Error matrices in quantum process tomography

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Outline:
- Basics of χ-matrix
- Error matrices χ^{err} and $\tilde{\chi}^{err}$
- Some properties (incl. interpretation)
- Composition of gates
- Unitary corrections
- Error from Lindblad-form decoherence
- SPAM identification and subtraction

Basics of the QPT matrix χ

Definition
$$\rho_{\rm fin} = \sum_{m,n} \chi_{mn} E_m \rho_{\rm in} E_n^\dagger,$$

Pauli basis
$$I \equiv \mathbb{1}, \;\; X \equiv \sigma_x, \;\; Y \equiv \sigma_y, \;\; Z \equiv \sigma_z,$$

two qubits: ***II, IX, IY, IZ, XI, XX, … ZZ***

Pauli basis is orthogonal
(almost orthonormal)
$$\langle E_m | E_n \rangle \equiv \mathrm{Tr}(E_m^\dagger E_n) = \delta_{mn} d, \quad d = 2^N$$

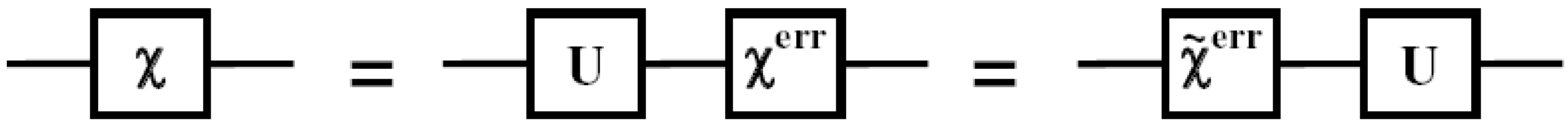
χ-matrix for unitary *U*
$$\chi_{mn} = u_m u_n^*, \;\; U = \sum u_n E_n, \;\; u_n = \frac{1}{d}\mathrm{Tr}(U E_n^\dagger)$$

Fidelity (unitary desired,
trace-preserving actual)
$$F_\chi = \mathrm{Tr}(\chi^{\rm des}\chi)$$

Relation to average state fidelity
(IBM term.: process fid. vs gate fid.)
$$1 - F_\chi = (1 - F_{\rm av})\,\frac{d+1}{d}$$
$$F_{\rm av} = \overline{\mathrm{Tr}(\rho_{\rm fin}\rho_{\rm fin}^{\rm des})} \qquad F_{\rm av} \geq F_\chi$$

Fidelity when compare with
a non-unitary process
$$F_\chi = \left(\mathrm{Tr}\sqrt{\sqrt{\chi}\,\chi^{\rm des}\sqrt{\chi}}\right)^2$$

Definition of error matrices



U is the desired unitary, the rest is "error"

$$\rho_{\rm fin} = \sum\nolimits_{m,n} \chi^{\rm err}_{mn} E_m U \rho_{\rm in} U^\dagger E_n^\dagger,$$

$$\rho_{\rm fin} = \sum\nolimits_{m,n} \tilde{\chi}^{\rm err}_{mn} U E_m \rho_{\rm in} E_n^\dagger U^\dagger .$$

Equivalent to the χ-matrix and to each other (two languages)

$$\chi^{\rm err} = V\chi V^\dagger, \;\; V_{mn} = {\rm Tr}(E_m^\dagger E_n U^\dagger)/d,$$

$$\tilde{\chi}^{\rm err} = \tilde{V}\chi \tilde{V}^\dagger, \;\; \tilde{V}_{mn} = {\rm Tr}(E_m^\dagger U^\dagger E_n)/d.$$

Same math. properties as for χ-matrix (Hermitian, positive, trace-one, etc.)

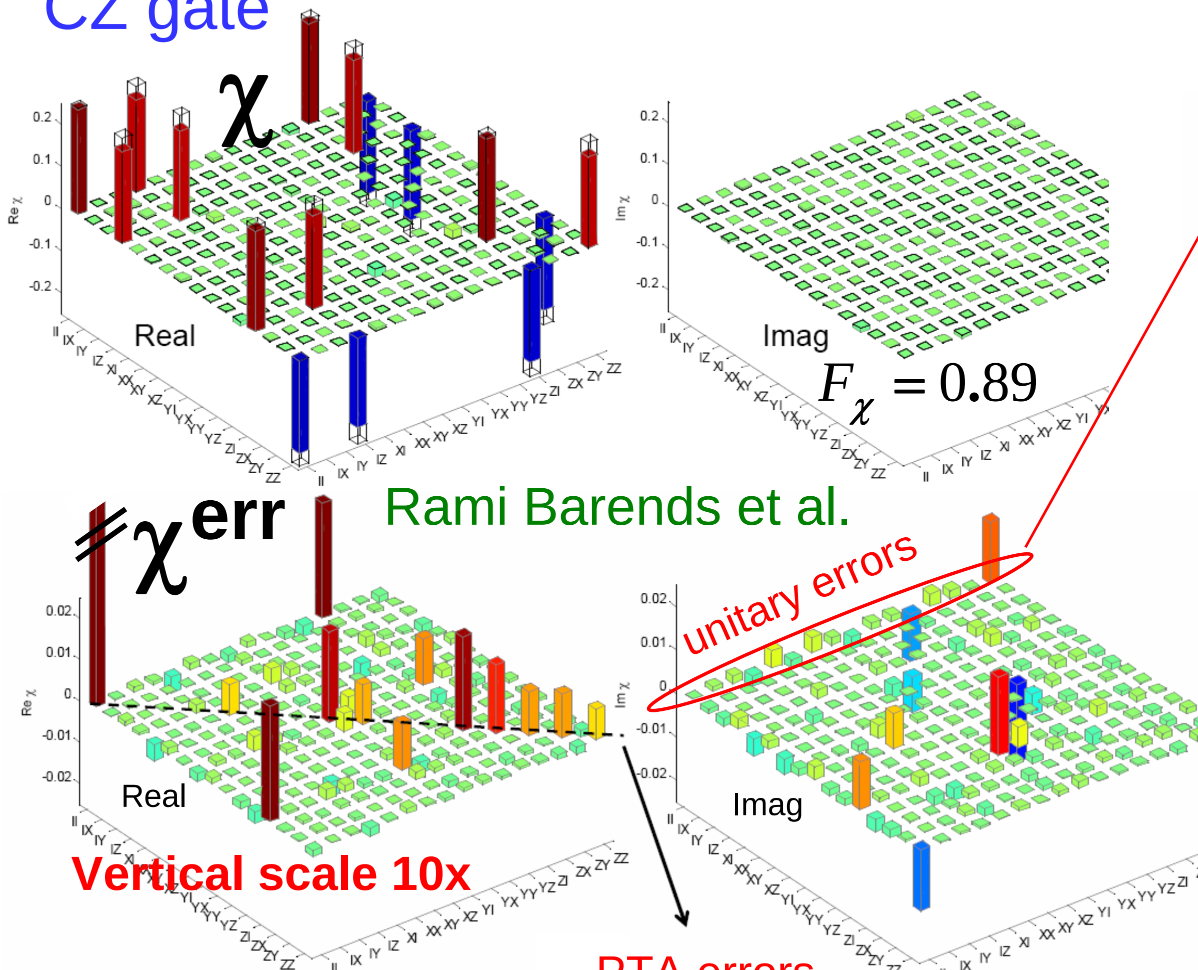
Convenience: only one big element at the top left corner, other non-zero elements indicate imperfections

$$F_\chi = \chi^{\rm err}_{00} = \tilde{\chi}^{\rm err}_{00},$$

$$0 \equiv I, II, ...$$

Experimental example

CZ gate



Unitary imperfection:

$$\mathrm{Im}(\chi_{n0}^{\text{err}}) \approx -iF_\chi u_n^{\text{err}}$$

$$U^{\text{err}} = \sum_n u_n^{\text{err}} E_n$$

Why imaginary part?

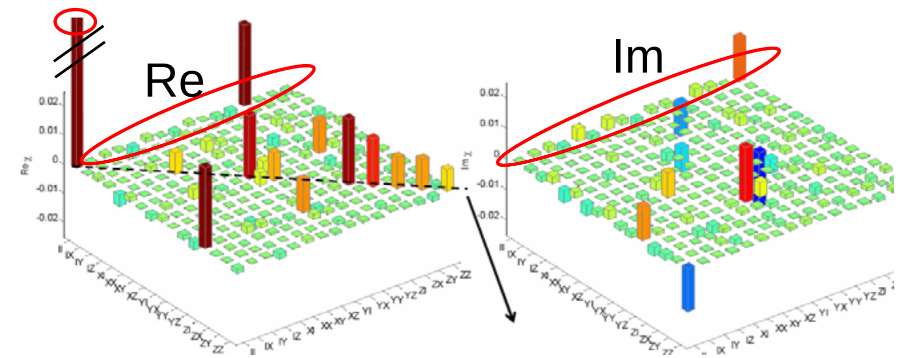
$$U^{\text{err}} = \cancel{e^{i\phi}} \exp(iH^{\text{err}})$$

$$\approx \mathbb{1} + \sum_{n\neq 0} i h_n^{\text{err}} E_n$$

real

Meaning of some elements

$$\chi_{mn}^{\text{err}}$$



χ_{00}^{err} - fidelity (top left)

$\mathbf{Im}(\chi_{0n}^{\text{err}}) = -\mathbf{Im}(\chi_{n0}^{\text{err}})$ - unitary imperfection (top row & left column); may be the biggest elements in χ^{err}

$$|\chi_{0n}^{err}| \le \sqrt{\chi_{nn}^{err}} \le \sqrt{1-F_\chi}, \qquad |\chi_{mn}^{err}| \le \sqrt{\chi_{mm}^{err}\chi_{nn}^{err}} \le (1-F_\chi)/2$$

$\mathbf{Re}(\chi_{0n}^{\text{err}}) = \mathbf{Re}(\chi_{n0}^{\text{err}})$ - non-unitary "Bayesian" evolution in the absence of "jumps" due to decoherence

Other elements (with $m\neq 0$, $n\neq 0$) originate from "strong jumps" due to decoherence

Diagonal elements ($n\neq 0$) have two contributions: from the "jumps" due to decoherence and second-order unitary imperfection, $\approx (\mathbf{Im}\,\chi_{0n}^{\text{err}})^2 / F_\chi$

The same applies to $\tilde{\chi}^{\text{err}}$

Decomposition into Kraus operators

Formal procedure: diagonalize χ^{err}

$$\chi^{\rm err} = TDT^{-1} \qquad D = {\rm diag}(\lambda_0, \lambda_1, ...) \qquad \lambda_0 \geq \lambda_1 \geq ...$$

One main eigenvalue λ_0 (≈1), other λ are small

$$F_\chi \leq \lambda_0 \leq 1 \qquad\qquad \sum\nolimits_n \lambda_n = 1$$

Decomposition

$$\rho_{fin} = \sum_{k=0}^{d^2-1} \lambda_k A_k (U\rho_{in}U^\dagger) A_k^\dagger, \;\; \sum_k \lambda_k A_k^\dagger A_k = 1\!\!1$$

$$A_k = \sum_n a_n^{(k)} E_n, \;\; a_n^{(k)} = T_{nk}, \;\; \chi_{mn}^{\rm err} = \sum\nolimits_k \lambda_k a_m^{(k)} (a_n^{(k)})^*$$

Kraus operators A_k form orthonormal basis

Interpretation: "apply A_k with probability λ_k" (there are caveats)

A_0≈**1** describes "coherent" (gradual) evolution, others are "strong jumps"

Actually $|\psi_{\rm in}\rangle \to \dfrac{A_k U |\psi_{\rm in}\rangle}{\rm Norm}$ with probability $P_k = {\rm Tr}(\lambda_k A_k^\dagger A_k U \rho_{\rm in} U^\dagger)$,

so λ_k is average probability, $\lambda_k = \overline{P_k}$

Intuitive (approximate) way to think

$$F_\chi \approx 1 - \mathcal{E}_U - \mathcal{E}_D$$

$\mathcal{E}_D \equiv 1 - \lambda_0$ (average) probability of “strong decoherence jump”

$\mathcal{E}_U \equiv \sum_{n>0} |u_n^{\rm err}|^2$ unitary error in the case of no jump

no-jump scenario: $$\sqrt{\lambda_0}\, A_0 \approx U^{\rm err} \underbrace{\left(1\!\!1 - \frac{1}{2} \sum_{k>0} \lambda_k A_k^\dagger A_k \right)}_{\text{(“Bayesian update”)}}$$

$\mathrm{Im}(\chi_{n0}^{\rm err})$ $\mathrm{Re}(\chi_{n0}^{\rm err})$

$$\chi^{\rm err} = \chi^{\rm coh} + \chi^{\rm dec}$$

“coherent” contribution to $\chi^{\rm err}$ is of the second order (except top row and left column), not important unless big unitary imperfection

Example: one-qubit T_1 and T_φ decoherence

$$\chi^{\rm err} = \tilde{\chi}^{\rm err} = \chi \qquad \text{(no unitary evolution)}$$

Energy relaxation

$$\begin{pmatrix} F_\chi & 0 & 0 & t/4T_1 \\ & t/4T_1 & -i\,t/4T_1 & 0 \\ & & t/4T_1 & 0 \\ & & & \sim t^2 \end{pmatrix}$$

$$A = \sqrt{t/4T_1}\,(X + iY)$$

$$A^\dagger A = (t/2T_1)(I - Z)$$

("jump" incl. prob.)

Markovian pure dephasing

$$\begin{pmatrix} F_\chi & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & t/2T_\varphi \end{pmatrix}$$

$$A = \sqrt{t/2T_\varphi}\, Z$$

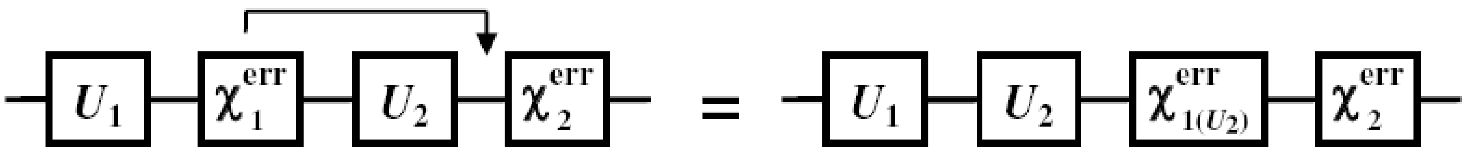
$A^\dagger A = (t/2T_\varphi)\, I$ (state-indep., no Bayes)

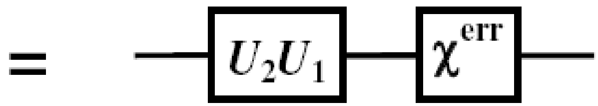
Non-Markovian pure dephasing

$$\chi_{ZZ} = \frac{1 - \langle \cos\varphi \rangle}{2}$$ same as in Ramsey $$P_R = \frac{1}{2} + \frac{1}{2}\, e^{-t/2T_1} \langle \cos\varphi \rangle \cos\phi_R$$

Very slow fluctuations (Gaussian Ramsey): $\chi_{ZZ} = t^2/2T_\varphi^2$

Composition of error processes





Small elements in χ^{err} $\Rightarrow$ first order is sufficient

If no *U*, then very simple: just add errors

$$\chi^{err} \approx \chi_1^{err} + \chi_2^{err} - \chi^{\mathbf{I}}$$

ideal

A little more accurate: $\chi_{mn}^{err} \approx F_2 \chi_{1,mn}^{err} + F_1 \chi_{2,mn}^{err}$

Even more accurate for diagonal elements: $\chi_{nn}^{err} \approx F_2 \chi_{1,nn}^{err} + F_1 \chi_{2,nn}^{err} + 2\,\mathrm{Im}(\chi_{1,0n}^{err})\,\mathrm{Im}(\chi_{2,0n}^{err})$

With *U* two steps: "jump over unitary", then add

$$\chi_{1(U_2)}^{err} = W_{(U_2)} \chi_1^{err} W_{(U_2)}^\dagger$$

$$W_{(U),mn} = \mathrm{Tr}(E_m^\dagger U E_n U^\dagger)/d$$

Same for jumping Krauses:

$$A_k \to U A_k U^\dagger$$

Unitary corrections

$\mathbf{Im}(\chi_{0n}^{\mathrm{err}})$ shows unitary imperfections

Can correct (at least some elements) by applying U^{corr}, then increase fidelity (only in the second order)

$$U^{\mathrm{corr}} = \sum_n u_n^{\mathrm{corr}} E_n \approx \mathbb{1}$$

$$\chi_{n0}^{\mathrm{err}} \to \chi_{n0}^{\mathrm{err}} + F_\chi u_n^{\mathrm{corr}}$$

choose $\mathrm{Im}(u_n^{\mathrm{corr}}) \approx -\mathrm{Im}(\chi_{n0}^{\mathrm{err}})/F_\chi$

fidelity improvement $\Delta F_\chi \approx \sum_{n\neq 0} (\mathrm{Im}\, \chi_{n0}^{\mathrm{err}})^2 / F_\chi$

$\chi^{\rm err}$ from Lindblad-form decoherence

$$\dot\rho = -\frac{i}{\hbar}[H,\rho] + \sum_j \Gamma_j (B_j\rho B_j^\dagger - \frac{1}{2}B_j^\dagger B_j \rho - \frac{1}{2}\rho B_j^\dagger B_j)$$

Contribution during Δt

$$\frac{\chi^{\rm err}_{mn}(t,\Delta t) - \chi^{\rm I}_{mn}}{\Gamma\,\Delta t} \equiv \mathcal{B}_{mn} = b_m b_n^* - \frac{c_m\delta_{n0} + c_n^*\delta_{m0}}{2},$$

$$b_n = {\rm Tr}(BE_n^\dagger)/d,$$

$$c_n = {\rm Tr}(B^\dagger B E_n^\dagger)/d = \sum_{p,q} b_p^* b_q {\rm Tr}(E_p^\dagger E_q E_n^\dagger)/d.$$

Jump over the unitary, then add

$$\tilde\chi^{\rm err} \approx \chi^{\rm I} + \int_0^{t_G} \Gamma\, W^\dagger(t)\, \mathcal{B}\, W(t)\, dt,$$

$$W_{mn}(t) = {\rm Tr}[E_m^\dagger U(t) E_n U^\dagger(t)]/d,$$

$$U(t) = \exp[\frac{-i}{\hbar}\int_0^t H(t)\, dt],$$

Equivalently, jump Kraus *B* over *U*

$$\tilde\chi^{\rm err} \approx \chi^{\rm I} + \int \Gamma \tilde{\mathcal{B}}(t)\, dt$$

$$\tilde B(t) \equiv U^\dagger(t) B U(t)$$

$\chi^{\rm err} \propto \Gamma$, "pattern" depends on $U(t)$

Infidelity accumulates

$$F_\chi \approx 1 - \int_0^{t_G} \Gamma \sum_{n\neq 0} |b_n|^2\, dt$$

$$1 - F_\chi = \frac{t_G}{2T_1^{(a)}} + \frac{t_G}{2T_1^{(b)}} + \frac{t_G}{2T_\varphi^{(a)}} + \frac{t_G}{2T_\varphi^{(b)}}$$

SPAM identification and subtraction

$$-\boxed{\chi^{\rm prep}}-\boxed{U}-\boxed{\chi^{\rm err}}-\boxed{\chi^{\rm meas}}- \;=\; -\boxed{U}-\boxed{\chi^{\rm err,exp}}-$$

Representation by error channels is an assumption

A way to check: $F_\chi \approx F_\chi^{\rm exp}/F_\chi^{I}$ (compare with randomized benchmarking)

SPAM contributions depend on *U*

$$\chi^{\rm err,exp} \approx \chi^{\rm err} + W_{(U)}(\chi^{\rm prep}-\chi^{\mathbf{I}})W_{(U)}^{\dagger} + (\chi^{\rm meas}-\chi^{\mathbf{I}})$$

Simple if SPAM is dominated by one component, then compare with no gate

mainly meas. error, then

no gate

$$\chi^{\rm err} \approx \chi^{\rm err,exp} - (\chi^{{\rm err},I} - \chi^{\mathbf{I}})$$

mainly prep. error, then

$$\tilde{\chi}^{\rm err} \approx \tilde{\chi}^{\rm err,exp} - (\chi^{{\rm err},I} - \chi^{\mathbf{I}})$$

In general, need to know $\chi^{\rm prep}$ and $\chi^{\rm meas}$ separately.

Idea: use high-fidelity single-qubit gates to separate the contributions.

X and *Y* gates flip the sign of some off-diagonal elements, $\sqrt{X}$ and $\sqrt{Y}$ exchange some diagonal elements. Lengthy procedure, but possible. Need it only for significant elements of $\chi^{{\rm err},I}$.

Conclusions

- Error matrices χ^{err} and $\tilde{\chi}^{err}$ are more convenient to use than χ, easy conversion between them
- More intuitive understanding of some elements, natural separation into “coherent” and “jump” contributions
- Since all elements are small (except one), the first-order calculations may be sufficient for composition of gates and accumulation of Lindblad-form decoherence
- Unitary imperfections are easily seen, simple analysis of unitary corrections
- SPAM is a serious problem, but there is (hopefully) a way to identify and subtract it. If SPAM is dominated by one type of error, then rather simple, otherwise quite lengthy.