Distributed Tracking Control for Linear Multiagent Systems With a Leader of Bounded Unknown Input

Zhongkui Li, *Member, IEEE*, Xiangdong Liu, Wei Ren, *Member, IEEE*, and Lihua Xie, *Fellow, IEEE*

***Abstract*—This technical note considers the distributed tracking control problem of multiagent systems with general linear dynamics and a leader whose control input is nonzero and not available to any follower. Based on the relative states of neighboring agents, two distributed discontinuous controllers with, respectively, static and adaptive coupling gains, are designed for each follower to ensure that the states of the followers converge to the state of the leader, if the interaction graph among the followers is undirected, the leader has directed paths to all followers, and the leader's control input is bounded. A sufficient condition for the existence of the distributed controllers is that each agent is stabilizable. Simulation examples are given to illustrate the theoretical results.**

***Index Terms*—Adaptive control, consensus, cooperative control, distributed tracking, multiagent system.**

I. Introduction

There has been a lot of interest in the consensus control problem of multiagent systems due to its potential applications in spacecraft formation flying, sensor networks, and so forth [1], [2]. Consensus means that a group of agents reaches an agreement on a physical quantity of interest by interacting with their local neighbors. Existing consensus algorithms can be roughly categorized into two classes, namely, consensus without a leader (i.e., leaderless consensus) and consensus with a leader. For more details on the leaderless consensus problem, the readers are referred to [3]–[8]. The case of consensus with a leader is also called leader-following consensus or distributed tracking.

In this technical note, we focus on the distributed tracking problem of multiagent systems, which has been studied from different perspectives. The authors in [9], [10] design a distributed neighbor-based estimator to track an active leader. Distributed tracking algorithms are proposed in [11] for a network of agents with first-order dynamics. In [12], discontinuous controllers are studied in the absence of velocity or acceleration measurements. In [13], the leader-following consensus in the presence of time-varying delays is considered. The authors in [14] address a distributed coordinated tracking problem for multiple Euler-Lagrange systems with a dynamic leader. In [15], a distributed tracking scheme with distributed estimators is developed for leader-following multiagent systems subject to measurement noises. The robust consensus tracking problem is studied in [16] for multiagent systems with integrator-type dynamics in the presence of unmodelled dynamics. Ref. [17] considers the consensus problem of multiple uncertain mechanical systems with respect to a desired trajectory. Distributed tracking controllers are designed in [6], [18], [19] for multiagent systems with general linear dynamics. It is worth noting that one common assumption in [6], [10], [18], [19] is that the leader's control input is either equal to zero or available to all the followers. In many circumstances, nonzero control actions might be implemented on the leader in order to achieve certain objectives, e.g., to reach a desirable consensus value or to avoid hazardous obstacles. However, it is restrictive and impractical to assume that all the followers know the leader's control input, especially when the scale of the network is large. Actually, the leader's control input might not be available to any follower, e.g., for the case where the leader is an uncooperative target.

In this technical note, we consider the distributed tracking control problem with a leader whose control input might be nonzero and time varying and not available to any follower for multiagent systems with general linear dynamics. Due to the nonzero control input, the dynamics of the leader are different from those of the followers. Thus, contrary to the homogeneous multiagent systems in [6], [10], [18], [19], the resulting multiagent system in this technical note is in essence heterogeneous. The distributed controllers in [6], [18], [19] are not applicable any more. Based on the relative states of neighboring agents, we propose two distributed discontinuous tracking controllers, namely, a static tracking controller with fixed coupling gains for the followers and an adaptive tracking controller with time-varying coupling gains. Using tools from nonsmooth analysis and Barbalat's Lemma, we show that the states of the followers under these distributed controllers converge to the state of the leader, if the interaction graph among the followers is undirected, the leader has directed paths to all followers, and the leader's control input is bounded. A sufficient condition for the existence of the distributed controllers is that each agent is stabilizable. Note that the design of the static tracking controller depends on the eigenvalues of the interaction graph and the upper bound of the leader's control input. This limitation is removed by the adaptive tracking controller at the cost of dynamically updating the coupling gains for different followers. It is pointed out that the results on consensus tracking in [12] for first-order and second-order integrators can be regarded as special cases of the results in this technical note.

Notation: Let $\mathbf{R}^{n\times n}$ be the set of $n\times n$ real matrices. I_N represents the identity matrix of dimension N. Denote by $\mathbf{1}$ a column vector with all entries equal to one. The matrix inequality $A>(\geq)B$ means that $A-B$ is positive (semi-)definite. $A\otimes B$ denotes the Kronecker product of matrices A and B. For a vector $x\in\mathbf{R}^n$, let $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$ denote its 1-norm, 2-norm, and ∞-norm, respectively.

II. Preliminaries

A. Graph Theory

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{v_1,\ldots,v_N\}$ is a nonempty finite set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is a set of edges, in which an edge is represented by an ordered pair of distinct nodes. For an edge (v_i,v_j), node v_i is called the parent node, node v_j the child node, and v_i is a neighbor of v_j. A graph with the property that $(v_i,v_j)\in\mathcal{E}$ implies $(v_j,v_i)\in\mathcal{E}$ for any $v_i,v_j\in\mathcal{V}$ is said to be undirected. A path from node v_{i_1} to node v_{i_l} is a sequence of ordered edges of the form $(v_{i_k},v_{i_{k+1}})$, $k=1,\ldots,l-1$. A subgraph $\mathcal{G}_s=(\mathcal{V}_s,\mathcal{E}_s)$ of $\mathcal{G}$ is a graph such that $\mathcal{V}_s\subseteq\mathcal{V}$ and $\mathcal{E}_s\subseteq\mathcal{E}$. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise is disconnected. A directed graph contains a directed spanning tree if there exists a node called the root, which has no parent node, such that the node has directed paths to all other nodes in the graph.

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Z. Li and X. Liu are with the Key Laboratory for Intelligent Control & Decision of Complex Systems, School of Automation, Beijing Institute of Technology, Beijing 100081, China (e-mail: zhongkli@gmail.com; xdliu@bit.edu.cn).

W. Ren is with the Department of Electrical Engineering, University of California, Riverside, CA 92521 USA (e-mail: ren@ee.ucr.edu).

L. Xie is with the School of Electrical and Electronic Engineering, Nanyang Technological University, 639798 Singapore (e-mail: elhxie@ntu.edu.sg).

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The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$ associated with the directed graph $\mathcal{G}$ is defined by $a_{ii} = 0$, $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}] \in \mathbf{R}^{N\times N}$ is defined as $\mathcal{L}_{ii} = \sum_{j\neq i} a_{ij}$ and $\mathcal{L}_{ij} = -a_{ij}$, $i \neq j$. For undirected graphs, both $\mathcal{A}$ and $\mathcal{L}$ are symmetric.

Lemma 1 ([4]): Zero is an eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ as a corresponding right eigenvector and all nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $\mathcal{L}$ if and only if the graph $\mathcal{G}$ has a directed spanning tree.

B. Nonsmooth Analysis

Consider the following differential equation with a discontinuous right-hand side:

$$\dot{z} = f(z,t) \tag{1}$$

where $f : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^m$ is Lebesgue measurable and locally essentially bounded. A vector function $x(\cdot)$ is called a Filippov solution of (1) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$ satisfies the following differential inclusion [20]: $z \in \mathscr{K}[f](z,t)$, where $\mathscr{K}[f] = \bigcap_{\delta>0}\bigcap_{\mu(\bar{N})=0} \overline{co}(f(B(z,\delta)-\bar{N}),t)$, $\bigcap_{\mu(\bar{N})=0}$ denotes the intersection over all sets $\bar{N}$ of Lebesgue measure zero, $\overline{co}(E)$ is the convex closure of set E, and $B(z,\delta)$ denotes the open ball of radius δ centered at z.

Let $V : \mathbf{R}^m \to \mathbf{R}$ be a locally Lipschitz continuous function. The Clarke's generalized gradient of V is given by $\partial V \triangleq co\{\lim \nabla V(z_i) | z_i \to z, z_i \in \Omega_v \cup \bar{N}\}$ [20], where co denotes the convex hull, Ω_v is the set of Lebesgue measure zero where ∇V does not exist, and $\bar{N}$ is an arbitrary set of zero measure. The set-valued Lie derivative of V with respect to (1) is defined as $\dot{\tilde{V}} \triangleq \bigcap_{\xi\in\partial V} \xi^T \mathscr{K}[f](z,t)$.

A Lyapunov stability theorem in terms of the set-valued map $\dot{\tilde{V}}$ is stated as follows.

Lemma 2 ([21]): For (1), let $f(z,t)$ be locally essentially bounded and $0 \in \mathscr{K}[f](0,t)$ in a region $Q \supset \{z \in \mathbf{R}^m | \|z\| < r\} \times \{t | t_0 \leq t < \infty\}$, where $r > 0$. Also, let $V : \mathbf{R}^m \to \mathbf{R}$ be a regular function satisfying $V(0,t) = 0$ and $0 < V_1(\|z\|) \leq V(z,t) \leq V_2(\|z\|)$, for $z \neq 0$, in Q for some V_1 and V_2 belonging to class $\mathcal{K}$. If there exists a class $\mathcal{K}$ functions $\omega(\cdot)$ in Q with the property $\dot{\tilde{V}}(z,t) \leq -\omega(z) < 0$, for $z \neq 0$, then the solution $z(t) \equiv 0$ is asymptotically stable.

III. Tracking With Distributed Static Controller

Consider a group of $N+1$ identical agents with general linear dynamics, consisting of N followers and a leader. The dynamics of the i-th agent are described by

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 0, \ldots, N \tag{2}$$

where $x_i \in \mathbf{R}^n$ is the state, $u_i \in \mathbf{R}^p$ is the control input, and A and B are constant matrices with compatible dimensions.

Without loss of generality, let the agent in (2) indexed by 0 be the leader and the agents indexed by $1, \ldots, N$, be the followers. It is assumed that the leader receives no information from any follower and the state (but not the control input) of the leader is available to only a subset of the followers. The interaction graph among the $N+1$ agents is represented by a directed graph $\mathcal{G}$, which satisfies the following assumption.

Assumption 1: The subgraph $\mathcal{G}_s$ associated with the N followers is undirected and in the graph $\mathcal{G}$ the leader has directed paths to all followers (Equivalently, $\mathcal{G}$ contains a directed spanning tree with the leader as the root).

Denote by $\mathcal{L}$ the Laplacian matrix associated with $\mathcal{G}$. Because the leader has no neighbors, $\mathcal{L}$ can be partitioned as

$$\mathcal{L} = \begin{bmatrix} 0 & 0_{1\times N} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix} \tag{3}$$

where $\mathcal{L}_2 \in \mathbf{R}^{N\times 1}$ and $\mathcal{L}_1 \in \mathbf{R}^{N\times N}$. Since $\mathcal{G}_s$ is undirected, $\mathcal{L}_1$ is symmetric.

The objective of this technical note is to solve the distributed tracking problem for the agents in (2), i.e., to design some distributed controllers under which the states of the N followers converge to the state of the leader in the sense of $\lim_{t\to\infty} \|x_i(t) - x_0(t)\|_2 = 0, \forall\, i = 1, \ldots, N$. Different from the existing literature, e.g., [6], [10], [18], [19], which assumes that the leader's control input u_0 is either equal to zero or available to all the followers, we consider here the general case where u_0 is possibly nonzero and time varying and not accessible to any follower, under the following mild assumption:

Assumption 2: The leader's control input u_0 is continuous and bounded, i.e., $\|u_0\|_\infty \leq \gamma$, where γ is a positive constant.

Based on the relative states of neighboring agents, the following distributed controller is proposed for each follower:

$$\begin{aligned} u_i = {} & c_1 K \sum_{j=0}^{N} a_{ij}(x_i - x_j) \\ & + c_2 \mathrm{sgn}\left(K \sum_{j=0}^{N} a_{ij}(x_i - x_j)\right), \quad i = 1, \ldots, N \end{aligned} \tag{4}$$

where $c_1 > 0$ and $c_2 > 0 \in \mathbf{R}$ are constant coupling gains, $K \in \mathbf{R}^{p\times n}$ is the feedback gain matrix, a_{ij} is the (i,j)-th entry of the adjacency matrix $\mathcal{A}$ associated with $\mathcal{G}$, and $\mathrm{sgn}(\cdot)$ is the signum function defined component-wise.

Let $\xi_i = x_i - x_0$, $i = 1, \ldots, N$. Using (4) for (2), we obtain the closed-loop network dynamics as

$$\begin{aligned} \dot{\xi}_i = {} & A\xi_i + c_1 BK \left[\sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j) + a_{i0}\xi_i\right] \\ & + c_2 B \mathrm{sgn}\left(K \left[\sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j) + a_{i0}\xi_i\right]\right) \\ & - Bu_0, \quad i = 1, \ldots, N. \end{aligned} \tag{5}$$

By letting $\xi = [\xi_1^T, \ldots, \xi_N^T]^T$, (5) can be rewritten in a compact form

$$\begin{aligned} \dot{\xi} = {} & (I_N \otimes A + c_1 \mathcal{L}_1 \otimes BK)\xi \\ & + c_2 (I_N \otimes B)\mathrm{sgn}((\mathcal{L}_1 \otimes K)\xi) - (\mathbf{1} \otimes B)u_0 \end{aligned} \tag{6}$$

where $\mathcal{L}_1$ is defined as in (3). Clearly, the distributed tracking problem is solved by (4) if the closed-loop system (6) is asymptotically stable.

Note that the right-hand side of (6) is discontinuous. Therefore, the stability of (6) will be analyzed by using differential inclusions and nonsmooth analysis [20], [21]. Because the signum function is measurable and locally essentially bounded, the Filippov solution for (6) exists [20]. Equation (6) is written in terms of differential inclusions as

$$\begin{aligned} \dot{\xi} \in^{a.e.} & \mathscr{K}[(I_N \otimes A + c_1 \mathcal{L}_1 \otimes BK)\xi \\ & + c_2 (I_N \otimes B)\mathrm{sgn}((\mathcal{L}_1 \otimes K)\xi) - (\mathbf{1} \otimes B)u_0] \end{aligned} \tag{7}$$

where a.e. stands for "almost everywhere".

Theorem 1: Suppose that Assumptions 1 and 2 hold. The distributed tracking control problem of the agents described by (2) is solved under the controller (4) with $c_1 \geq 1/\lambda_1$, $c_2 \geq \gamma$, and $K = -B^T P^{-1}$, where

$\lambda_1 \leq \cdots \leq \lambda_N$ are the eigenvalues of $\mathcal{L}_1$ and $P > 0$ is a solution to the following linear matrix inequality (LMI):

$$AP + PA^T - 2BB^T < 0. \tag{8}$$

Proof: Consider the following Lyapunov function candidate:

$$V_1 = \xi^T(\mathcal{L}_1 \otimes P^{-1})\xi. \tag{9}$$

For an interaction graph $\mathcal{G}$ satisfying Assumption 1, it follows from Lemma 1 and (3) that $\mathcal{L}_1 > 0$. Clearly, V_1 is positive definite and continuously differentiable.

Using the properties of $\mathscr{K}[\cdot]$ [20], we can obtain the set-valued Lie derivative of V_1 along (7) as follows:

$$\begin{aligned}\dot{\tilde{V}}_1 = \mathscr{K}[&2\xi^T(\mathcal{L}_1 \otimes P^{-1}A + c_1\mathcal{L}_1^2 \otimes P^{-1}BK)\xi \\ &+ 2c_2\xi^T(\mathcal{L}_1 \otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}_1 \otimes K)\xi) \\ &- 2\xi^T(\mathcal{L}_1\mathbf{1} \otimes P^{-1}B)u_0].\end{aligned} \tag{10}$$

Let $\tilde{\xi} = (I_N \otimes P^{-1})\xi$. Substituting $K = -B^TP^{-1}$ into (10) gives

$$\begin{aligned}\dot{\tilde{V}}_1 &= \mathscr{K}[\tilde{\xi}^T(\mathcal{L}_1 \otimes (AP + PA^T) \\ &\qquad - 2c_1\mathcal{L}_1^2 \otimes BB^T)\tilde{\xi} \\ &\qquad - 2c_2\tilde{\xi}^T(\mathcal{L}_1 \otimes B)\mathrm{sgn}((\mathcal{L}_1 \otimes B^T)\tilde{\xi}) \\ &\qquad - 2\tilde{\xi}^T(\mathcal{L}_1\mathbf{1} \otimes B)u_0] \\ &= \tilde{\xi}^T[\mathcal{L}_1 \otimes (AP + PA^T) - 2c_1\mathcal{L}_1^2 \otimes BB^T]\tilde{\xi} \\ &\qquad - 2c_2\|(\mathcal{L}_1 \otimes B^T)\tilde{\xi}\|_1 - 2\tilde{\xi}^T(\mathcal{L}_1\mathbf{1} \otimes B)u_0\end{aligned} \tag{11}$$

where we have used the facts that $x^T\mathrm{sgn}(x) = \|x\|_1$ and $\mathscr{K}[f] = \{f\}$ if f is continuous [20] to obtain the last equality. Clearly, the set-valued Lie derivative $\dot{\tilde{V}}_1$ is a singleton. By using Assumption 2 and the Hölder's inequality, it follows from (11) that

$$\begin{aligned}\dot{\tilde{V}}_1 &\leq \tilde{\xi}^T[\mathcal{L}_1 \otimes (AP + PA^T) - 2c_1\mathcal{L}_1^2 \otimes BB^T]\tilde{\xi} \\ &\qquad - 2c_2\|(\mathcal{L}_1 \otimes B^T)\tilde{\xi}\|_1 + 2\|u_0\|_\infty\|(\mathcal{L}_1 \otimes B^T)\tilde{\xi}\|_1 \\ &\leq \tilde{\xi}^T(\mathcal{L}_1 \otimes (AP + PA^T) - 2c_1\mathcal{L}_1^2 \otimes BB^T)\tilde{\xi} \\ &\qquad - 2(c_2 - \gamma)\|(\mathcal{L}_1 \otimes B^T)\tilde{\xi}\|_1.\end{aligned} \tag{12}$$

Let $U \in \mathbf{R}^{N\times N}$ be such a unitary matrix that $U^T\mathcal{L}_1U = \Lambda \triangleq \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $\bar{\xi} \triangleq [\bar{\xi}_1^T, \ldots, \bar{\xi}_N^T]^T = (U^T \otimes I_n)\tilde{\xi}$. Then

$$\begin{aligned}&\tilde{\xi}^T\left[\mathcal{L}_1 \otimes (AP + PA^T) - 2c_1\mathcal{L}_1^2 \otimes BB^T\right]\tilde{\xi} \\ &\quad = \bar{\xi}^T[\Lambda \otimes (AP + PA^T) - 2c_1\Lambda^2 \otimes BB^T]\bar{\xi} \\ &\quad = \sum_{i=1}^N \lambda_i\bar{\xi}_i^T(AP + PA^T - 2c_1\lambda_iBB^T)\bar{\xi}_i.\end{aligned} \tag{13}$$

By noting that $c_1\lambda_i \geq 1$, $i = 1, \ldots, N$, it follows from (8) that $AP + PA^T - 2c_1\lambda_iBB^T < 0$. Then, we get from (12) and (13) that $\dot{\tilde{V}}_1 < 0$. In light of Lemma 2, it follows that $\xi(t) \equiv 0$ of (7) is asymptotically stable, i.e., the distributed tracking control problem is solved. ∎

Remark 1: As shown in [6], a necessary and sufficient condition for the existence of a $P > 0$ to the LMI (8) is that (A, B) is stabilizable. Therefore, a sufficient condition for the existence of (4) satisfying Theorem 1 is that (A, B) is stabilizable. It is worth noting that a favorable feature in Theorem 1 is that the parameters c_1, c_2, and K of (4) can be independently designed. Theorem 1 is proved by using tools from nonsmooth analysis. The readers can refer to [22], [23] and references therein for further applications of nonsmooth analysis in the area of multiagent systems.

Remark 2: Different from the distributed controllers in [6], [10], [18], [19], the proposed controller (4) in this technical note can solve the distributed tracking problem for the general case where u_0 is bounded and not available to any follower. Due to the nonzero control input u_0, the dynamics of the leader are different from those of the followers. Thus, contrary to the homogeneous multiagent systems in [6], [10], [18], [19], the resulting multiagent system in this technical note is in essence heterogeneous. Furthermore, by letting $c_2 = 0$, (4) is reduced to the distributed controller in [6], [18], [19] for the special case where $u_0 = 0$.

IV. Tracking With Distributed Adaptive Controller

In the last section, the design of the distributed controller (4) depends on the minimal eigenvalue λ_1 of $\mathcal{L}_1$ and the upper bound γ of the leader's control input u_0. However, it is not easy to compute λ_1 especially when the multiagent network is of a large scale. Furthermore, the value of γ might not be explicitly known for each follower in some applications. The objective of this section is to solve the distributed tracking problem without requiring that λ_1 and γ be explicitly known. To this end, we propose the following distributed controller with an adaptive law for updating the coupling gain for each follower:

$$\begin{aligned}u_i &= e_iK\sum_{j=0}^N a_{ij}(x_i - x_j) + e_i\mathrm{sgn}(K\sum_{j=0}^N a_{ij}(x_i - x_j)) \\ \dot{e}_i &= \tau_i\left[\sum_{j=0}^N a_{ij}(x_i - x_j)\right]^T\Gamma\left[\sum_{j=0}^N a_{ij}(x_i - x_j)\right] \\ &\quad + \tau_i\|K\sum_{j=0}^N a_{ij}(x_i - x_j)\|_1, \quad i = 1, \ldots, N\end{aligned} \tag{14}$$

where $K \in \mathbf{R}^{p\times n}$ and a_{ij} are defined in (4), $\Gamma \in \mathbf{R}^{n\times n}$ is a constant gain matrix, τ_i is a positive scalar, and $e_i(t)$ denotes the time-varying coupling gain associated with the i-th follower.

Let $\zeta_i = x_i - x_0$, $\zeta = [\zeta_1^T, \ldots, \zeta_N]^T$, $E = \mathrm{diag}(e_1, \ldots, e_N)$. From (2) and (14), we can get the closed-loop network dynamics as

$$\begin{aligned}\dot{\zeta} &= (I_N \otimes A + E\mathcal{L}_1 \otimes BK)\zeta \\ &\quad + (E \otimes B)\mathrm{sgn}((\mathcal{L}_1 \otimes K)\zeta) \\ &\quad - (\mathbf{1} \otimes B)u_0 \\ \dot{e}_i &= \tau_i\left[\sum_{j=1}^N a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i\right]^T \\ &\quad \times \Gamma\left[\sum_{j=1}^N a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i\right] \\ &\quad + \tau_i\|K(\sum_{j=1}^N a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i)\|_1 \\ i &= 1, \ldots, N\end{aligned} \tag{15}$$

where $\mathcal{L}_1$ is defined as in (3). Obviously, the distributed tracking problem is solved by the adaptive controller (14) if the state ζ of (15) converges to zero.

Theorem 2: Suppose that Assumptions 1 and 2 hold. Then, the distributed tracking control problem of the agents described by (2) is solved under the adaptive controller (14) with $K = -B^TP^{-1}$ and $\Gamma = P^{-1}BB^TP^{-1}$, where $P > 0$ is a solution to the LMI (8). Moreover, each coupling gain e_i converges to some finite steady-state value.

Proof: Consider the following Lyapunov function candidate

$$V_2 = \zeta^T(\mathcal{L}_1 \otimes P^{-1})\zeta + \sum_{i=1}^{N} \frac{1}{\tau_i}(e_i - \alpha)^2 \tag{16}$$

where α is a positive constant. As shown in the proof of Theorem 1, $V_2 \geq 0$. The time derivative of V_2 along (15) can be obtained as

$$\begin{aligned} \dot{V}_2 =& 2\zeta^T(\mathcal{L}_1 \otimes P^{-1})\dot{\zeta} + \sum_{i=1}^{N} \frac{2}{\tau_i}(e_i - \alpha)\dot{e}_i \\ =& 2\zeta^T(\mathcal{L}_1 \otimes P^{-1}A + \mathcal{L}_1 E \mathcal{L}_1 \otimes P^{-1}BK)\zeta \\ &+ 2\zeta^T(\mathcal{L}_1 E \otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}_1 \otimes K)\zeta) \\ &- 2\zeta^T(\mathcal{L}_1 \mathbf{1} \otimes P^{-1}B)u_0 + \sum_{i=1}^{N} \frac{2}{\tau_i}(e_i - \alpha)\dot{e}_i. \end{aligned} \tag{17}$$

By noting $K = -B^T P^{-1}$, it is not difficult to get that

$$\begin{aligned} &\zeta^T(\mathcal{L}_1 E \mathcal{L}_1 \otimes P^{-1}BK)\zeta \\ &= -\sum_{i=1}^{N} e_i \left[\sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i\right]^T \\ &\quad \times P^{-1}BB^T P^{-1}\left[\sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i\right] \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\zeta^T(\mathcal{L}_1 E \otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}_1 \otimes K)\zeta) \\ &= -\sum_{i=1}^{N} e_i \left[\sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i\right]^T P^{-1}B \\ &\quad \times \mathrm{sgn}\left(B^T P^{-1}\left[\sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i\right]\right) \\ &= -\sum_{i=1}^{N} e_i(t)\left\| B^T P^{-1}\left[\sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j) + a_{i0}\zeta_i\right]\right\|_1. \end{aligned} \tag{19}$$

Then, by substituting $\Gamma = P^{-1}BB^T P^{-1}$, (18), (19), the second equation in (15) into (17), and letting $\tilde{\zeta} = (I_N \otimes P^{-1})\zeta$, we can obtain that

$$\begin{aligned} \dot{V}_2 =& \tilde{\zeta}^T[\mathcal{L}_1 \otimes (AP + PA^T) - 2\alpha\mathcal{L}_1^2 \otimes BB^T]\tilde{\zeta} \\ &- 2\alpha\|(\mathcal{L}_1 \otimes B^T)\tilde{\zeta}\|_1 - 2\tilde{\zeta}^T(\mathcal{L}_1\mathbf{1} \otimes B)u_0 \\ \leq& \tilde{\zeta}^T[\mathcal{L}_1 \otimes (AP + PA^T) - 2\alpha\mathcal{L}_1^2 \otimes BB^T]\tilde{\zeta} \\ &- 2(\alpha - \gamma)\|(\mathcal{L}_1 \otimes B^T)\tilde{\zeta}\|_1. \end{aligned} \tag{20}$$

As shown in the proof of Theorem 1, by selecting α sufficiently large such that $\alpha \geq \gamma$ and $\alpha\lambda_i \geq 1, i = 1, \ldots, N$, we get from (20) that

$$\begin{aligned} \dot{V}_2 \leq& 2\tilde{\zeta}^T[\mathcal{L}_1 \otimes (AP + PA^T) - 2\alpha\mathcal{L}_1^2 \otimes BB^T]\tilde{\zeta} \\ \triangleq& -W(\tilde{\zeta}) \end{aligned} \tag{21}$$

and from (13) that $W(\tilde{\zeta})$ is positive definite. Then, it follows that $\dot{V}_2 \leq 0$, implying that $V_2(t)$ is nonincreasing. Therefore, in view of (16), we know that e_i, ζ, and $\tilde{\zeta}$ are bounded. Since by Assumption 2, u_0 is bounded, this implies from the first equation in (15) that $\dot{\zeta}$ and $\dot{\tilde{\zeta}}$ are bounded. As $V_2(t)$ is nonincreasing and bounded from below by zero, it has a finite limit V_2^∞ as $t \to \infty$. Integrating (21), we have $\int_0^\infty W(\tilde{\zeta}(\tau))d\tau \leq V_2(\zeta(t_0)) - V_2^\infty$. Thus, $\int_0^\infty W(\tilde{\zeta}(\tau))d\tau$ exists and is finite. Because $\tilde{\zeta}$ and $\dot{\tilde{\zeta}}$ are bounded, it is easy to see from (21) that $\dot{W}(\tilde{\zeta})$ is also bounded, which in turn guarantees the uniform continuity of $W(\tilde{\zeta})$. Therefore, by Barbalat's Lemma [24], we get that $W(\tilde{\zeta}) \to 0$ as $t \to \infty$, i.e., $\zeta(t) \to 0$ as $t \to \infty$. By noting that $\Gamma \geq 0$ and $\tau_i > 0$, it follows from (15) that e_i is monotonically increasing. Thus, the boundedness of e_i implies that each e_i converges to some finite value. ■

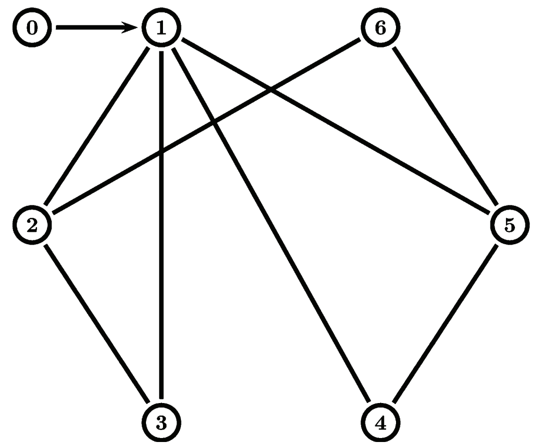

Fig. 1. Interaction graph.

Remark 3: It is worth mentioning that the adaptive scheme in (14) for updating the coupling gains is partly inspired by the adaptive strategies in [25]–[27], which however are applicable only to the case without a leader or the case of a leader with zero control input. Compared to the static controller (4), the adaptive controller (14) requires neither the minimal eigenvalue λ_1 of $\mathcal{L}_1$ nor the upper bound γ of u_0, as long as u_0 is bounded. On the other hand, the coupling gains need to be dynamically updated in (14), implying that the adaptive controller (14) is more complex than the static controller (4).

Remark 4: In the related work [12], distributed controllers are designed to ensure that a group of first-order (or second-order) integrators track a leader with a bounded first-order (or second-order) derivative. Compared to [12], the contribution of this technical note is at least twofold. First, we have presented a systematic framework in terms of LMI to design a distributed static tracking controller (4) for the agents with general linear dynamics. The results on consensus tracking in [12] can be regarded as special cases of Theorem 1. Second, it is required in [12] that each follower knows the bound of the leader's control input or the eigenvalues of the communication topology, which however might not be practical in many applications. In Theorem 2, we have removed this limitation by proposing a distributed adaptive controller (14).

Remark 5: Although it is theoretically proved in Theorem 2 that the states of the followers under the adaptive controller (14) converge to the state of the leader, a practical issue in implementing (14) is that the coupling gains $e_i(t)$ in (14) might always slowly increase in the presence of possible measurement noises, external disturbances or the chattering phenomenon. In the following, we propose one practical way to tackle this problem. For agent i, assume that at time t its tracking error $\upsilon_i \triangleq \sum_{j=0}^{N} a_{ij}(x_i - x_j)$ gets into a desirable bound $\kappa_i > 0$, namely, $\|\upsilon_i(t)\|_2 \leq \kappa_i$. Agent i will uses its coupling gain $e_i(t)$ at time t for the next iteration $t+1$, instead of updating it as in (14). If $\|\upsilon_i(t+1)\|_2 \leq \kappa_i$, agent i still use $e_i(t)$ for the iteration $t+2$; otherwise, it will update its coupling strength using the second equation in (14) which implies that $e_i(t+1) > e_i(t)$. After a number of iterations, say, after time t^*, the coupling strength of each agent will be larger than $\varrho \triangleq \max\{1/\lambda_1, \gamma\}$. By the proposition stated as below, even subject to external disturbances, the coupling gains $e_i(t)$ in (14) will

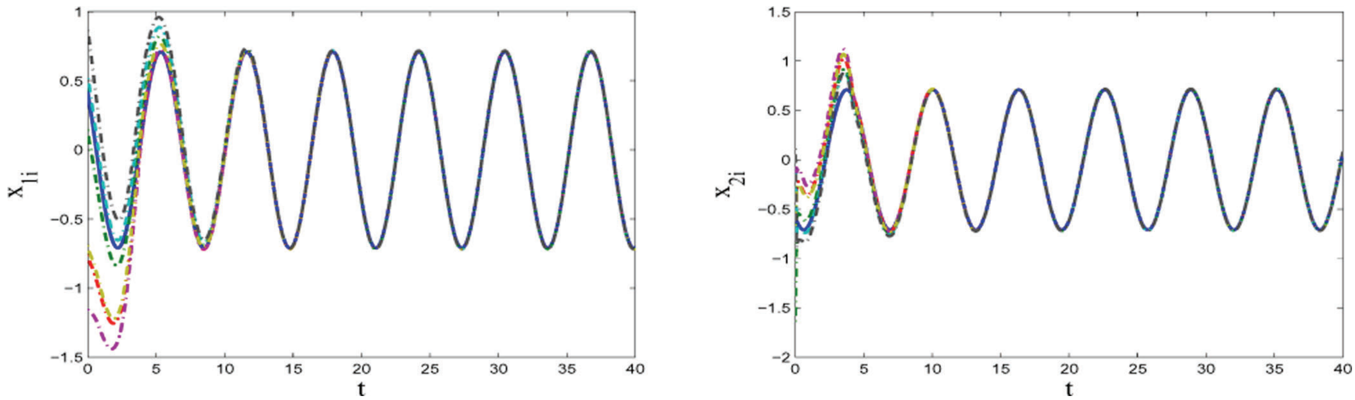

Fig. 2. State trajectories of the agents. The solid and dash-dotted lines denote, respectively, the trajectories of the leader and the followers.

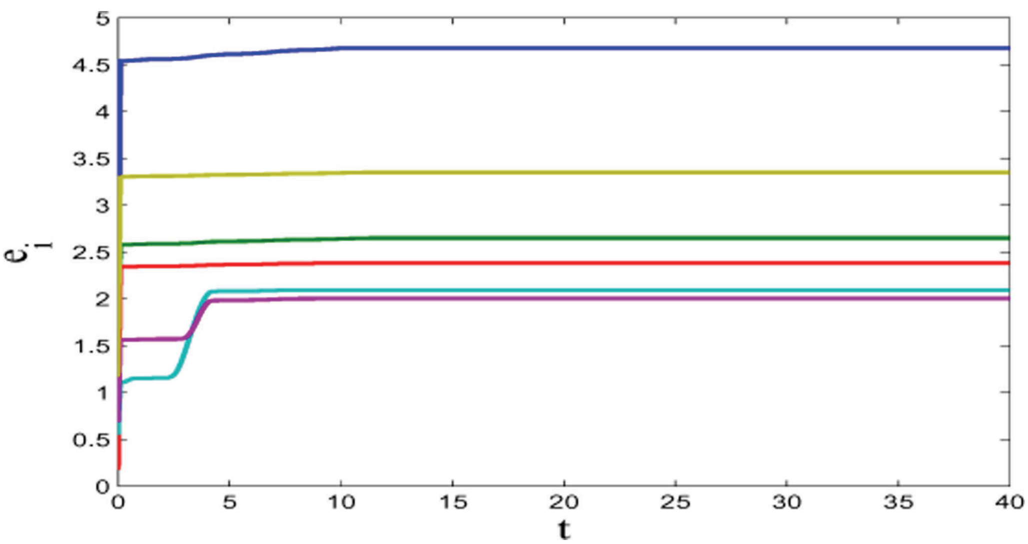

Fig. 3. Coupling gains e_i in (14).

be fixed to be $e_i(t^*)$ after time t^* and the tracking error of each agent will converge into a desirable bound.

Proposition 1: Suppose that Assumptions 1 and 2 hold. The distributed tracking control problem of the agents in (2) is solved by the following controller:

$$u_i = d_i K \sum_{j=0}^{N} a_{ij}(x_i - x_j) + d_i \mathrm{sgn}\left(K \sum_{j=0}^{N} a_{ij}(x_i - x_j)\right)$$

where $d_i \geq \varrho$, $i = 1, \ldots, N$, and $K = -B^T P^{-1}$.

Proposition 1 can be shown by using similar steps in the proofs of Theorems 1 and 2, which is omitted here for brevity.

V. Simulation

In this section, a simulation example is provided to validate the effectiveness of the theoretical results.

The dynamics of the agents are given by (2), with

$$x_i = \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Clearly, all the agents are unstable without control. Design the control input $u_0 = K_2 x_0$ for the leader with $K_2 = [\,0 \quad -2\,]$. Thus, the closed-loop dynamics of the leader is $\dot{x}_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_0$, whose state trajectory is an oscillator. In this case, u_0 is bounded. However, the bound γ, for which $\|u_0\|_\infty \leq \gamma$, depends on the initial state $x_0(0)$, which thereby might not be known to the followers. Here we use the adaptive control (14) to ensure that the followers track the leader.

Solving the LMI (8) by using the LMI toolbox of Matlab gives the gain matrices K and Γ in (14) as

$$K = -[\,1.2983 \quad 3.3878\,], \quad \Gamma = \begin{bmatrix} 1.6857 & 4.3984 \\ 4.3984 & 11.4769 \end{bmatrix}.$$

To illustrate Theorem 2, let the interaction graph be given as in Fig. 1, where the node indexed by 0 is the leader and the rest are followers. Let $\tau_i = 1, i = 1, \ldots, 6$, in (14) and $e_i(0)$ be randomly chosen. When implementing (14), we adopt the scheme in Remark 5 and choose the tracking error bound for each agent to be 0.05. The state trajectories of the agents under (14) with K and Γ as above are depicted in Fig. 2, implying that the followers indeed track the leader. The coupling gains e_i associated with the followers are drawn in Fig. 3, from which we can see that e_i remain unchanged after about $t = 10$.

VI. Conclusion

In this technical note, we have considered the distributed tracking control problem of multiagent systems with general linear dynamics and a leader whose control input is nonzero and not available to any follower. Based on the relative states of neighboring agents, two distributed controllers with static and adaptive coupling gains have been designed, under which the states of the followers approach the state of the leader, if the interaction graph among the followers is undirected, the leader has directed paths to all followers, and the leader's control input is bounded. A sufficient condition for the existence of the distributed controllers is that each agent is stabilizable. An interesting future topic is to consider the distributed tracking problem for the case with only relative output information of neighboring agents.

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Output Feedback Stabilization Using Small-Gain Method and Reduced-Order Observer for Stochastic Nonlinear Systems

Cong-Ran Zhao and Xue-Jun Xie

***Abstract*—This technical note further investigates more general stochastic nonlinear systems with stochastic integral input-to-state stability (SiISS) inverse dynamics and focuses on solving the following problems: 1) a class of counterexamples which are different from that in [14] are reconstructed to discuss the relationship between two small-gain type conditions on SiISS and 2) under the weaker conditions on nonlinearities, based on a reduced-order observer, small-gain type condition on SiISS and stochastic LaSalle theorem, an output feedback controller is constructed to guarantee the global asymptotical stability in probability of the closed-loop stochastic system.**

***Index Terms*—Output feedback stabilization, reduced-order observer, small-gain method, stochastic LaSalle theorem, stochastic nonlinear systems.**

I. INTRODUCTION

It is well known that the concepts of input-to-state stability (ISS) in [1] and integral input-to-state stability (iISS) in [2] and [3] play very important roles in feedback design and stability analysis of deterministic nonlinear systems. In view of the importance of these two concepts, it is natural that one tries to generalize them to the stochastic setting.

[4]and [5] proposed two kinds of stochastic input-to-state stability (SISS). Input-to-state practical stability (ISpS) in probability was introduced by [6]. Recently, [7]–[9] gave a sufficient condition using Lyapunov function on SISS and discussed different control problems for

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The authors are with the Institute of Automation, Qufu Normal University, Shandong Province, Qufu 273165, China (e-mail: congran369@126.com; xuejunxie@126.com, xxj@mail.qfnu.edu.cn).

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