Bayesian formalism for continuous measurement of solid-state qubits

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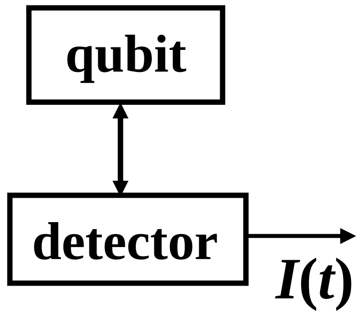
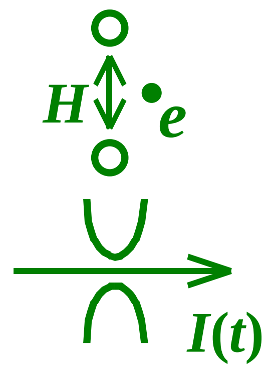
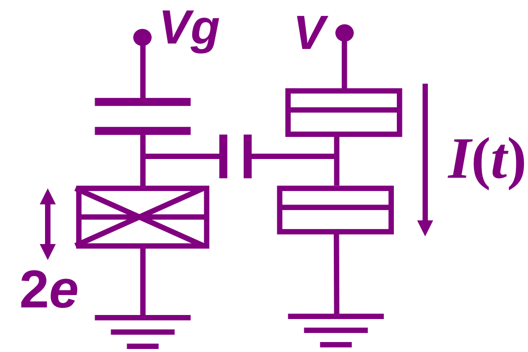


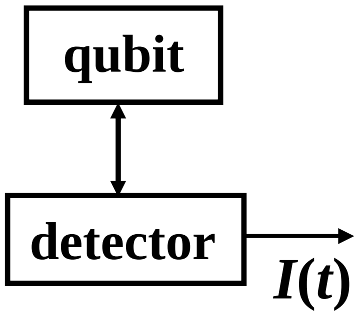


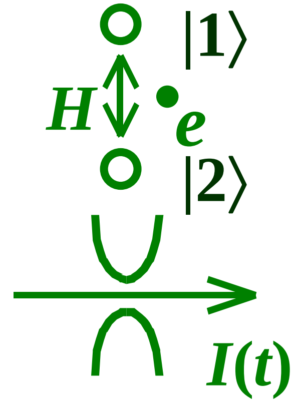


Preliminary remarks:

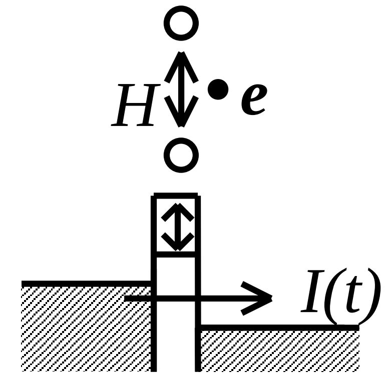
- Today: formalism (+derivations), tomorrow: some applications
- Very simple, almost trivial formalism, but new for solid state (at least 9 years ago), that is why many predictions
- Special case of a general formalism (POVM, optics, etc.); but simple and important for understanding (same essence)
- Ideology: the standard Copenhagen QM (almost)

The system we consider: qubit + detector





Double-quantum-qot (DQD) and quantum point contact (QPC)



$H = H_{\text{QB}} + H_{\text{DET}} + H_{\text{INT}}$

$H_{\text{QB}} = (\varepsilon/2)(c_1^+c_1 - c_2^+c_2) + H(c_1^+c_2 + c_2^+c_1)$ ε – asymmetry, H – tunneling

$$H_{DET} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} T(a_r^\dagger a_l + a_l^\dagger a_r)$$

$$H_{INT} = \sum_{l,r} \Delta T\,(c_1^\dagger c_1 - c_2^\dagger c_2)(a_r^\dagger a_l + a_l^\dagger a_r)$$

Assume real T and ΔT (for simplicity)

Two levels of average detector current: I_1 for qubit state $|1\rangle$, I_2 for $|2\rangle$

$$I_1 = 2\pi\,|T + \Delta T|^2\,\rho_l\rho_r\,e^2V/\hbar, \quad I_2 = 2\pi\,|T - \Delta T|^2\,\rho_l\rho_r\,e^2V/\hbar$$

Response: $\Delta I = I_1 - I_2$ Assume $|\Delta I| \ll I_0 = (I_1+I_2)/2$

Detector noise: white, spectral density S_I $S_I = 2eI$

What happens to a qubit state during measurement?

Start with density matrix evolution due to measurement only ($H=\varepsilon=0$)

"Orthodox" answer

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{matrix} \nearrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \searrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix}$$

|1> or |2>, depending on the result

"Conventional" (decoherence) answer (Leggett, Zurek)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} & \frac{\exp(-\Gamma t)}{2} \\ \frac{\exp(-\Gamma t)}{2} & \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

no measurement result! **(ensemble averaged)**

The two answers apparently contradict each other!

(different questions: instantaneous evolution of a single system vs. continuous evolution of ensemble of systems)

"Conventional" (ensemble averaged) evolution:

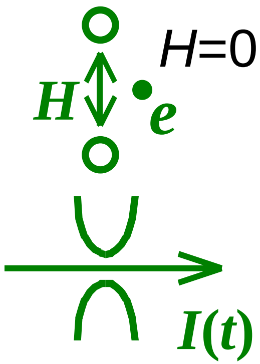
$$d\rho_{11}/dt = -d\rho_{22}/dt = -2(H/\hbar)\,\mathrm{Im}\,\rho_{12}$$

$$d\rho_{12}/dt = i(\varepsilon/\hbar)\rho_{12} + i(H/\hbar)(\rho_{11} - \rho_{22}) - \Gamma\rho_{12}$$

$$\Gamma = (\Delta I)^2 / 4S_I$$

(Gurvitz-97, Aleiner et al.-97, etc.; experimentally verified: Buks et al.-98)

Bayesian formalism for DQD-QPC (qubit-detector) system



Qubit evolution due to continuous measurement:

1) Diagonal matrix elements of the qubit density matrix evolve as classical probabilities (i.e. according to the classical Bayes rule)

2) Non-diagonal matrix elements evolve so that the degree of purity ("murity") $\rho_{ij}/[\rho_{ii}\,\rho_{jj}]^{1/2}$ is conserved

(A.K., 1998)

Bayes rule:

$$P(A_i\,|\,R) = \frac{P(A_i)\,P(R\,|\,A_i)}{\sum_k P(A_k)\,P(R\,|\,A_k)}$$

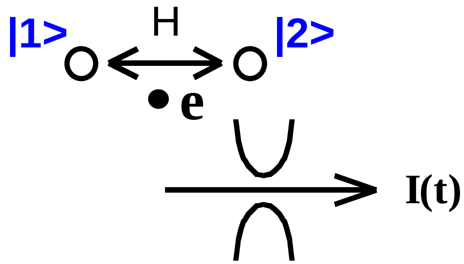
So simple because:

1) QPC happens to be an ideal detector
2) no Hamiltonian evolution of the qubit

Similar formalisms developed earlier. Key words: Imprecise, weak, selective, or conditional measurements, POVM, Quantum trajectories, Quantum jumps, Restricted path integral, etc.

Names: Davies, Kraus, Holevo, Belavkin, Mensky, Caves, Gardiner, Carmichael, Knight, Walls, Gisin, Percival, Milburn, Wiseman, Habib, etc. (very incomplete list)

"Quantum Bayes theorem" (ideal detector assumed)

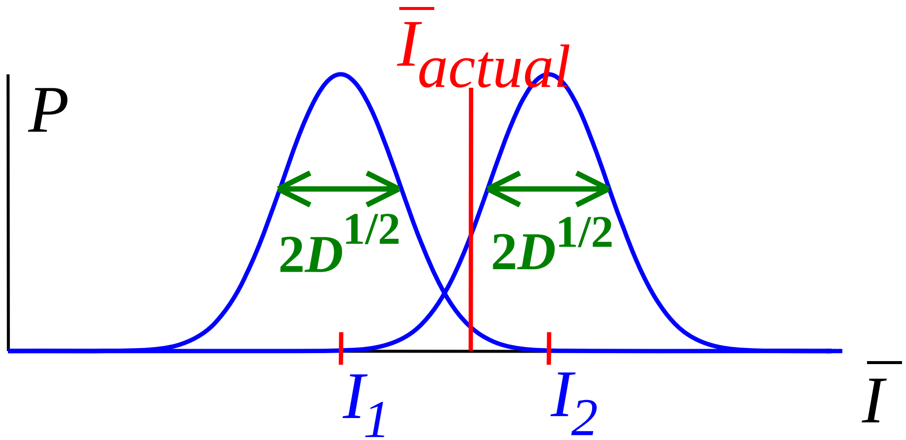


$H = \varepsilon = 0$

("frozen" qubit)

Initial state: $\begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}$

Measurement (during time τ):



$$\bar{I} \equiv \frac{1}{\tau}\int_0^\tau I(t)\,dt$$

$$P(\bar{I},\tau) = \rho_{11}(0)\, P_1(\bar{I},\tau) + \rho_{22}(0)\, P_2(\bar{I},\tau)$$

$$P_i(\bar{I},\tau) = \frac{1}{\sqrt{2\pi D}} \exp[-(\bar{I} - I_i)^2 / 2D],$$

$$D = S_I / 2\tau, \quad |I_1 - I_2| \ll I_i, \quad \tau \gg S_I / I_i^2$$

After the measurement during time τ, the probabilities should be updated using the standard Bayes formula:

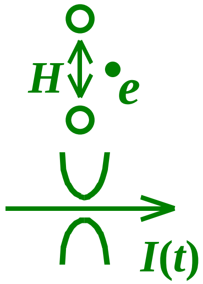
$$P(B_i \mid A) = \frac{P(B_i)P(A \mid B_i)}{\sum_k P(B_k)P(A \mid B_k)}$$

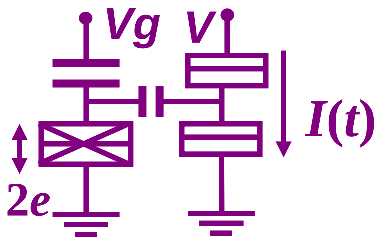
Quantum Bayes formulas:

$$\rho_{11}(\tau) = \frac{\rho_{11}(0)\exp[-(\bar{I} - I_1)^2/2D]}{\rho_{11}(0)\exp[-(\bar{I} - I_1)^2/2D] + \rho_{22}(0)\exp[-(\bar{I} - I_2)^2/2D]}$$

$$\frac{\rho_{12}(\tau)}{[\rho_{12}(\tau)\,\rho_{22}(\tau)]^{1/2}} = \frac{\rho_{12}(0)}{[\rho_{12}(0)\,\rho_{22}(0)]^{1/2}}, \qquad \rho_{22}(\tau) = 1 - \rho_{11}(\tau)$$

Bayesian formalism for a single qubit





$$\hat{H}_{QB} = \frac{\varepsilon}{2}(c_1^\dagger c_1 - c_2^\dagger c_2) + H(c_1^\dagger c_2 + c_2^\dagger c_1)$$

$|1\rangle \rightarrow I_1$, $|2\rangle \rightarrow I_2$, $\Delta I = I_1 - I_2$, $I_0 = (I_1 + I_2)/2$

S_I – detector noise

(Stratonovich form)

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2(H/\hbar)\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}(2\Delta I/S_I)[\underline{\underline{I(t)}} - I_0]$$

$$\dot{\rho}_{12} = i(\varepsilon/\hbar)\rho_{12} + i(H/\hbar)(\rho_{11} - \rho_{22}) + \rho_{12}(\rho_{11} - \rho_{22})(\Delta I/S_I)[\underline{\underline{I(t)}} - I_0] - \gamma\rho_{12}$$

$$I(t) - I_0 = (\rho_{22} - \rho_{11})\Delta I/2 + \xi(t), \quad S_\xi = S_I$$

(A.K., 1998)

$\gamma = \Gamma - (\Delta I)^2/4S_I$, Γ – ensemble decoherence, $\Gamma > (\Delta I)^2/4S_I$

$\eta = 1 - \gamma/\Gamma = (\Delta I)^2/4S_I\Gamma$ – detector ideality (efficiency), $\eta \leq 100\%$

Ideal detector (η=1, as QPC) does not decohere a qubit, then random evolution of qubit *wavefunction* can be monitored

Averaging over result $I(t)$ leads to conventional master equation:

$$d\rho_{11}/dt = -d\rho_{22}/dt = -2(H/\hbar)\,\mathrm{Im}\,\rho_{12}$$

$$d\rho_{12}/dt = i(\varepsilon/\hbar)\rho_{12} + i(H/\hbar)(\rho_{11} - \rho_{22}) - \Gamma\rho_{12}$$

Assumptions needed for the Bayesian formalism

- Detector voltage is much larger than the qubit energies involved

  $eV >> \hbar\Omega$, $eV >> \hbar\Gamma$ (no coherence in the detector, $\hbar/eV << (1/\Omega, 1/\Gamma)$; Markovian approximation)

- Small detector response, $|\Delta I| << I_0$, $\Delta I = I_1 - I_2$, $I_0 = (I_1 + I_2)/2$

  Many electrons pass through detector before qubit evolves noticeably.

  (Not a really important condition, but simplifies formalism.)

Coupling $C \sim \Gamma/\Omega$ is arbitrary [we define $C = \hbar(\Delta I)^2/S_I H$]

$$\frac{d}{dt}\rho_{11} = -\frac{d}{dt}\rho_{22} = -2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}\frac{2\Delta I}{S_I}[I(t) - I_0]$$

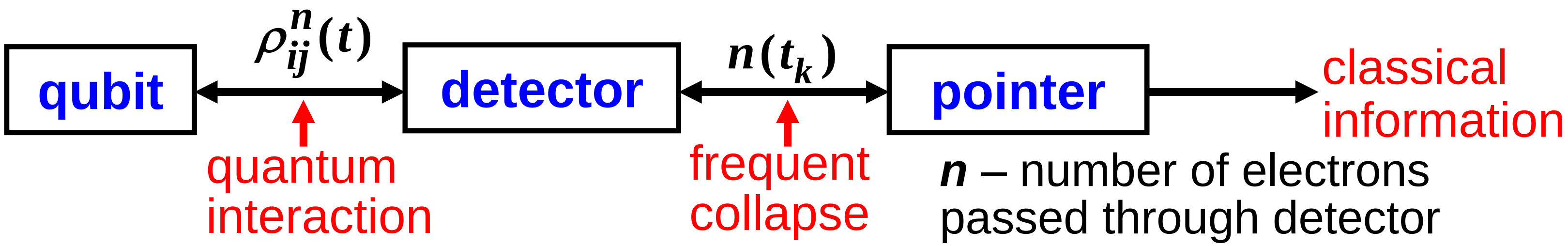
$$\frac{d}{dt}\rho_{12} = i\frac{\varepsilon}{\hbar}\rho_{12} + i\frac{H}{\hbar}(\rho_{11} - \rho_{22}) + \rho_{12}(\rho_{11} - \rho_{22})\frac{\Delta I}{S_I}[I(t) - I_0] - \gamma\rho_{12}$$

Notice: ρ_{11} changes despite $c_1^+ c_1$ commutes with Hamiltonian (H=ε=0)

Derivations

1) "logical": via correspondence principle and comparison with decoherence approach (A.K., 1998)

2) "microscopic": Schr. eq. + collapse of the detector (A.K., 2000)



3) from "quantum trajectory" formalism developed for quantum optics (Goan-Milburn, 2001; also: Wiseman, Sun, Oxtoby, etc.)

4) from POVM formalism (Jordan-A.K., 2006)

"Logical" ("informational") derivation of the Bayesian formalism

Step 1. Assume $H = \varepsilon = 0$, "frozen" qubit
Since ρ_{12} is not involved, evolution of ρ_{11} and ρ_{22} should be the same as in the classical case, i.e. Bayes formula (correspondence principle).

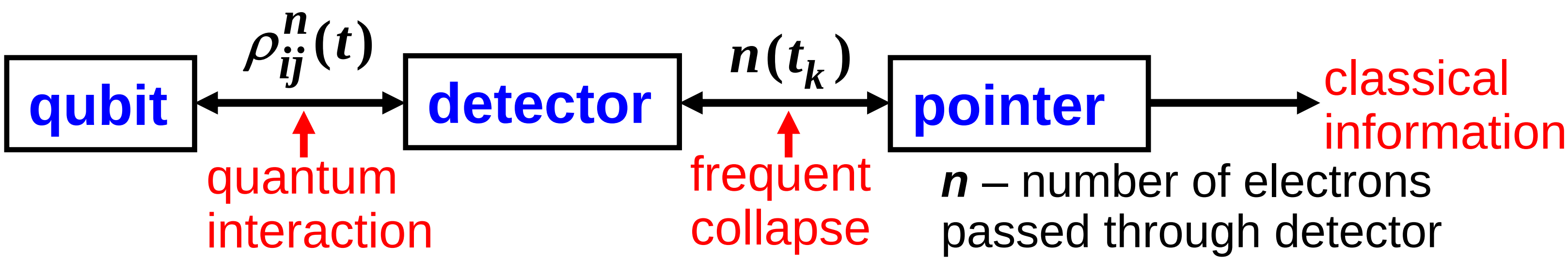
Step 2. Assume $H = \varepsilon = 0$ and pure initial state, $\rho_{12}(0) = [\rho_{11}(0)\, \rho_{22}(0)]^{1/2}$
For any realization $|\rho_{12}(t)| \leq [\rho_{11}(t)\, \rho_{22}(t)]^{1/2}$. Hence, averaging over ensemble of realizations gives $|\rho_{12}^{av}(t)| \leq \rho_{12}^{av}(0) \exp[-(\Delta I^2/4S_I)\, t]$
However, conventional (ensemble) result (Gurvitz-1997, Aleiner *et al*.-1997) for QPC is exactly the upper bound: $\rho_{12}^{av}(t) = \rho_{12}^{av}(0) \exp[-(\Delta I^2/4S_I)\, t]$.
Therefore, pure state remains pure: $\rho_{12}(t) = [\rho_{11}(t)\, \rho_{22}(t)]^{1/2}$.

Step 3. Account of a mixed initial state
Result: the degree of purity $\rho_{12}(t) / [\rho_{11}(t)\, \rho_{22}(t)]^{1/2}$ is conserved.

Step 4. Add qubit evolution due to H and ε.

Step 5. Add extra dephasing due to detector nonideality (i.e., for SET).

“Microscopic” derivation of the Bayesian formalism



Schrödinger evolution of “qubit + detector” for a low-T QPC as a detector (Gurvitz, 1997)

$$\frac{d}{dt}\rho_{11}^{n} = -\frac{I_1}{e}\rho_{11}^{n} + \frac{I_1}{e}\rho_{11}^{n-1} - 2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12}^{n}$$

$$\frac{d}{dt}\rho_{22}^{n} = -\frac{I_2}{e}\rho_{22}^{n} + \frac{I_2}{e}\rho_{22}^{n-1} + 2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12}^{n}$$

$$\frac{d}{dt}\rho_{12}^{n} = i\frac{\varepsilon}{\hbar}\rho_{12}^{n} + i\frac{H}{\hbar}(\rho_{11}^{n} - \rho_{22}^{n}) - \frac{I_1 + I_2}{2e}\rho_{12}^{n} + \frac{\sqrt{I_1 I_2}}{e}\rho_{12}^{n-1}$$

Detector collapse at $t = t_k$

Particular n_k is chosen at t_k

$$P(n) = \rho_{11}^{n}(t_k) + \rho_{22}^{n}(t_k)$$

$$\rho_{ij}^{n}(t_k + 0) = \delta_{n,nk}\,\rho_{ij}(t_k + 0)$$

$$\rho_{ij}(t_k + 0) = \frac{\rho_{ij}^{nk}(t_k)}{\rho_{11}^{nk}(t_k) + \rho_{22}^{nk}(t_k)}$$

If $H = \varepsilon = 0$, this leads to

$$\rho_{11}(t) = \frac{\rho_{11}(0)P_1(n)}{\rho_{11}(0)P_1(n) + \rho_{22}(0)P_2(n)}, \quad \rho_{22}(t) = \frac{\rho_{22}(0)P_2(n)}{\rho_{11}(0)P_1(n) + \rho_{22}(0)P_2(n)}$$

$$\rho_{12}(t) = \rho_{12}(0)\frac{[\rho_{11}(t)\rho_{22}(t)]^{1/2}}{[\rho_{11}(0)\rho_{22}(0)]^{1/2}}, \quad P_i(n) = \frac{(I_i t/e)^n}{n!}\exp(-I_i t/e),$$

which are exactly quantum Bayes formulas

Derivation via POVM formalism

(Jordan-A.K., 2006)

General theory of quantum measurement (POVM):

If $\quad |\psi\rangle_A |0\rangle_B \rightarrow \sum_Q M_Q |\psi\rangle_A |Q\rangle_B \qquad$ (result Q)

then $\quad P(Q) = Tr(\rho_A M_Q^\dagger M_Q), \quad \sum_Q M_Q^\dagger M_Q = 1, \quad$ **positive** $M_Q^\dagger M_Q$

$$\rho_A^{after} = \frac{M_Q \rho_A M_Q^\dagger}{Tr(M_Q \rho_A M_Q^\dagger)}$$

$$|in\rangle_B (\alpha |1\rangle_A + \beta |2\rangle_B) \rightarrow \alpha (t_1 |T\rangle_B + r_1 |R\rangle_B)|1\rangle_A + \beta (t_2 |T\rangle_B + r_2 |R\rangle_B)|2\rangle_A$$

$$M_T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \quad M_R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

If transmitted: $\quad \rho_{11}^{after} = \dfrac{|t_1|^2 \rho_{11}}{|t_1|^2 \rho_{11} + |t_2|^2 \rho_{22}}, \quad \rho_{12}^{after} = \dfrac{t_1 t_2^* \rho_{12}}{|t_1|^2 \rho_{11} + |t_2|^2 \rho_{22}}$

If reflected: $\quad \rho_{11}^{after} = \dfrac{|r_1|^2 \rho_{11}}{|r_1|^2 \rho_{11} + |r_2|^2 \rho_{22}}, \quad \rho_{12}^{after} = \dfrac{r_1 r_2^* \rho_{12}}{|r_1|^2 \rho_{11} + |r_2|^2 \rho_{22}}$

Bayesian equations for the case with classical correlation between output and back-action noises

(A.K., 1999-2002)

$$\frac{d}{dt}\rho_{11} = -\frac{d}{dt}\rho_{22} = -2H\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}\frac{2\Delta I}{S_I}[I(t)-I_0]$$

$$\frac{d}{dt}\rho_{12} = i\varepsilon\rho_{12} + iH(\rho_{11}-\rho_{22}) + \rho_{12}(\rho_{11}-\rho_{22})\frac{\Delta I}{S_I}[I(t)-I_0] + \underline{\underline{iK[I(t)-I_0]}} - \tilde{\gamma}\rho_{12}$$

Relation between the Bayesian formalism and general quantum measurement (POVM)

$$\rho_A^{after} = \frac{M_Q \rho_A M_Q^\dagger}{Tr(M_Q \rho_A M_Q^\dagger)} \qquad \text{(result } Q\text{)}$$

Theorem: $M_Q = U_Q \sqrt{M_Q^\dagger M_Q}$

General = Bayesian + classical back-action, depending on *Q*

(Even more general: add partial averaging over result *Q*)

Stratonovich and Ito forms for nonlinear stochastic differential equations

Definitions of the derivative:

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t/2) - f(t-\Delta t/2)}{\Delta t} \qquad \text{(Stratonovich)}$$

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \qquad \text{(Ito)}$$

Why matters? Usually $(f+df)^2 \approx f^2 + 2f\,df, \;\; (df)^2 \ll df$

But if $df = \xi\, dt$ (white noise ξ), then $(df)^2 = \xi^2 dt^2 \approx \frac{S_\xi}{2} dt$

Simple translation rule:

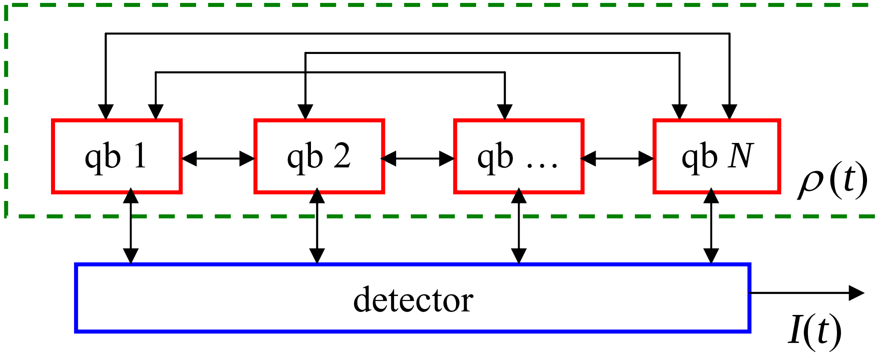
$$\dot{x}_i(t) = G_i(\vec{x},t) + F_i(\vec{x},t)\xi(t) \qquad \text{(Stratonovich)}$$

$$\dot{x}_i(t) = G_i(\vec{x},t) + F_i(\vec{x},t)\xi(t) + \frac{S_\xi}{4}\sum_k \frac{\partial F_i(\vec{x},t)}{dx_k} F_k(\vec{x},t) \qquad \text{(Ito)}$$

Advantage of Stratonovich: usual calculus rules (intuition)

Advantage of Ito: simple averaging

Bayesian formalism for *N* entangled qubits measured by one detector



Up to 2^N levels of current

$$\frac{d}{dt}\rho_{ij} = \frac{-i}{\hbar}[\hat{H}_{qb},\rho]_{ij} + \rho_{ij}\frac{1}{S}\sum_k \rho_{kk}\Big[\big(I(t) - \frac{I_k + I_i}{2}\big)(I_i - I_k) + \big(I(t) - \frac{I_k + I_j}{2}\big)(I_j - I_k)\Big] - \gamma_{ij}\rho_{ij}$$

(Stratonovich form)

$$\gamma_{ij} = (\eta^{-1} - 1)(I_i - I_j)^2 / 4S_I \qquad I(t) = \sum_i \rho_{ii}(t) I_i + \xi(t)$$

Averaging over ξ(t) ⟹ master equation

No measurement-induced dephasing between states $|i\rangle$ and $|j\rangle$ if $I_i = I_j$!

A.K., PRA 65 (2002), PRB 67 (2003)

Conclusions