[3] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Trans. Autom. Control*, vol. 36, no. 11, pp. 1241–1253, 1991.

[4] J. Hauser, S. Sastry, and P. Kokotovic, "Nonlinear control via approximate input-output linearization: The ball and beam example," *IEEE Trans. Autom. Control*, vol. 37, no. 3, pp. 392–398, 1992.

[5] W. Kang, "Approximate linearization of nonlinear control systems," *Syst. Control Lett.*, vol. 23, no. 1, pp. 43–52, 1994.

[6] J. Ghosh and B. Paden, "Nonlinear repetitive control," *IEEE Trans. Autom. Control*, vol. 45, no. 5, pp. 949–954, 2000.

[7] G. O. Guardabassi and S. E. Savaresi, "Approximate linearization via feedback—An overview," *Automatica*, vol. 37, no. 1, pp. 1–15, 2001.

[8] D. Cheng, X. Hu, and Y. Wang, "Non-regular feedback linearization of nonlinear systems via a normal form algorithm," *Automatica*, vol. 40, no. 3, pp. 439–447, 2004.

[9] K. Nam, A. Araposthathis, and K. K. Lee, "Some numerical aspects of approximate lineariztion of single input nonlinear systems," *Int. J. Control*, vol. 57, no. 2, pp. 463–472, 1993.

[10] S. A. Bortoff, "Approximate state-feedback linearization using spline functions," *Automatica*, vol. 33, no. 8, pp. 1449–1458, 1997.

[11] T. A. Johansen and K. J. Hunt, "A computational approach to approximate input/state feedback linearization," in *Proc. 39th CDC*, Sydney, Australia, Dec. 2000, pp. 4467–4472.

[12] K. Nam, "Stabilization of feedback linearizable systems using a radial basis function network," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 1026–1031, 1999.

[13] H.-L. Choi and J.-T. Lim, "Stabilizing a class of nonlinear system based on approximate feedback linearization," *IEICE Trans. Fund.*, vol. E88-A, no. 6, pp. 1626–1630, 2005.

[14] J. Deutscher, "Input-output linearization of nonlinear systems using multivariable Legendre polynomials," *Automatica*, vol. 41, no. 2, pp. 299–304, 2005.

[15] B. Muller and J. Deutscher, " Approximate input-output linearization using $\boldsymbol{L}_2$-optimal bilinearization," in *Proc. 44th CDC-ECC*, Seville, Spain, Dec. 2005, pp. 4903–4908.

[16] K. D. Young, P. V. Kokotovic, and V. I. Utkin, "A singular perturbation analysis of high-gain feedback systems," *IEEE Trans. Autom. Control*, vol. 22, no. 6, pp. 931–938, 1977.

[17] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control Analysis and Design*. Philadelphia, PA: SIAM, 1999.

[18] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: Wiley, 1989.

[19] F. L. Lewis, C. T. Abdallah, and D. M. Dawson, *Control of Robot Manipulators*. New York: Macmillan, 1993.

[20] Z. H. Shao, "Robust stability of two-time-scale systems with nonlinear uncertainties," *IEEE Trans. Autom. Control*, vol. 49, no. 2, pp. 258–261, 2004.

[21] J.-W. Son and J.-T. Lim, "Robust stability of nonlinear singularly perturbed system with uncertainties," *Inst. Elect. Eng. Proc.-Control Theory Appl.*, vol. 153, no. 1, pp. 104–110, 2006.

[22] H. J. Sussmann and P. V. Kokotovic, "The peaking phenomena and the global stabilization of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 36, no. 4, pp. 424–440, 1991.

[23] A. N. Atassi and H. K. Khalil, "A separation principle for the stabilization of a class of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 44, no. 9, pp. 1672–1687, 1999.

On Consensus Algorithms for Double-Integrator Dynamics

Wei Ren, *Member, IEEE*

***Abstract*—This note considers consensus algorithms for double-integrator dynamics. We propose and analyze consensus algorithms for double-integrator dynamics in four cases: 1) with a bounded control input, 2) without relative velocity measurements, 3) with a group reference velocity available to each team member, and 4) with a bounded control input when a group reference state is available to only a subset of the team. We show that consensus is reached asymptotically for the first two cases if the undirected interaction graph is connected. We further show that consensus is reached asymptotically for the third case if the directed interaction graph has a directed spanning tree and the gain for velocity matching with the group reference velocity is above a certain bound. We also show that consensus is reached asymptotically for the fourth case if and only if the group reference state flows directly or indirectly to all of the vehicles in the team.**

***Index Terms*—Consensus, cooperative control, coordination, graph theory, multivehicle systems.**

I. Introduction

Consensus means that a team of agents reaches an agreement on a common value by negotiating with their neighbors. Consensus algorithms have a historical perspective represented in [2]–[4], to name a few, and have recently been studied extensively in the context of cooperative control of multivehicle systems (see [5] and references therein). Some results in consensus algorithms can be understood in the context of connective stability [6].

Consensus algorithms are primarily studied for single-integrator kinematics (see [5] and references therein). Recent works also deal with nonholonomic unicycles [7] and rigid body attitude dynamics [8]. This note focuses on consensus algorithms for double-integrator dynamics, which are more challenging than those for single-integrator kinematics. Consensus algorithms for double-integrator dynamics are studied in [9]–[11]. Variants of the algorithms are applied to formation control [12]–[17] and flocking [18], [19]. Related to this note are [9], [10], [12], [17]. In particular, [9] proposes and analyzes consensus algorithms for double-integrator dynamics and shows that unlike the single-integrator case, both the interaction graph and the coupling strength of relative velocities between neighboring vehicles affect the convergence result in the general case of directed interaction. However, the issues of actuator saturation and lack of relative velocity measurements are not addressed. In [12], formation keeping strategies accounting for actuator saturation and lack of relative velocity measurements are addressed for multirobot formation maneuvers. However, the results are restricted to a bidirectional ring interaction graph and rely on the assumption that every robot has the knowledge of its desired location. A consensus algorithm for double-integrator dynamics is also considered in [10], where a damping term for velocity

Manuscript received March 14, 2007; revised August 31, 2007. Published August 29, 2008 (projected). Recommended by Associate Editor F. Bullo. This work was supported in part by a National Science Foundation CAREER Award (ECCS-0748287) and the Utah Water Research Laboratory.

The author is with the Electrical and Computer Engineering Department, Utah State University, Logan, UT 84332 USA (e-mail: wren@engineering.usu.edu).

Digital Object Identifier 10.1109/TAC.2008.924961

is introduced. However, the analysis is based on undirected interaction. Reference [17] extends [9] to incorporate a group reference velocity. However, the algorithm does not explicitly take into account actuator saturation.

The main purpose of the current note is to extend some existing results in consensus algorithms for double-integrator dynamics in four aspects. First, we propose and analyze a consensus algorithm for double-integrator dynamics with a bounded control input under an undirected interaction graph. Second, we propose and analyze a consensus algorithm for double-integrator dynamics without relative velocity measurements under an undirected interaction graph. The first two aspects extend [9] to account for, respectively, actuator saturation and lack of relative velocity measurements while extending [12] to any undirected connected interaction graph. Third, we analyze a consensus algorithm for double-integrator dynamics with a group reference velocity available to each team member under a directed interaction graph. This aspect extends [10] to the case of directed interaction. Finally, we propose and analyze a consensus algorithm for double-integrator dynamics with a bounded control input that allows a group reference state to be available to only a subset of the team under a directed interaction graph. This aspect extends [17] to account for actuator saturation. A preliminary version of the work has appeared in [1].

II. Background and Preliminaries

A. Graph Theory Notions

A weighted graph consists of a node set $\mathcal{V} = \{1, \ldots, p\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A_p = [a_{ij}] \in \mathbb{R}^{p \times p}$. An edge (i, j) in a weighted directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. In contrast, the pairs of nodes in a weighted undirected graph are unordered, where an edge (i, j) denotes that vehicles i and j can obtain information from one another. The weighted adjacency matrix A_p of a weighted directed graph is defined such that a_{ij} is a positive weight if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The weighted adjacency matrix A_p of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}, \forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$.

A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. An undirected path in an undirected graph is defined analogously. A directed graph has a directed spanning tree if there exists at least one node having a directed path to all of the other nodes. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Let the (nonsymmetric) Laplacian matrix $L_p = [\ell_{ij}] \in \mathbb{R}^{p \times p}$ associated with A_p be defined as

$$\ell_{ii} = \sum_{j=1, j\neq i}^{p} a_{ij}, \quad \ell_{ij} = -a_{ij}, \quad i \neq j. \tag{1}$$

For an undirected graph, L_p is symmetric positive semidefinite. However, L_p for a directed graph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of L_p with the associated eigenvector $\mathbf{1}_p$, where $\mathbf{1}_p$ is a $p \times 1$ column vector of all ones. In the case of undirected graphs, 0 is a simple eigenvalue of L_p and all of the other eigenvalues are positive if and only if the undirected graph is connected [20]. In the case of directed graphs, 0 is a simple eigenvalue of L_p and all of the other eigenvalues have positive real parts if and only if the directed graph has a directed spanning tree [21].

B. Existing Consensus Algorithms for Double-Integrator Dynamics

Consider vehicles with double-integrator dynamics given by

$$\dot{r}_i = v_i, \quad \dot{v}_i = u_i, \quad i \in \mathcal{I}_n \tag{2}$$

where $r_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ are, respectively, the position and velocity of the ith vehicle, $u_i \in \mathbb{R}^m$ is the control input, and $\mathcal{I}_n \triangleq \{1, \ldots, n\}$. A consensus algorithm for (2) is proposed in [9] as

$$u_i = -\sum_{j=1}^{n} a_{ij}[(r_i - r_j) + \gamma(v_i - v_j)], \quad i \in \mathcal{I}_n \tag{3}$$

where a_{ij} is the (i, j)th entry of the weighted adjacency matrix $A_n \in \mathbb{R}^{n \times n}$ characterizing the interaction graph for r_i and v_i, and γ is a positive gain. In the presence of a group reference velocity $v^d \in \mathbb{R}^m$, a consensus algorithm for (2) is proposed in [11] as

$$u_i = \dot{v}^d - \alpha(v_i - v^d) - \sum_{j=1}^{n} a_{ij}[(r_i - r_j) + \gamma(v_i - v_j)], \quad i \in \mathcal{I}_n \tag{4}$$

where a_{ij} is defined as in (3), and α and γ are positive gains.

Consensus is reached for (3) if for all $r_i(0)$ and $v_i(0), r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t)$ asymptotically as $t \to \infty$. Consensus is reached for (4) if for all $r_i(0)$ and $v_i(0), r_i(t) \to r_j(t)$ and $v_i(t) \to v^d(t)$ asymptotically as $t \to \infty$.

III. Consensus With a Bounded Control Input

Note that (3) does not explicitly take into account actuator saturation. We propose a consensus algorithm for (2) with a bounded control input as

$$u_i = -\sum_{j=1}^{n} \{a_{ij} \tanh[K_r(r_i - r_j)] + b_{ij} \tanh[K_v(v_i - v_j)]\}, \quad i \in \mathcal{I}_n \tag{5}$$

where $K_r \in \mathbb{R}^{m \times m}$ and $K_v \in \mathbb{R}^{m \times m}$ are positive-definite diagonal matrices, a_{ij} and b_{ij} are, respectively, the (i, j)th entry of the weighted adjacency matrices $A_n \in \mathbb{R}^{n \times n}$ and $B_n \in \mathbb{R}^{n \times n}$ characterizing, respectively, the undirected interaction graphs for r_i and v_i, and $\tanh(\cdot)$ is defined component-wise. That is, $\tanh([x_1, \ldots, x_m]^T) = [\tanh(x_1), \ldots, \tanh(x_m)]^T$, where $x_i \in \mathbb{R}$. Note that A_n and B_n are allowed to be chosen differently. Also note that with (5) $\|u_i\|_\infty \leq \sum_{j=1}^{n}(a_{ij} + b_{ij})$, which is independent of the initial positions and velocities of the vehicles.

Before moving on, we need the following lemma.

Lemma 3.1: Suppose $\varsigma \in \mathbb{R}^m, \varphi \in \mathbb{R}^m, K \in \mathbb{R}^{m \times m}$, and $C = [c_{ij}] \in \mathbb{R}^{n \times n}$. If C is symmetric, then

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(\varsigma_i - \varsigma_j)^T \tanh[K(\varphi_i - \varphi_j)] = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\varsigma_i^T \tanh[K(\varphi_i - \varphi_j)].$$

Proof:

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(\varsigma_i - \varsigma_j)^T \tanh[K(\varphi_i - \varphi_j)] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\varsigma_i^T \tanh[K(\varphi_i - \varphi_j)] \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\varsigma_j^T \tanh[K(\varphi_i - \varphi_j)] \qquad (6) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\varsigma_i^T \tanh[K(\varphi_i - \varphi_j)] \\
&\quad + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ji}\varsigma_j^T \tanh[K(\varphi_j - \varphi_i)] \qquad (7) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\varsigma_i^T \tanh[K(\varphi_i - \varphi_j)]
\end{aligned}$$

where we have used the fact that $c_{ij} = c_{ji}$ and $\tanh[K(\varphi_j - \varphi_i)] = -\tanh[K(\varphi_i - \varphi_j)]$ and have switched the order of the summation signs in the second term in (6) to obtain (7), and have switched the dummy variables i and j in the second term in (7) to obtain the last equality. ■

Theorem 3.1: With (5), $r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t)$ asymptotically as $t \to \infty$ if the undirected graphs of both A_n and B_n are connected.

Proof: Note that with (5), (2) can be written as

$$\begin{aligned} \dot{r}_{ij} &= v_i - v_j \\ \dot{v}_i &= -\sum_{j=1}^{n}\{a_{ij}\tanh(K_r r_{ij}) + b_{ij}\tanh[K_v(v_i - v_j)]\} \end{aligned} \tag{8}$$

where $r_{ij} \overset{\triangle}{=} r_i - r_j$. Consider a Lyapunov function candidate for (8) as

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\mathbf{1}_m^T K_r^{-1}\log(\cosh(K_r r_{ij})) + \frac{1}{2}\sum_{i=1}^{n} v_i^T v_i, \tag{9}$$

where $\cosh(\cdot)$ and $\log(\cdot)$ are defined component-wise. Note that V is positive definite and radially unbounded with respect to $r_{ij}, \forall i \neq j$, and v_i if the undirected graph associated with A_n is connected. Differentiating V, gives

$$\begin{aligned} \dot{V} &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(v_i - v_j)^T\tanh(K_r r_{ij}) \\ &\quad - \sum_{i=1}^{n} v_i^T\left(\sum_{j=1}^{n}\{a_{ij}\tanh(K_r r_{ij}) + b_{ij}\tanh[K_v(v_i - v_j)]\}\right) \\ &= \quad -\sum_{i=1}^{n} v_i^T\left\{\sum_{j=1}^{n} b_{ij}\tanh[K_v(v_i - v_j)]\right\} \\ &= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(v_i - v_j)^T\tanh[K_v(v_i - v_j)] \le 0 \end{aligned}$$

where we have used the fact that $(d\log(\cosh(x)))/(dt) = \dot{x}\tanh(x)$ with $x \in \mathbb{R}$ and have used (8) to obtain the first equality, have used Lemma 3.1 to obtain the second equality by noting that $r_{ij} = r_i - r_j$, have used Lemma 3.1 again to obtain the third equality, and have used the fact that x and $\tanh(Kx)$ have the same sign component-wise when x is a vector and K is a positive-definite diagonal matrix to obtain the last inequality.

Let $S = \{(r_{ij}, v_i) \mid \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $v_i \equiv v_j, \forall i \neq j$, when the undirected graph associated with B_n is connected, which, in turn, implies that $\dot{v}_i \equiv \dot{v}_j, \forall i \neq j$. Therefore, it follows that $\dot{v} \in \operatorname{span}(\mathbf{1}_n \otimes \eta)$, where $\dot{v} = [\dot{v}_1^T, \ldots, \dot{v}_n^T]^T$ and η is some $m \times 1$ real vector, when the undirected graph associated with B_n is connected. Because $v_i \equiv v_j$, it follows from (8) that

$$\dot{v}_i \equiv -\sum_{j=1}^{n} a_{ij}\tanh(K_r r_{ij}), \quad i \in \mathcal{I}_n. \tag{10}$$

Note from (10) that $(\mathbf{1}_n \otimes \eta)^T\dot{v} \equiv \sum_{i=1}^{n}\eta^T[-\sum_{j=1}^{n} a_{ij}\tanh(K_r r_{ij})] \equiv -\eta^T\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\tanh(K_r r_{ij})$. Noting that $a_{ij} = a_{ji}$ and $\tanh(K_r r_{ij}) = -\tanh(K_r r_{ji})$, it follows that $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\tanh(K_r r_{ij}) \equiv 0$, which implies that $(\mathbf{1}_n \otimes \eta)^T\dot{v} \equiv 0$. Thus it follows that $\dot{v}$ is orthogonal to $\mathbf{1}_n \otimes \eta$. Therefore, we conclude that $\dot{v} \equiv 0$, which in turn implies that $-\sum_{j=1}^{n} a_{ij}\tanh(K_r r_{ij}) \equiv 0$ from (10). As a result, it follows that $-\sum_{i=1}^{n} r_i^T[\sum_{j=1}^{n} a_{ij}\tanh(K_r r_{ij})] \equiv 0$, which in turn implies that $-(1/2)\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} r_{ij}^T\tanh(K_r r_{ij}) \equiv 0$ from Lemma 3.1 by noting that $r_{ij} = r_i - r_j$. Since $\tanh(K_r r_{ij})$ is an odd function, it follows that $a_{ij} r_{ij}^T\tanh(K_r r_{ij}) \ge 0$. Combing the above arguments, gives $a_{ij} r_{ij}^T\tanh(K_r r_{ij}) \equiv 0, \forall i, j$. When the undirected graph associated with A_n is connected, it follows that $r_{ij} \equiv 0$, i.e., $r_i \equiv r_j, \forall i \neq j$. By LaSalle's Invariance principle, it follows that $r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t), \forall i \neq j$, asymptotically as $t \to \infty$. ■

Note that (5) guarantees that $r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t)$ asymptotically as $t \to \infty$. When it is desirable that $r_i(t) \to r_j(t)$ and $v_i(t) \to 0$ asymptotically as $t \to \infty$, we propose a consensus algorithm for (2) with a bounded control input as

$$u_i = -\sum_{j=1}^{n} a_{ij}\tanh[K_r(r_i - r_j)] - \tanh(K_{vi}v_i), \quad i \in \mathcal{I}_n \tag{11}$$

where $K_r \in \mathbb{R}^{m \times m}$ and $K_{vi} \in \mathbb{R}^{m \times m}$, $i \in \mathcal{I}_n$, are positive-definite diagonal matrices.

Corollary 3.2: With (11), $r_i(t) \to r_j(t)$ and $v_i(t) \to 0$ asymptotically as $t \to \infty$ if the undirected graph associated with A_n is connected.

Proof: The proof is similar to that of Theorem 3.1 and is omitted here. ■

Note that the result for saturated control in [12] requires each robot to know its desired position and is restricted to a bidirectional ring graph for convergence analysis. The algorithms (5) and (11) guarantee consensus convergence under any undirected connected interaction graph.

IV. Consensus Without Relative Velocity Measurement

Note that (3) requires measurements of relative velocities between neighboring vehicles. Motivated by [12] and [22], we propose a consensus algorithm without relative velocity measurements based on a passivity approach as

$$\begin{aligned} \dot{\hat{x}}_i &= \Gamma\hat{x}_i + \sum_{j=1}^{n} a_{ij}(r_i - r_j) \quad y_i = P\Gamma\hat{x}_i + P\sum_{j=1}^{n} a_{ij}(r_i - r_j) \\ u_i &= -\sum_{j=1}^{n} a_{ij}(r_i - r_j) - y_i, \quad i \in \mathcal{I}_n \end{aligned} \tag{12}$$

where $\Gamma \in \mathbb{R}^{m \times m}$ is Hurwitz, a_{ij} is defined as in (5), $P \in \mathbb{R}^{m \times m}$ is a symmetric positive-definite matrix and is the solution to the Lyapunov equation $\Gamma^T P + P\Gamma = -Q$ with $Q \in \mathbb{R}^{m \times m}$ being a symmetric positive-definite matrix. Note that $\hat{x}_i(0) \in \mathbb{R}^m$ can be chosen arbitrarily.

Theorem 4.1: With (12), $r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t)$ asymptotically as $t \to \infty$ if the undirected graph associated with A_n is connected.

Proof: Let $r = [r_1^T, \ldots, r_n^T]^T$, $v = [v_1^T, \ldots, v_n^T]^T$, $y = [y_1^T, \ldots, y_n^T]^T$, $\hat{x} = [\hat{x}_1^T, \ldots, \hat{x}_n^T]^T$, and $u = [u_1^T, \ldots, u_n^T]^T$. The control law (12) can be written as

$$\dot{\hat{x}} = (I_n \otimes \Gamma)\hat{x} + (L_n \otimes I_m)r \tag{13a}$$

$$y = (I_n \otimes P)\dot{\hat{x}} \tag{13b}$$

$$u = -(L_n \otimes I_m)r - y, \tag{13c}$$

where $\otimes$ denotes the Kronecker product, I_n is the $n \times n$ identity matrix, and L_n is the Laplacian matrix defined in (1) with $p = n$. Note that L_n is symmetric positive semidefinite since the graph associated with A_n is undirected.

Note that with (12), (2) can be written as

$$\begin{aligned}\dot{r}_{ij} &= v_{ij}, \quad \dot{v}_{ij} = -\sum_{j=1}^{n} a_{ij} r_{ij} \\ &\quad - P\dot{\hat{x}}_i + \sum_{k=1}^{n} a_{jk} r_{jk} + P\dot{\hat{x}}_j \\ \ddot{\hat{x}}_i &= \Gamma \dot{\hat{x}}_i + \sum_{j=1}^{n} a_{ij} v_{ij} \end{aligned} \tag{14}$$

where $r_{ij} \triangleq r_i - r_j$ and $v_{ij} \triangleq v_i - v_j$. Consider a Lyapunov function candidate for (14) as

$$\begin{aligned} V &= \frac{1}{2} r^T (L_n \otimes I_m)^2 r \\ &\quad + \frac{1}{2} v^T (L_n \otimes I_m) v \\ &\quad + \frac{1}{2} \dot{\hat{x}}^T (I_n \otimes P) \dot{\hat{x}}. \end{aligned}$$

Note that from the property of L_n, V is positive definite and radially unbounded with respect to $r_{ij}, v_{ij}, \forall i \neq j$, and $\dot{\hat{x}}_i$ when the undirected graph associated with A_n is connected. Differentiating V, gives

$$\begin{aligned} \dot{V} &= v^T (L_n \otimes I_m)^2 r + v^T (L_n \otimes I_m) u \\ &\quad + \frac{1}{2} \ddot{\hat{x}}^T (I_n \otimes P) \dot{\hat{x}} + \frac{1}{2} \dot{\hat{x}}^T (I_n \otimes P) \ddot{\hat{x}} \\ &= v^T [(L_n \otimes I_m)^2 r + (L_n \otimes I_m) u] + \frac{1}{2} \dot{\hat{x}}^T (I_n \otimes \Gamma^T)(I_n \otimes P) \dot{\hat{x}} \\ &\quad + \frac{1}{2} v^T (L_n^T \otimes I_m)(I_n \otimes P) \dot{\hat{x}} \\ &\quad + \frac{1}{2} \dot{\hat{x}}^T (I_n \otimes P)(I_n \otimes \Gamma) \dot{\hat{x}} + \frac{1}{2} \dot{\hat{x}}^T (I_n \otimes P)(L_n \otimes I_m) v \\ &= v^T [(L_n \otimes I_m)^2 r + (L_n \otimes I_m) u] - \frac{1}{2} \dot{\hat{x}}^T (I_n \otimes Q) \dot{\hat{x}} \\ &\quad + v^T (L_n \otimes I_m)(I_n \otimes P) \dot{\hat{x}} \\ &= -\frac{1}{2} \dot{\hat{x}}^T (I_n \otimes Q) \dot{\hat{x}} \leq 0 \end{aligned}$$

where we have used (13a)–(13b), $L_n = L_n^T$, and properties of the Kronecker product. In particular, given real matrices E, F, G, and H, $(E \otimes F)^T = E^T \otimes F^T$ and $(E \otimes F)(G \otimes H) = EF \otimes GH$.

Let $S = \{(r_{ij}, v_{ij}, \dot{\hat{x}}_i) \mid \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\dot{\hat{x}} \equiv 0$, which in turn implies that $\ddot{\hat{x}} \equiv 0$, $(L_n \otimes I_m) v \equiv 0$ by differentiating (13a), and $y \equiv 0$ from (13b). Because $(L_n \otimes I_m) v \equiv 0$, it follows that $v_i \equiv v_j$, i.e., $v_{ij} \equiv 0, \forall i \neq j$, when the undirected graph associated with A_n is connected. It also follows that $(L_n \otimes I_m) \dot{v} \equiv 0$, which implies that $\dot{v} \in \mathrm{span}(\mathbf{1}_n \otimes \eta)$, where η is some $m \times 1$ real vector, when the undirected graph associated with A_n is connected. Because $y \equiv 0$, from (2) and (13c), it follows that

$$\dot{v} \equiv -(L_n \otimes I_m) r. \tag{15}$$

Note that $(\mathbf{1}_n \otimes \eta)^T \dot{v} \equiv -(\mathbf{1}_n \otimes \eta)^T (L_n \otimes I_m) r \equiv -(\mathbf{1}_n^T L_n \otimes \eta^T I_m) \equiv 0$ because $\mathbf{1}_n^T L_n = 0$ when the graph associated with A_n is undirected. Thus $\dot{v}$ is orthogonal to $\mathbf{1}_n \otimes \eta$. We then conclude that $\dot{v} \equiv 0$, which in turn implies that $(L_n \otimes I_m) r \equiv 0$ from (15). If the undirected graph associated with A_n is connected, $(L_n \otimes I_m) r \equiv 0$ implies that $r_i \equiv r_j$, i.e., $r_{ij} \equiv 0, \forall i \neq j$. By LaSalle's Invariance principle, it follows that $r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t), \forall i \neq j$, as $t \to \infty$. ■

When it is desirable that $r_i(t) \to r_j(t)$ and $v_i(t) \to 0$ asymptotically as $t \to \infty$, we propose an algorithm as

$$\begin{aligned} \dot{\hat{x}}_i &= \Gamma \hat{x}_i + r_i, \quad y_i = P\Gamma \hat{x}_i + P r_i \\ u_i &= -\sum_{j=1}^{n} a_{ij}(r_i - r_j) - y_i, \quad i \in \mathcal{I}_n \end{aligned} \tag{16}$$

where Γ, P, and a_{ij} are defined as in (12).

Corollary 4.2: With (16), $r_i(t) \to r_j(t)$ and $v_i(t) \to 0$ asymptotically as $t \to \infty$ if the undirected graph associated with A_n is connected.

Proof: Consider a Lyapunov function candidate as $V = (1/2) r^T (L_n \otimes I_m) r + (1/2) v^T v + (1/2) \dot{\hat{x}}^T (I_n \otimes P) \dot{\hat{x}}$, which is positive definite and radially unbounded with respect to $r_{ij}, \forall i \neq j, v_i$, and $\dot{\hat{x}}_i$. Following the proof of Theorem 4.1, the derivative of V is given as

$$\begin{aligned} \dot{V} &= v^T [(L_n \otimes I_m) r + u] - \frac{1}{2} \dot{\hat{x}}^T (I_n \times Q) \dot{\hat{x}} + v^T (I_n \otimes P) \dot{\hat{x}} \\ &= -\frac{1}{2} \dot{\hat{x}}^T (I_n \times Q) \dot{\hat{x}} \leq 0. \end{aligned}$$

A similar proof to that of Theorem 4.1 shows that $r_i(t) \to r_j(t)$ and $v_i(t) \to 0, \forall i \neq j$, as $t \to \infty$. ■

Again, the algorithms (12) and (16) extend the result for passivity-based interrobot damping in [12] to consensus convergence under any undirected connected interaction graph.

V. Consensus With a Group Reference Velocity

In this section, we consider a consensus algorithm with a group reference velocity as

$$u_i = \dot{v}^d - \alpha(v_i - v^d) - \sum_{j=1}^{n} a_{ij}(r_i - r_j), \quad i \in \mathcal{I}_n \tag{17}$$

where α is a positive gain, a_{ij} is (i,j)th entry of the weighted adjacency matrix $A_n \in \mathbb{R}^{n \times n}$ characterizing the possibly directed interaction graph for r_i, and $v^d \in \mathbb{R}^m$ denotes the possibly time-varying group reference velocity. In contrast to (4), (17) removes the coupling between relative velocities. Reference [10] studies a special case of (17) where $v^d \equiv 0$. However, the analysis is restricted to undirected graphs. The following theorem considers the general case of directed interaction among vehicles, which generalizes [10].

Theorem 5.1: Let μ_i denotes the ith eigenvalue of $-L_n$ with L_n given by (1), where $p = n$, and $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent, respectively, the real and imaginary parts of a number. With (17), $r_i(t) \to r_j(t)$ and $v_i(t) \to v^d(t)$ asymptotically as $t \to \infty$ if the directed graph associated with A_n has a directed spanning tree and

$$\alpha > \bar{\alpha} \tag{18}$$

where $\bar{\alpha} \triangleq 0$ if all of the $n-1$ nonzero eigenvalues of $-L_n$ are negative and

$$\bar{\alpha} \triangleq \max_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)>0} |\mu_i| \sqrt{\frac{2}{-\mathrm{Re}(\mu_i)}}$$

otherwise.

Proof: Let $r = [r_1^T, \ldots, r_n^T]^T$, $\tilde{r} = r - \mathbf{1}_n \otimes \int_0^t v^d(\tau) d\tau$, $v = [v_1^T, \ldots, v_n^T]^T$, and $\tilde{v} = v - \mathbf{1}_n \otimes v^d$. With (17), (2) can be written in matrix form as $\begin{bmatrix} \dot{\tilde{r}} \\ \dot{\tilde{v}} \end{bmatrix} = (\Gamma \otimes I_m) \begin{bmatrix} \tilde{r} \\ \tilde{v} \end{bmatrix}$, where $\Gamma \triangleq \begin{bmatrix} 0_{n \times n} & I_n \\ -L_n & -\alpha I_n \end{bmatrix}$ with L_n given by (1), where $p = n$.

Noting that $L_n \mathbf{1}_n = 0$, it follows that $[\mathbf{1}_n^T, \mathbf{0}_n^T]^T$, where $\mathbf{0}_n$ denotes the $n \times 1$ column vector of all zeros, is an eigenvector for Γ associated with an eigenvalue 0, which implies that $\mathrm{span}(\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix})$ is contained in

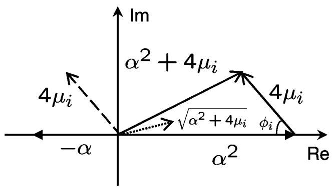

Fig. 1. Graphical view of notations used in the proof.

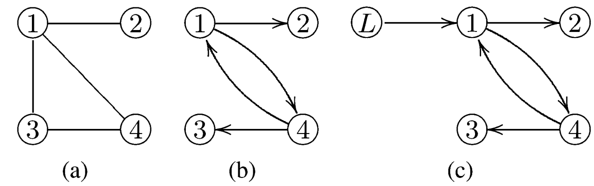

Fig. 2. Interaction graphs for Cases I–IV.

the kernel of Γ. If Γ has a simple zero eigenvalue and all of the other eigenvalues have negative real parts, then $\begin{bmatrix} \tilde{r}(t) \\ \tilde{v}(t) \end{bmatrix} \to \operatorname{span}(\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} \otimes \eta)$ asymptotically as $t \to \infty$, where η is an $m \times 1$ vector, which is equivalent to $\tilde{r}_i(t) \to \tilde{r}_j(t)$ and $\tilde{v}_i \to 0$ asymptotically as $t \to \infty$.

Next, we show that if the directed graph associated with A_n has a directed spanning tree and the inequality (18) is satisfied, then Γ has a simple zero eigenvalue and all of the other eigenvalues have negative real parts. Let λ be an eigenvalue of Γ and $s = [p^T, q^T]^T$ be its associated eigenvector, where p and q are $n \times 1$ column vectors. Note that

$$\begin{aligned} \Gamma s = \lambda s &\Longleftrightarrow \begin{bmatrix} 0_{n\times n} & I_n \\ -L_n & -\alpha I_n \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \\ &= \lambda \begin{bmatrix} p \\ q \end{bmatrix} \Longleftrightarrow q \\ &= \lambda p \text{ and } -L_n p - \alpha q \\ &= \lambda q \Longleftrightarrow -L_n p = (\lambda^2 + \alpha\lambda) p \end{aligned}$$

which implies that $\lambda^2 + \alpha\lambda$ is an eigenvalue of $-L_n$ with an associated eigenvector p. Letting $\mu \triangleq \lambda^2 + \alpha\lambda$, gives $\lambda^2 + \alpha\lambda - \mu = 0$, which implies that given each μ, there are two roots for λ, denoted by $\lambda_\pm = ((-\alpha \pm \sqrt{\alpha^2 + 4\mu})/2)$. As a result, each eigenvalue of $-L_n$, denoted by $\mu_i, i = 1, \ldots, n$, corresponds to two eigenvalues of Γ, denoted by λ_{2i-1} and λ_{2i}.

If the directed graph associated with A_n has a directed spanning tree, then L_n has a simple zero eigenvalue and all of the other eigenvalues have positive real parts, which implies that $-L_n$ has a simple zero eigenvalue and all of the other eigenvalues have negative real parts. Without loss of generality, let $\mu_1 = 0$ and $\operatorname{Re}(\mu_i) < 0, i = 2, \ldots, n$. Then it follows that $\lambda_1 = 0$ and $\lambda_2 = -\alpha$. Note that if $\mu_i < 0$, then $\operatorname{Re}((-\alpha \pm \sqrt{\alpha^2 + 4\mu_i})/(2)) < 0$ for any $\alpha > 0$. It is left to show that the inequality (18) guarantees that all of the eigenvalues of Γ corresponding to μ_i that satisfies $\operatorname{Re}(\mu_i) < 0$ and $\operatorname{Im}(\mu_i) \neq 0$ have negative real parts. Motivated by [9], [15], we use Fig. 1 to show the notations used in the proof. We only need to consider μ_i that satisfies $\operatorname{Re}(\mu_i) < 0$ and $\operatorname{Im}(\mu_i) > 0$ since any μ_i that satisfies $\operatorname{Re}(\mu_i) < 0$ and $\operatorname{Im}(\mu_i) < 0$ is a complex conjugate of some μ_i that satisfies $\operatorname{Re}(\mu_i) < 0$ and $\operatorname{Im}(\mu_i) > 0$. Consider the triangle formed by vectors $\alpha^2, 4\mu_i$, and $\alpha^2 + 4\mu_i$. According to the law of cosines, $|\alpha^2 + 4\mu_i|^2 = (\alpha^2)^2 + (4|\mu_i|)^2 - 8\alpha^2|\mu_i|\cos(\phi_i)$, where $\cos(\phi_i) = (-\operatorname{Re}(\mu_i))/(|\mu_i|)$. Note that if $\alpha > |\mu_i|\sqrt{(2/(-\operatorname{Re}\mu_i)}$, then $|\alpha^2 + 4\mu_i|^2 < \alpha^4$, which implies that $|\sqrt{\alpha^2 + 4\mu_i}| < \alpha$. Therefore, it follows that $|\operatorname{Re}(\sqrt{\alpha^2 + 4\mu_i})| < \alpha$, which in turn implies that $\operatorname{Re}(\lambda_{2i-1,2i}) = \operatorname{Re}((-\alpha \pm \sqrt{\alpha^2 + 4\mu_i})/(2)) < 0$.

Combing the above arguments, it follows that if the directed graph associated with A_n has a directed spanning tree and the inequality (18) is valid, then $\tilde{r}_i(t) \to \tilde{r}_j(t)$ and $\tilde{v}_i(t) \to 0$ asymptotically as $t \to \infty$, which in turn implies that $r_i(t) \to r_j(t)$ and $v_i(t) \to v^d(t)$ asymptotically as $t \to \infty$. ■

VI. Consensus With a Bounded Control Input and With Partial Access to a Group Reference State

Note that (4) and (17) require that the group reference velocity be available to each vehicle in the team. Next, we propose a consensus algorithm with a bounded control input that allows the group reference position r^d, velocity v^d, and acceleration $\dot{v}^d$ to be available to only a subset of the team as

$$\begin{aligned} u_i = {} & \frac{1}{\kappa_i}\left(\sum_{j=1}^n a_{ij}\dot{v}_j + a_{i(n+1)}\dot{v}^d\right) \\ & - \frac{1}{\kappa_i}K_{ri}\tanh\left[\sum_{j=1}^n a_{ij}(r_i - r_j) + a_{i(n+1)}(r_i - r^d)\right] \\ & - \frac{1}{\kappa_i}K_{vi}\tanh\left[\sum_{j=1}^n a_{ij}(v_i - v_j) + a_{i(n+1)}(v_i - v^d)\right], \\ & \qquad i \in \mathcal{I}_n \qquad (19) \end{aligned}$$

where $a_{ij}, i, j \in \mathcal{I}_n$, is the (i,j)th entry of the weighted adjacency matrix $A_n \in \mathbb{R}^{n\times n}$ defined as in (17), $a_{i(n+1)} > 0, i \in \mathcal{I}_n$, if vehicle i has access to r^d, v^d, and $\dot{v}^d$, $\kappa_i \triangleq \sum_{j=1}^{n+1} a_{ij}$, $\dot{v}^d$ is bounded, and K_{ri} and K_{vi} are $m \times m$ positive-definite diagonal matrices. Note that each control input depends on not only its local neighbors' positions and velocities but also their accelerations. In practical implementation, the accelerations can be calculated by numerical differentiation of the velocities. The algorithm (19) extends the result in [17] to explicitly account for actuator saturation.

Theorem 6.1: Let $A_{n+1} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be the adjacency matrix, where $a_{ij}, i \in \mathcal{I}_n, j \in \mathcal{I}_{n+1}$, is defined in (19) and $a_{(n+1)j} = 0, j \in \mathcal{I}_{n+1}$. With (19), there exists a unique bounded solution for u_i and $r_i(t) \to r^d(t)$ and $v_i(t) \to v^d(t)$ asymptotically as $t \to \infty$ if and only if the directed graph associated with A_{n+1} has a directed spanning tree.

Proof: We first show that (19) has a unique solution for u_i if and only if the directed graph associated with A_{n+1} has a directed spanning tree and the solution is bounded. Noting that all entries of the last row of A_{n+1} are zero and the directed graph associated with A_{n+1} has a directed spanning tree, it follows that no other row of A_{n+1} can have all zero entries. It thus follows that $\kappa_i = \sum_{j=1}^{n+1} a_{ij} \neq 0, i = 1, \ldots, n$. Define $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ as $w_{ij} = -a_{ij}, i \neq j$, and $w_{ii} = \sum_{j=1, j\neq i}^{n+1} a_{ij}$. Also define $b = [b_1, \ldots, b_n]^T \in \mathbb{R}^{n\times 1}$ with $b_i = -a_{i(n+1)}$, and $d = [d_1^T, \ldots, d_n^T]^T \in \mathbb{R}^{mn\times 1}$ with $d_i = -K_{ri}\tanh[\sum_{j=1}^n a_{ij}(r_i - r_j) + a_{i(n+1)}(r_i - r^d)] - K_{vi}\tanh[\sum_{j=1}^n a_{ij}(v_i - v_j) + a_{i(n+1)}(v_i - v^d)]$. With (19), (2) can be written as $(W \otimes I_m)u = (-b \otimes I_m)\dot{v}^d + d$, where $u = [u_1^T, \ldots, u_n^T]^T$, by noting that $\dot{v}_j = u_j$. Note that $b, \dot{v}^d$, and d are all bounded. If W has full rank, then it is straightforward to show that there is a unique solution for u and the solution is bounded. Let $L_{n+1} = \begin{bmatrix} W \mid b \\ 0_{1\times n} \mid 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$, which is the nonsymmetric Laplacian matrix associated with A_{n+1}. Note that $\operatorname{Rank}(L_{n+1}) = \operatorname{Rank}(W|b)$ and $W\mathbf{1}_n + b = \mathbf{0}_n$ (i.e., b is a linear combination of the n columns of W). It follows that $\operatorname{Rank}(W) = \operatorname{Rank}(W \mid b) = \operatorname{Rank}(L_{n+1})$. Also note that $\operatorname{Rank}(L_{n+1}) = n$ if and only if the directed graph associated with A_{n+1} has a directed spanning tree ([23], Lemma 2.10). Therefore, $\operatorname{Rank}(W) = n$ (i.e., full rank) if and only if the directed graph

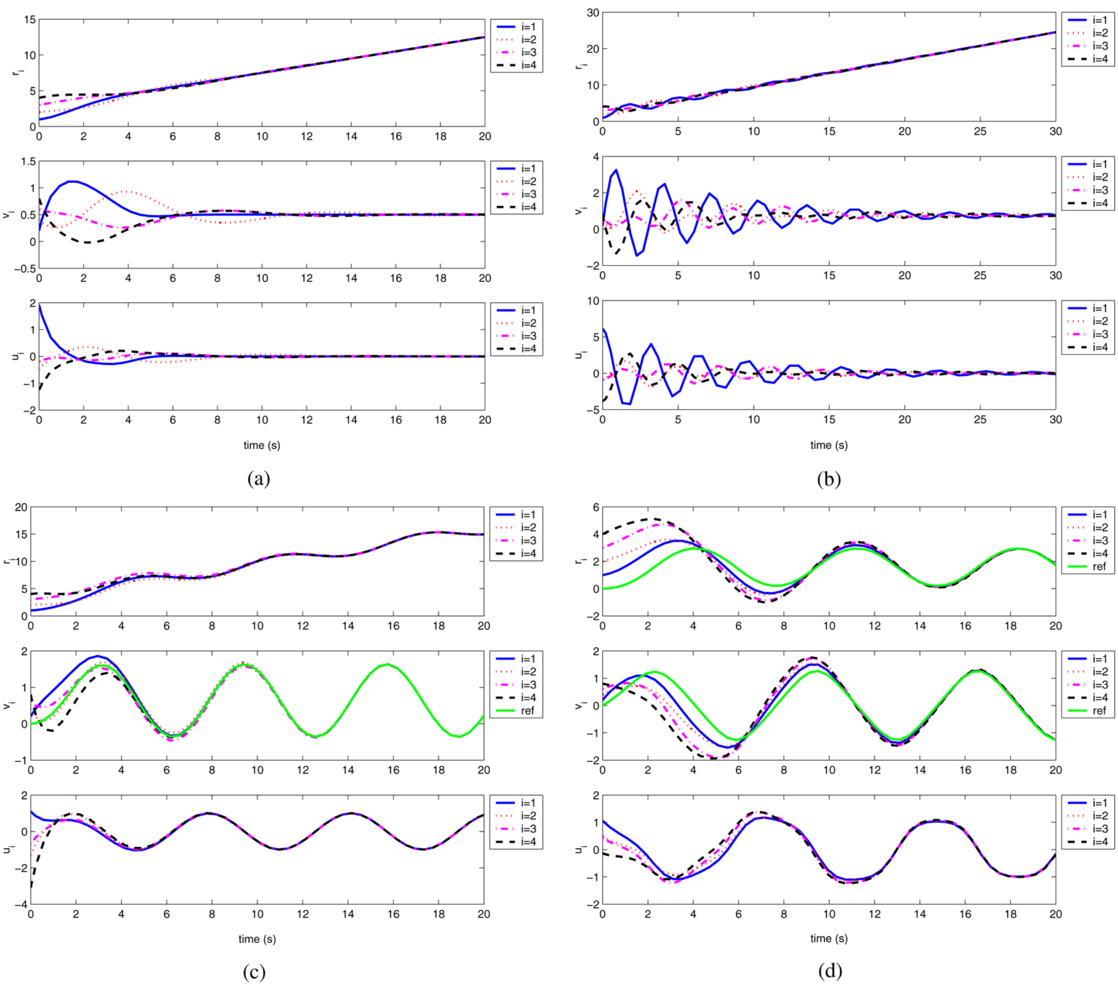

Fig. 3. Simulation results of Cases I–IV: (a) Case I, (b) Case II, (c) Case III, and (d) Case IV.

associated with A_{n+1} has a directed spanning tree. This proves the first argument of the theorem.

Note that with (19), (2) can be written as

$$\ddot{e}_i = -K_{ri}\tanh(e_i) - K_{vi}\tanh(\dot{e}_i) \tag{20}$$

where $e_i = \sum_{j=1}^n a_{ij}(r_i - r_j) + a_{i(n+1)}(r_i - r^d)$. Consider a Lyapunov function candidate $V = \sum_{i=1}^n \{\mathbf{1}_m^T K_{ri}\log[\cosh(e_i)] + (1/2)\dot{e}_i^T\dot{e}_i\}$, which is positive definite and radially unbounded with respect to e_i and $\dot{e}_i$. Differentiating V, gives

$$\begin{aligned}\dot{V} &= \sum_{i=1}^n \Big(\dot{e}_i^T K_{ri}\tanh(e_i) \\ &\quad + \dot{e}_i^T[-K_{ri}\tanh(e_i) - K_{vi}\tanh(\dot{e}_i)]\Big) \\ &= -\sum_{i=1}^n \dot{e}_i^T K_{vi}\tanh(\dot{e}_i) \le 0.\end{aligned}$$

Let $S = \{(e_i, \dot{e}_i) \mid \dot{V} = 0\}$. Note that $\dot{V} \equiv 0$ implies that $\dot{e}_i \equiv 0$, which in turn implies that $\ddot{e}_i \equiv 0$. Because $\dot{e}_i \equiv 0$ and $\ddot{e}_i \equiv 0$, it follows that $e_i \equiv 0$ from (20). By LaSalle's Invariance principle, it follows that $e_i(t) \to 0$ and $\dot{e}_i(t) \to 0$ asymptotically as $t \to \infty$. Note that $e = (W \otimes I_m)r + (b \otimes I_m)r^d$, where $e = [e_1^T, \ldots, e_n^T]^T$ and $r = [r_1^T, \ldots, r_n^T]^T$. Because $W\mathbf{1} + b = \mathbf{0}_n$ and $\mathrm{Rank}(W) = n$ (i.e., $W^{-1}b = -\mathbf{1}_n$) if and only if the directed graph associated with A_{n+1} has a directed spanning tree, it follows that $e(t) \to 0$ asymptotically as $t \to \infty$ is equivalent to $r_i(t) \to r^d(t)$ asymptotically as $t \to \infty$ under the same assumption. Similarly, it follows that $\dot{e}(t) \to 0$ asymptotically as $t \to \infty$ is equivalent to $v_i(t) \to v^d(t)$ asymptotically as $t \to \infty$ under the same assumption. ■

VII. Simulation

In this section, we demonstrate simulation results for Cases I–IV using (5), (12), (17), and (19), respectively. The undirected graph associated with A_n in (5) and (12) is shown by Fig. 2(a). For simplicity, we assume that $B_n = A_n$ in (5). The directed graph associated with A_n in (17) is shown in Fig. 2(b) while the directed graph associated with A_{n+1} in (19) is shown in Fig. 2(c), where an arrow from node L to node i denotes that vehicle i has access to r^d, v^d, and $\dot{v}^d$. In all cases, we let $m = 1$ and choose the nonzero entries of the weighted adjacency matrices to be 0.5 for simplicity. In Case I, we let $K_r = K_v = 1$. In Case II, we let $\Gamma = -0.5$ and $P = 1$. In Case III, we let $\dot{v}^d = (\sin(t))/(1+e^{-t})$. In Case IV, we let $K_{ri} = K_{vi} = 1$ and have r^d and v^d satisfy $\dot{r}^d = v^d$ and $\dot{v}^d = (\cos(r^d))/(1+e^{-t})$. Fig. 3 shows the simulation results for Cases I–IV. Note that consensus is reached in each case.

VIII. Conclusion and Future Work

We have extended some existing results in consensus algorithms for double-integrator dynamics to account for actuator saturation, remove the requirement for relative velocity measurements, introduce a group reference velocity to each vehicle, and incorporate a group reference state to a subset of the team and account for actuator saturation. We have shown convergence conditions for consensus in each case. Future work will consider the effects of time delay and switching interaction graphs in those algorithms.

REFERENCES

[1] W. Ren, "On consensus algorithms for double-integrator dynamics," in *Proc. IEEE Conf. Decision Control*, New Orleans, LA, Dec. 2007.

[2] M. H. DeGroot, "Reaching a consensus," *J. Amer. Stat. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974.

[3] J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Trans. Autom. Control*, vol. 31, no. 9, pp. 803–812, 1986.

[4] N. A. Lynch, *Distributed Algorithms*. San Francisco, CA: Morgan Kaufmann, 1996.

[5] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control: Collective group behavior through local interaction," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, Apr. 2007.

[6] D. D. Siljak, *Decentralized Control of Complex Systems*. New York: Academic, 1991.

[7] D. V. Dimarogonas and K. J. Kyriakopoulos, "On the rendezvous problem for multiple nonholonomic agents," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 916–922, 2007.

[8] W. Ren, "Distributed attitude alignment in spacecraft formation flying," *Int. J. Adapt. Contr. Signal Process.*, vol. 21, no. 2–3, pp. 95–113, Mar.–Apr. 2007.

[9] W. Ren and E. M. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *Int. J. Robust Nonlinear Contr.*, vol. 17, no. 10–11, pp. 1002–1033, Jul. 2007.

[10] G. Xie and L. Wang, "Consensus control for a class of networks of dynamic agents," *Int. J. Robust Nonlinear Contr.*, vol. 17, no. 10–11, pp. 941–959, Jul. 2007.

[11] W. Ren, "Consensus strategies for cooperative control of vehicle formations," *IET Control Theory Appl.*, vol. 1, no. 2, pp. 505–512, Mar. 2007.

[12] J. R. Lawton, R. W. Beard, and B. Young, "A decentralized approach to formation maneuvers," *IEEE Trans. Robot. Autom.*, vol. 19, no. 6, pp. 933–941, Dec. 2003.

[13] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.

[14] W. Ren and R. W. Beard, "Decentralized scheme for spacecraft formation flying via the virtual structure approach," *J. Guidance, Contr., Dynam.*, vol. 27, no. 1, pp. 73–82, Jan.–Feb. 2004.

[15] S. Roy, A. Saberi, and K. Herlugson, , S. Phoha, T. LaPorta, and C. Griffin, Eds., "Formation and alignment of distributed sensing agents with double-integrator dynamics and actuator saturation," in *Sensor Network Applications*. Piscataway, NJ: IEEE Press, 2007.

[16] G. Lafferriere, A. Williams, J. Caughman, and J. J. P. Veerman, "Decentralized control of vehicle formations," *Syst. Contr. Lett.*, vol. 54, no. 9, pp. 899–910, 2005.

[17] W. Ren, "Formation keeping and attitude alignment for multiple spacecraft through local interactions," *J. Guidance, Contr., Dynam.*, vol. 30, no. 2, pp. 633–638, Mar.–Apr. 2007.

[18] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 401–420, Mar. 2006.

[19] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 863–868, May 2007.

[20] R. Merris, "Laplacian matrices of graphs: A survey," *Linear Algebra Appl.*, vol. 197–198, pp. 143–176, 1994.

[21] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[22] F. Lizarralde and J. Wen, "Attitude control without angular velocity measurement: A passivity approach," *IEEE Trans. Autom. Control*, vol. 41, no. 3, pp. 468–472, Mar. 1996.

[23] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*, ser. Communications and Control Engineering. London, U.K.: Springer-Verlag, 2008.

Perfect Elimination of Regulation Transients in Discrete-Time LPV Systems via Internally Stabilizable Robust Controlled Invariant Subspaces

Elena Zattoni, *Senior Member, IEEE*

***Abstract*—This note introduces a geometric solution to the problem of perfect elimination of regulation transients in discrete-time, linear systems subject to swift and wide, *a priori*-known, parameter variations. The constructive proof of the conditions for problem solvability requires a preliminary, strictly geometric interpretation of the multivariable autonomous regulator problem, specifically aimed at discrete-time, linear systems. The novel concept of internal stabilizability of a robust controlled invariant subspace plays a key role in the formulation of those conditions as well as in the synthesis of the control scheme.**

***Index Terms*—Geometric approach, linear parameter varying systems, robust controlled invariant subspaces.**

I. INTRODUCTION

Robust asymptotic regulation, achieved through the internal model principle, as was first established in [1] and [2], is very effective in those situations where the systems involved are subject to sufficiently small parameter variations. Conversely, the problem of handling sudden, relevant changes occurring in the regulated system dynamics has been the object of a fair number of contributions in the more recent literature: linear parameter varying (or LPV) systems, jump linear systems, switching systems are definitions extensively used to denote specific classes of systems somehow affected by significant modifications in their parameters and/or structure. A variety of techniques has been proposed to cope with those kinds of systems. However, as to LPV systems, the geometric approach has proved to be a particularly congenial methodology (see, e.g., [3]–[8]). Indeed, several aspects of control and observation in LPV systems have been deeply analyzed, from a geometric perspective, in the abovementioned articles. Nonetheless, the investigation of the problem of perfect elimination of regulation transients in discrete-time linear systems with *a priori*-known switching laws, which is the scope of this work, is still lacking.

Linear parameter varying systems are adopted in many areas of control systems technology to model regulated systems susceptible to important variations in their dynamics. In fact, LPV systems are widely used in flight control (see, e.g., [9]–[11]), road vehicle control (see, e.g., [12] and [13]), process control (see, e.g., [14]), power plant control (see, e.g., [15]), machine tool control (see, e.g., [16]), etc. In most of the cases considered in the literature mentioned above, the parameter variations are measurable in real time. Nevertheless, in some circumstances, the variations in the system dynamics and the time of their occurrence are known in advance. For instance, in aircraft flight control, some manoeuvres are predetermined, which implies that switching between the linear time-invariant systems modeling the aircraft dynamics at the different points of interest throughout the operational envelope

Manuscript received February 21, 2007; revised November 20, 2007. Published August 29, 2008 (projected). Recommended by Associate Editor F. Wu.

The author is with the Department of Electronics, Computer Science, and Systems, University of Bologna, 40136 Bologna, Italy (e-mail: elena.zattoni@unibo.it).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2008.928334