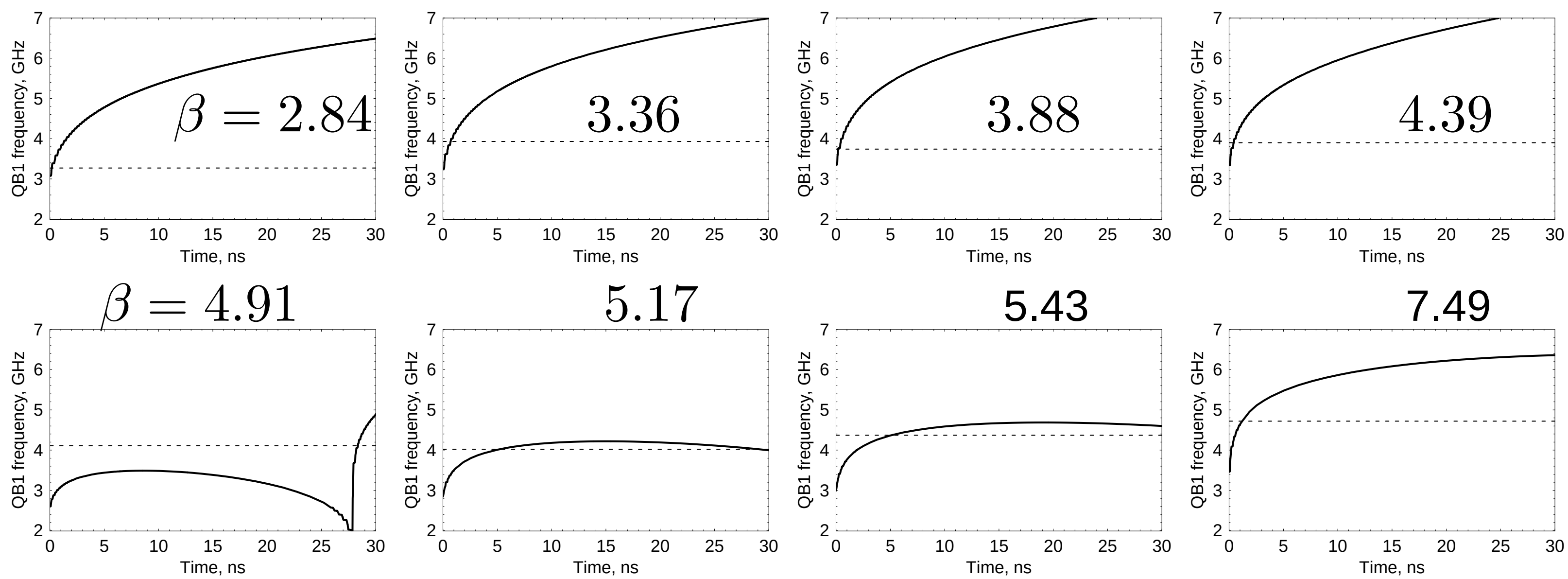


$$1007 \leq n \leq 1014$$

Crosstalk (switching probability) M_c: Large dots – $C_x = 3$ fF, small dots – $C_x = 5$ fF.

The pulse duration is 5 ns or 10 ns, as denoted in the plot

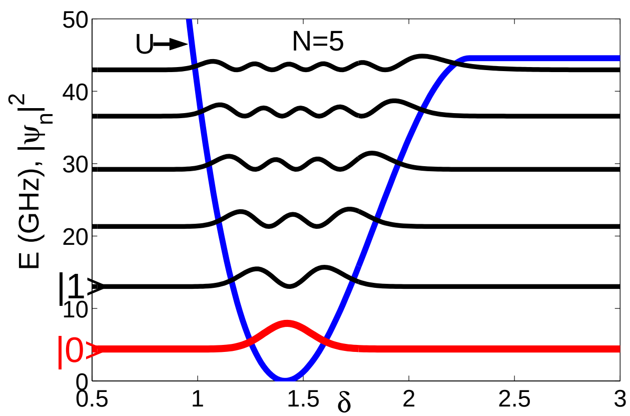


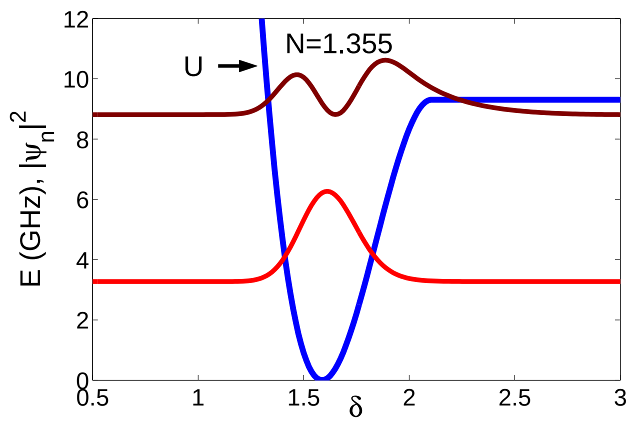


One-qubit errors during the measurement pulse

3. Nonadiabatic effects

$$C = 700 \text{ fF},\ L = 0.72 \text{ nH},\ I_0 = 1.7 \text{ mkA}.$$





$$H(t) = \frac{\hat{p}^2}{2m} + U(\delta, t),\ \ U(\delta, t) = E_J \left\{ \frac{[\delta - \phi(t)]^2}{2\lambda} - \cos\delta \right\}$$

Pulse of $\phi(t)$, the external magnetic flux (in units of the flux quantum).

Theory

2 types of nonadiabatic errors:

(a) One of the levels is populated: $P_e \equiv 1 - P_{00}(\tau) \approx P_{10}(\tau)$.

(b) Both levels are populated: $\simeq \sqrt{P_e} = \sqrt{P_{00}(\tau)}$ - error is increased due to quantum interference.

Roughly (PT): $P_e \approx \dfrac{E_J^2}{2\hbar\beta^2 m\omega_{10}^{3/2}} \left| \int_0^\tau dt \dot{\phi}(t) e^{i\omega_{10} t} \right|^2$ $[\omega_{10} \to \omega_{10}(0)$ or $\omega_{10}(\tau)]$.

More exactly, adiabatic PT (taking into account 2 states):

$$P_e \approx \left(\frac{E_J}{\hbar\beta}\right)^2 \left| \int_0^\tau dt \frac{\dot{\phi}(t)\delta_{10}(t)}{\omega_{10}(t)} e^{i\int_0^t \omega_{10}(t')dt'} \right|^2 .$$

$$\delta_{10} \approx \sqrt{\hbar/(2m\omega_{10})} \Rightarrow P_e \approx \frac{E_J^2}{2\hbar\beta^2 m} \left| \int_0^\tau \frac{dt\dot{\phi}(t)}{\omega_{10}^{3/2}(t)} e^{i\int_0^t \omega_{10}(t')dt'} \right|^2 .$$

Theory: Long-τ behavior

Dependence on the pulse duration τ: $\phi(t) \to \phi(t/\tau) = \phi(\theta)$.

Let $\omega_{10}(t=\tau)\tau \gg 1$. Then

$$P_e \approx \frac{E_J^2}{\hbar^2 \beta^2 \tau^{2j}} \left| \frac{\phi^{(j_0)}(0)\delta_{10}(0)}{\omega_{10}^{j_0+1}(0)\tau^{j_0-j}} - \frac{i^{j_1-j_0}\phi^{(j_1)}(1)\delta_{10}(1)}{\omega_{10}^{j_1+1}(1)\tau^{j_1-j}} e^{i\bar{\omega}_{10}\tau} \right|^2,$$

$$\bar{\omega}_{ni} = \int_0^1 \omega_{ni}(\theta)d\theta$$

j_0, j_1 - lowest orders of derivatives at $t=0$ and $t=\tau$, respectively.
$\phi^{(j)}(\theta)$ is the jth derivative of $\phi(\theta)$.

If $j_0 = j_1 = j$,

$$P_e \approx \frac{E_J^2}{\hbar^2 \lambda^2 \tau^{2j}} \left| \frac{\phi^{(j)}(0)\delta_{10}(0)}{\omega_{10}^{j+1}(0)} - \frac{\phi^{(j)}(1)\delta_{10}(1)}{\omega_{10}^{j+1}(1)} e^{i\bar{\omega}_{10}\tau} \right|^2.$$

- oscillations. If $j_0 \neq j_1$, oscillations vanish for long τ.

Numerical simulations

Method

$$t_m = m\Delta t \; (m = 0, 1, \ldots, N_\tau - 1)$$

We obtain eigenvalues and eigenfunctions of $H(t_m)$ by Fourier grid Hamiltonian (or, equivalently, periodic pseudospectral) method.

$$\Psi(\delta, t) = \sum_n a_n^m \psi_n^m(\delta) e^{-iE_n^m(t-t_m)/\hbar} \;\; (t_m \leq t \leq t_{m+1}),$$

$$a_n^m = \int_{-\infty}^{\infty} \psi_n^{m*}(\delta)\Psi(\delta, t_m) d\delta.$$

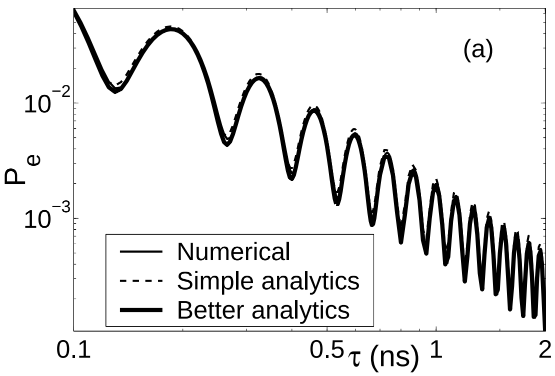
The population of the qubit state n, subject to the condition that initially state i is populated, is given by

$$P_{ni}(t) = |a_n^m|^2 \;\; (t_m \leq t \leq t_{m+1}).$$

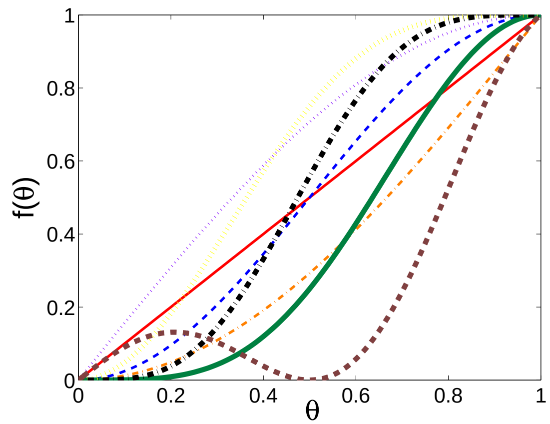
Results

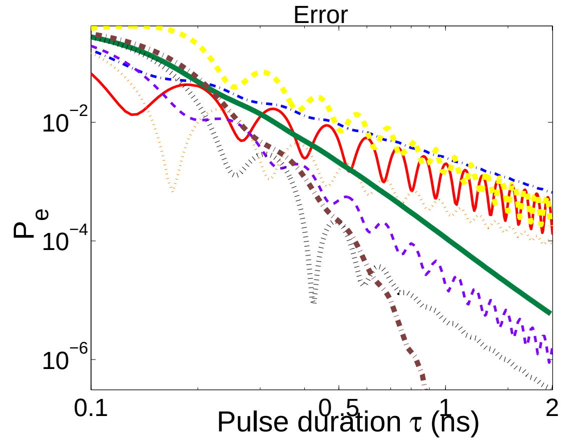
Measurement pulse shape $f(\theta)$: $\phi(t/\tau) = \phi_0 + (\phi_1 - \phi_0) f(t/\tau)$

Linear pulse, $f(\theta) = \theta$:



Pulse shapes





$f(\theta) = \theta$ (thin solid line), $\sin(\pi\theta/2)$ (thin dotted line), $1 - \cos(\pi\theta/2)$ (thin dash-dot line), $\sin^2(\pi\theta/2)$ (thin dashed line), $\sin^4(\pi\theta/2)$ (thick solid line), $1 - \cos^4(\pi\theta/2)$ (thick dotted line), $\theta\cos^2(\pi\theta)$ (thick dashed line), $[1 + \pi^2(1-\theta)^2/2]\sin^4(\pi\theta/2)$ (thick dash-dot line).

Good agreement with the theory.

4. Left-well repopulation

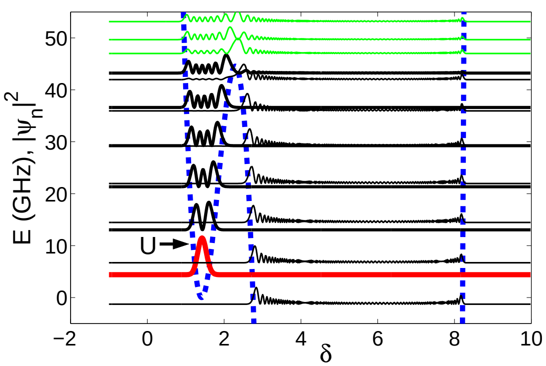
No relaxation ($T_1 \to \infty$)

Case I (above case): $C = 700$ fF, $L = 0.72$ nH, $I_0 = 1.7\ \mu$A, $N_r = 174$ (no. of right-well levels).

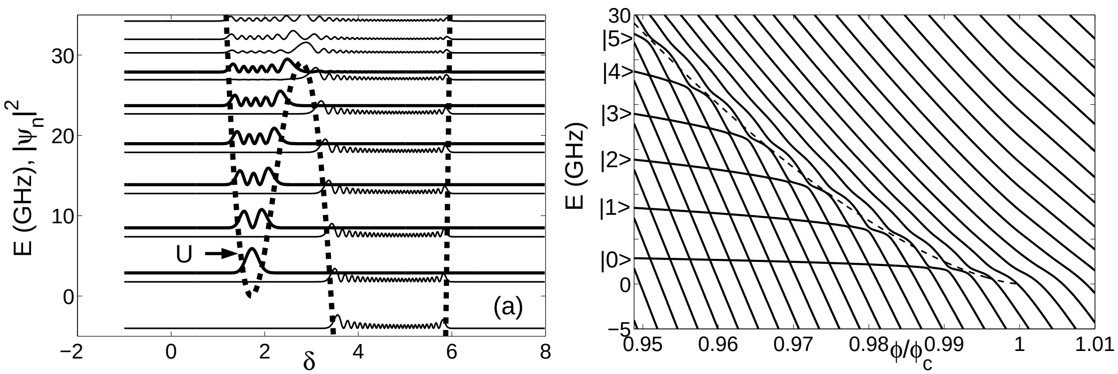
Case II: $C = 790$ fF, $L = 0.720$ nH, $I_0 = 0.764\ \mu$A, $N_r = 30$.

Parabolic pulse: $\phi(t/\tau) = \phi_0 + 4(\phi_1 - \phi_0)(t/\tau)(1 - t/\tau)\ \ (0 \le t \le \tau)$.

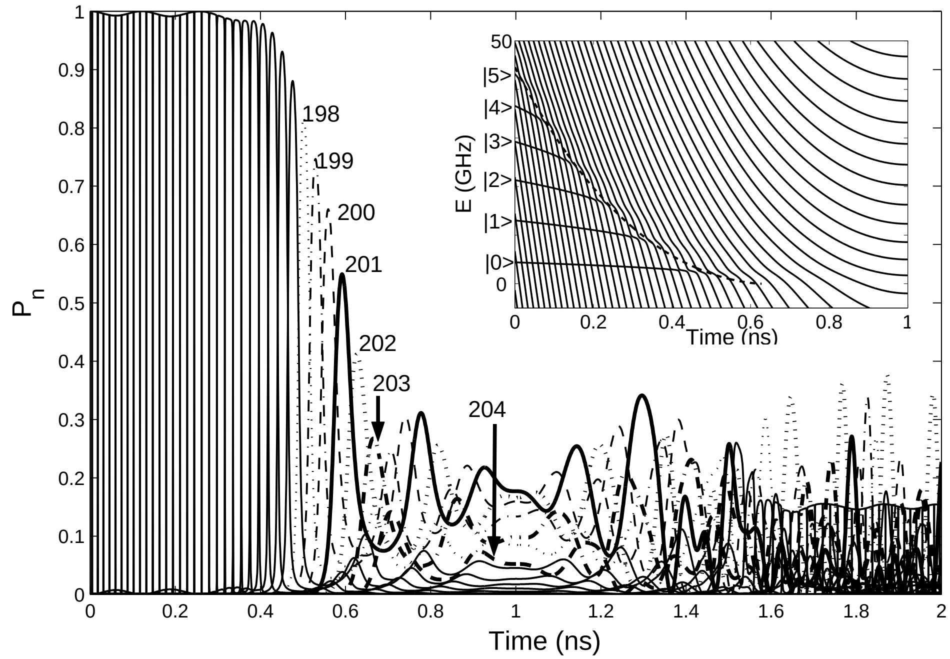
Case I

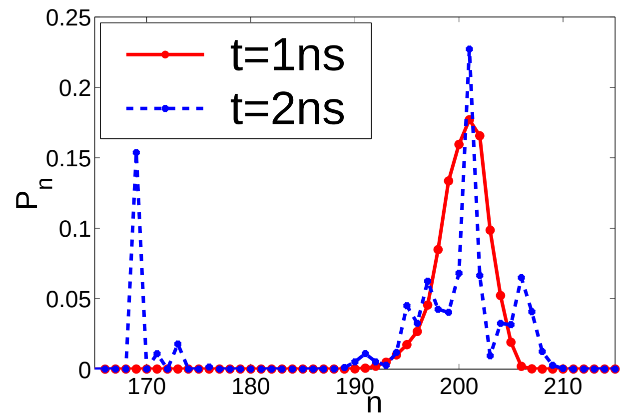


Case II



Case I: Level populations $P_n(t)$ $(169 \leq n \leq 204)$ for $\phi_1 = 1.01\phi_c$ and $\tau = 2$ ns





$t < 0.63$ ns ($t < 0.4$ ns), population of $|0\rangle$ ($|1\rangle$) are the upper (lower) envelope of $P_n(t)$.

$t < 0.3$ ns, they vary due to above nonadiabatic effects.

$0.4 < t < 0.63$ ns, Landau-Zener transitions, due to crossings of left- and right-well levels.

$t < 0.5$ ns, $|0\rangle$ is below the barrier, transitions represent quantum tunneling.

$t \geq 0.5$, level splittings are comparable to level separations, > 2 levels are coupled simultaneously.

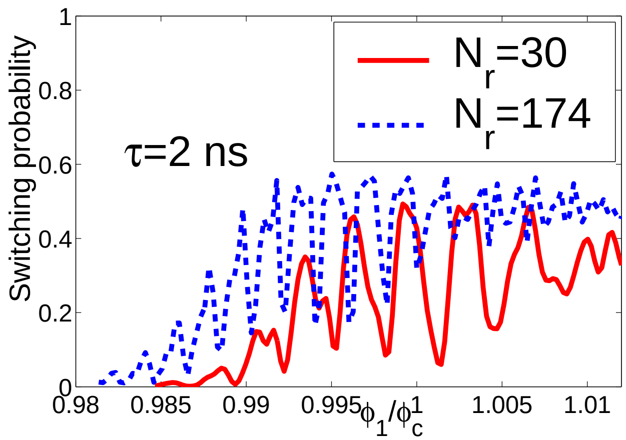
$t \geq 0.63$, the left well disappears.

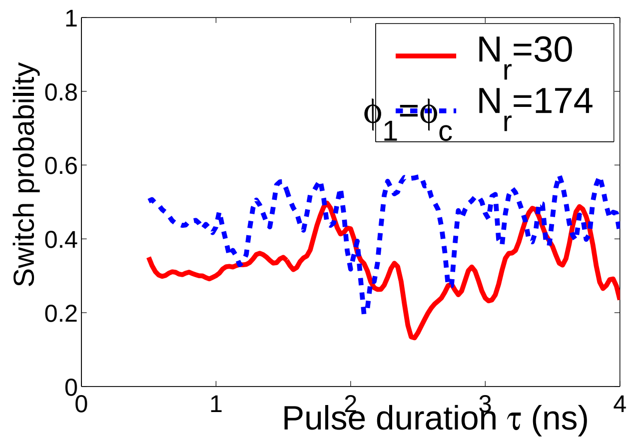
Probability for the system to be to the right of the barrier top: $P_r(t) = \int_{\delta_m}^{\infty} |\Psi(\delta, t)|^2 d\delta$

(where δ_m is barrier-top position). It depends on time, due to excited delocalized levels.

The switching probability is the time average of $P_r(t)$: $P_s = \sum_n P_n \int_{\delta_m}^{\infty} |\psi_n(\delta)|^2 d\delta,$

where P_n is the population of level n after the pulse; the integral is the probability for the system in state n to be in the right well.





The measurement pulse **excites** the system!

Relaxation is obligatory for switching.

5. Effect of relaxation on qubit switching

A rough estimate of the minimal pulse duration τ_m necessary for switching:

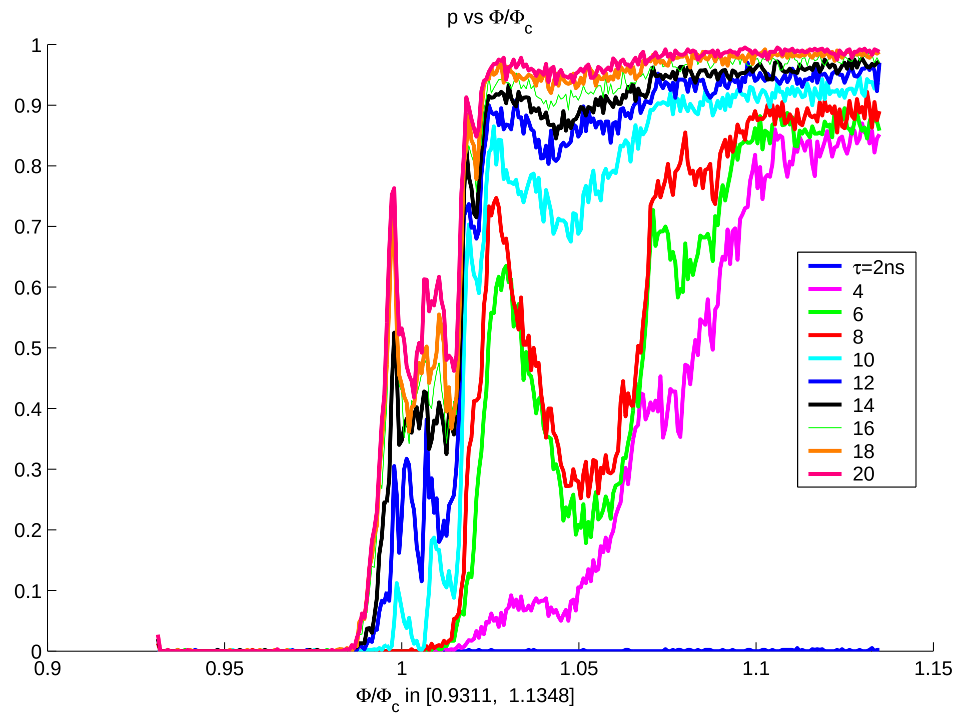
$$\tau_m > \frac{n_i - n_f}{n_i} T_1.$$

Here n_i is the highest state populated immediately after the pulse rise, n_f is the first level above the barrier top after the pulse is over.

In case I, $n_i \approx 200$, $n_f = 181$, hence $\tau_m \approx 0.1 T_1$.

For experiments, $C = 1.1$ pF, $L = 850$ pH, and $I_0 = 1.118$ μA, $T_1 = 110$ ns.
Then we estimate $n_i = 156$, $n_f = 135$ and get $\tau_m \gtrsim 14$ ns.

This estimation seems to correlate with the experimental data. Indeed, in the plot, at $\Phi \approx 1.01\Phi_c$ there is a sharp increase of P to a value close to 1 for $\tau \geq 10$ ns.



The escape probability P vs. Φ/Φ_c.

Conclusions

- Crosstalk of phase qubits after the measurement pulse has been analyzed both classically (for several models) and quantum-mechanically. The theory provides limits on the coupling capacitance and the operating frequency.
- Crosstalk during the measurement pulse has been analyzed by the quantum-mechanical approach. The results agree qualitatively with recent experiments. A region of the qubit parameters, where the crosstalk is reduced, has been found.
- Nonadiabatic errors during the measurement pulse have been studied both analytically (using several approaches with different levels of complexity) and numerically by solving the time-dependent Schrödinger equation.
- Qubit switching in the absence of relaxation has been studied by numerical simulations of the Schrödinger equation. The left-well repopulation effect has been observed and its mechanism has been discussed.
- A relation between the measurement-pulse duration and the relaxation time has been suggested and compared with recent experiments.