International Journal of Control
Vol. 82, No. 11, November 2009, 2137–2149

Distributed leaderless consensus algorithms for networked Euler–Lagrange systems

Wei Ren*

Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322, USA

(*Received 22 August 2008; final version received 4 April 2009*)

This article proposes and analyses distributed, leaderless, model-independent consensus algorithms for networked Euler–Lagrange systems. We propose a fundamental consensus algorithm, a consensus algorithm accounting for actuator saturation, and a consensus algorithm accounting for unavailability of measurements of generalised coordinate derivatives, for systems modelled by Euler–Lagrange equations. Due to the fact that the closed-loop interconnected Euler–Lagrange equations using these algorithms are non-autonomous, Matrosov's theorem is used for convergence analysis. It is shown that consensus is reached on the generalised coordinates and their derivatives of the networked Euler–Lagrange systems as long as the undirected communication topology is connected. Simulation results show the effectiveness of the proposed algorithms.

Keywords: consensus; synchronisation; leaderless coordination; Euler–Lagrange system

1. Introduction

Consensus problems study how a team of agents can reach an agreement on a common value by negotiating with their neighbours and have received significant attention in the area of cooperative control recently. Current consensus algorithms primarily focus on vehicles with single-integrator kinematics (see Ren, Beard, and Atkins (2007), Olfati-Saber, Fax, and Murray (2007) and references therein), double-integrator dynamics (e.g. Lafferriere, Williams, Caughman, and Veerman 2005; Veerman, Lafferriere, Caughman, and Williams 2005; Olfati-Saber 2006; Lee and Spong 2007; Tanner, Jadbabaie, and Pappas 2007; Ren and Atkins 2007; Xie and Wang 2007) and rigid-body attitude dynamics with attitudes represented by Euler parameters (e.g. Ren 2007).

Euler–Lagrange equations can be used to model a class of mechanical systems including robotic manipulators and rigid bodies. A related problem to consensus is synchronisation of Euler–Lagrange systems (e.g. Krogstad and Gravdahl 2006; Su, Sun, Ren, and Mills 2006; Sun, Shao, and Feng 2007; Chung and Slotine 2007) or general passive systems (Chopra and Spong 2006). In Krogstad and Gravdahl (2006), a synchronisation scheme is developed for formation-flying spacecraft. The results in Krogstad and Gravdahl (2006) rely on an all-to-all communication topology. In Su et al. (2006) and Sun et al. (2007), position synchronisation of multi-axis motions is addressed via a cross-coupling technique. Chung and Slotine (2007) use contraction analysis to study synchronisation of Lagrangian systems. The results in Su et al. (2006), Sun et al. (2007) and Chung and Slotine (2007) rely on a bidirectional or unidirectional ring communication topology. In Chopra and Spong (2006), output synchronisation is studied under a passivity-based framework. Chopra and Spong (2006) address both fixed and switching communication topologies and unify several existing results on consensus or synchronisation in the literature. To use the passivity property, the control law on synchronisation of Euler–Lagrange systems derived in Chopra and Spong (2006) is model dependent in the sense that it requires the knowledge of the inertial matrix and the Coriolis and centrifugal torques. In addition, the control law requires measurements of generalised coordinate derivatives.

1.1 *Motivation of the current article*

The objective of the current article is to propose and analyse distributed, leaderless, model-independent consensus algorithms for networked Euler–Lagrange systems. We are motivated to derive distributed, leaderless, model-independent consensus algorithms that guarantee that the networked Euler–Lagrange systems reach consensus on their states when the systems have only local interaction with their neighbours, none of them has the knowledge of the group reference trajectory (i.e. none of them is a leader), and the models of the systems are not accurately known. The distributed feature of the algorithms makes them

*Email: wren@engineering.usu.edu

ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207170902948027
http://www.informaworld.com

scalable to a large number of systems while the model-independent feature of the algorithms makes them not to rely on accurate knowledge of the systems. The leaderless feature of the algorithms makes them suitable for applications where the particular consensus equilibrium is not what is important, but rather that each system in the team converges to an identical state. While there are many applications where there exists a group reference trajectory (i.e. leader-following case), there are also numerous applications where leaderless algorithms are important. Examples include rendezvous, flocking, and attitude synchronisation. For example, the proposed algorithms have potential applications in automated rendezvous and docking. In addition, rigid-body attitude dynamics can be written in the form of Euler–Lagrange equations. The proposed algorithms can be used for attitude synchronisation of multiple spacecraft with local interaction. Furthermore, when there is a team of networked mobile vehicles equipped with robotic arms that hold sensors (e.g. iRobot PackBot Explorer), the robotic arms on each mobile vehicle can be modelled by Euler–Lagrange equations. The proposed algorithms can be used to synchronise the robotic arms and the sensors equipped on different mobile vehicles so that a team of mobile vehicles can scan an area cooperatively. For Euler–Lagrange systems, there exists actuator saturation. Also it is often the case that only measurements of generalised coordinates instead of generalised coordinate derivatives are available. Unfortunately, these constraints are often neglected in the existing results for networked Euler–Lagrange systems. We are therefore motivated to also derive distributed, leaderless, model-independent consensus algorithms that account for actuator saturation or unavailability of measurements of generalised coordinate derivatives.

1.2 *Contributions of the current article*

The current article complements some results in existing consensus algorithms for single-integrator, double-integrator, and rigid-body attitude dynamics and synchronisation approaches for Euler–Lagrange systems or general passive systems in the following aspects. First, we propose and analyse a fundamental distributed, leaderless, model-independent consensus algorithm for Euler–Lagrange systems. Second, we propose and analyse a distributed, leaderless, model-independent consensus algorithm for Euler–Lagrange systems that accounts for actuator saturation. Third, we propose and analyse a distributed, leaderless, model-independent consensus algorithm for Euler–Lagrange systems that accounts for unavailability of measurements of generalised coordinate derivatives. In our prior work (Ren 2007, 2008), LaSalle's invariance principle combined with algebraic graph theory and Lyapunov theory is used to analyse consensus algorithms for double-integrator dynamics and rigid-body attitude dynamics with attitudes represented by Euler parameters. However, the closed-loop interconnected Euler–Lagrange equations using the proposed algorithms are non-autonomous, implying that LaSalle's invariance principle is no longer applicable and the convergence analysis is more challenging in this case. Instead, we resort to the interplay of tools from algebraic graph theory, Lyapupov theory and Matrosov's theorem for convergence analysis of the proposed algorithms. While each tool can be used in some contexts, the interplay of these tools poses significant theoretical challenges. The analysis techniques that we use in this article themselves are novel and will offer insights for stability analysis of other interconnected systems.

1.3 *Comparison with the existing results*

The novelty of the proposed algorithms in this article compared with those reported in the existing literature includes (i) study of distributed, leaderless consensus algorithms for non-linear Euler–Lagrange equations, (ii) independence on the knowledge of system models, (iii) explicit consideration of actuator saturation, and (iv) reduced requirement on communication/sensing. In particular, in contrast to existing consensus algorithms (e.g. Olfati-Saber et al. (2007), Ren et al. (2007) and references therein), the algorithms proposed in this article take into account more challenging non-linear Euler–Lagrange equations. In contrast to existing synchronisation approaches for Euler–Lagrange systems (e.g. Krogstad and Gravdahl 2006; Su et al. 2006; Sun et al. 2007; Chung and Slotine 2007), the algorithms proposed in this article are leaderless in the sense that there does not exist a group reference trajectory for each system. In addition, the proposed algorithms do not rely on restrictive all-to-all or ring communication topologies but allow for arbitrary undirected connected communication topologies. In contrast to the passivity-based approach for output synchronisation (Chopra and Spong 2006), the algorithms proposed in this article do not rely on the passivity property of the systems and are model independent in the sense that the algorithms do not require the knowledge of the inertial matrix and the Coriolis and centrifugal torques. In contrast to existing synchronisation approaches for Euler–Lagrange systems (e.g. Krogstad and Gravdahl 2006; Su et al. 2006; Sun et al. 2007; Chung and Slotine 2007), our second algorithm explicitly accounts for actuator saturation by introduction of bounded non-linear functions. In contrast to all the above-mentioned references in synchronisation of Euler–Lagrange systems or general passive systems, our third algorithm does not require measurements of generalised coordinate derivatives, which reduces communication/sensing.

2. Problem statement and background

Euler–Lagrange systems are represented by

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i = \tau_i, \quad i = 1, \dots, n, \tag{1}$$

where $q_i \in \mathbb{R}^p$ is the vector of generalised coordinates, $M_i(q_i) \in \mathbb{R}^{p \times p}$ is the symmetric positive-definite inertia matrix, $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques and τ_i is the vector of torques produced by the actuators associated with the i-th system. Here we have omitted the vector of gravitational torques for simplicity. However, this term can be compensated straightforwardly when designing τ_i. We further assume that $0 < k_{\underline{m}} \leq \|M_i(q_i)\| \leq k_{\bar{m}}$, $\|C_i(q_i, \dot{q}_i)\| \leq k_c \|\dot{q}_i\|$, where $k_c > 0$, and that $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric (i.e. $y^T[\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)]y = 0$ for all $y \in \mathbb{R}^p$).

Weighted undirected graph $\mathcal{G}$ is used to model communication among the n systems. Graph $\mathcal{G}$ consists of a node set $\mathcal{V} = \{1, \dots, n\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{C} = [c_{ij}] \in \mathbb{R}^{n \times n}$. Weighted adjacency matrix $\mathcal{C}$ is defined such that $c_{ij} = c_{ji}$ is a positive weight if $(j, i) \in \mathcal{E}$, while $c_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. Note that $\mathcal{C}$ is symmetric. Let Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n \times n}$ associated with $\mathcal{C}$ be defined as $\ell_{ii} = \sum_{j=1, j \neq i}^{n} c_{ij}$ and $\ell_{ij} = -c_{ij}$, where $i \neq j$. Note that $\mathcal{L}$ is symmetric positive semidefinite. In addition, 0 is a simple eigenvalue of $\mathcal{L}$ with the associated eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is the $n \times 1$ column vector of all ones, and all other eigenvalues of $\mathcal{L}$ are positive if and only if graph $\mathcal{G}$ is connected (Merris 1994). Accordingly, if $\mathcal{G}$ is connected, then $(\mathcal{L} \otimes I_p)x = 0$ or $x^T (\mathcal{L} \otimes I_p)x = 0$ if and only if $x_i = x_j$, where $x_i \in \mathbb{R}^p$, $x = [x_1^T, \dots, x_n^T]^T$, $\otimes$ denotes the Kronecker product and I_p denotes the $p \times p$ identity matrix. Note that $(\mathcal{L} \otimes I_p)x$ is a column stack vector of all $\sum_{j=1}^{n} c_{ij}(x_i - x_j)$, $i = 1, \dots, n$. Also note that $x^T(\mathcal{L} \otimes I_p)x = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\|x_i - x_j\|^2$.

The objective of the current article is to design distributed, leaderless, model-independent consensus algorithms for (1) such that $q_i(t) \to q_j(t)$ and $\dot{q}_i(t) \to 0$ as $t \to \infty$. Before moving on, we need the following theorem and lemmas:

Theorem 2.1 (Matrosov's theorem restated in Paden and Panja (1988) and Krogstad and Gravdahl (2006)**:** *Given the system*

$$\dot{x} = f(t, x), \tag{2}$$

where $f(t, 0) \equiv 0$ and f is such that solutions exist and are unique. Let $V(x, t)$ and $W(x, t)$ be continuous functions on domain $\mathbb{D}$ and satisfy the following four conditions:

1. *$V(x, t)$ is positive definite and decrescent.*
2. *$\dot{V}(x, t) \leq U(x) \leq 0$, where $U(x)$ is continuous.*
3. *$|W(x, t)|$ is bounded.*
4. $\max(d(x, M), |\dot{W}(x, t)|) \geq \gamma(\|x\|)$, *where $M = \{x | U(x) = 0\}$, $d(x, M)$ denotes the distance from x to set M, and $\gamma(\cdot)$ is a class $\mathcal{K}$ function.*

Then the equilibrium of (2) *is uniformly asymptotically stable on $\mathbb{D}$.*

Lemma 2.2 Paden and Panja (1988)**:** *Condition* 4 *in Theorem* 2.1 *is satisfied if the following two conditions are satisfied*:

1. *The function $\dot{W}(x, t)$ is continuous in both arguments and $\dot{W}(x, t) = g(x, \beta(t))$, where g is continuous in both arguments and $\beta(t)$ is continuous and bounded.*
2. *There exists a class $\mathcal{K}$ function, α, such that $|\dot{W}(x, t)| \geq \alpha(\|x\|)$ for all $x \in M$, where M is the set defined in Theorem 2.1.*

Lemma 2.3 Graham (1981)**:** *Suppose that $U \in \mathbb{R}^{p \times p}$, $V \in \mathbb{R}^{q \times q}$, $X \in \mathbb{R}^{p \times p}$ and $Y \in \mathbb{R}^{q \times q}$. The following arguments are valid*:

(i) $(U \otimes V)(X \otimes Y) = UX \otimes VY$, (ii) *Suppose that U and V are invertible. Then $(U \otimes V)^{-1} = U^{-1} \otimes V^{-1}$*, (iii) *If U and V are symmetric, so is $U \otimes V$* and (iv) *If U and V are symmetric positive definite, so is $U \otimes V$.*

Lemma 2.4 Ren (2008)**:** *Suppose that $\varsigma_i \in \mathbb{R}^m$, $\varphi_i \in \mathbb{R}^m$, $K \in \mathbb{R}^{m \times m}$ and $D = [d_{ij}] \in \mathbb{R}^{n \times n}$. If D is symmetric, then*

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(\varsigma_i - \varsigma_j)^T \tanh[K(\varphi_i - \varphi_j)] = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\varsigma_i^T \tanh[K(\varphi_i - \varphi_j)].$$

3. Distributed, leaderless, model-independent consensus algorithms for networked Euler–Lagrange systems

We consider three distributed, leaderless, model-independent consensus algorithms for networked Euler–Lagrange systems. Sections 3.1, 3.2, and 3.3 introduce, respectively, a fundamental algorithm, an algorithm accounting for actuator saturation, and an algorithm accounting for unavailability of measurements of generalised coordinate derivatives.

3.1 *Fundamental algorithm*

In this section, we consider a fundamental consensus algorithm as

$$\tau_i = -\sum_{j=1}^{n} a_{ij}(q_i - q_j) - \sum_{j=1}^{n} b_{ij}(\dot{q}_i - \dot{q}_j) - K_i\dot{q}_i, \qquad (3)$$

where $i = 1, \ldots, n$, a_{ij} is the (i,j) entry of weighted adjacency matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}_A$ for q_i, b_{ij} is the (i,j) entry of weighted adjacency matrix $\mathcal{B} \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}_B$ for $\dot{q}_i$ and $K_i \in \mathbb{R}^{p\times p}$ is symmetric positive definite. Note that here $\mathcal{G}_A$ and $\mathcal{G}_B$ are allowed to be different.

Theorem 3.1: *Using* (3) *for* (1), $q_i(t) \to q_j(t)$ *and* $\dot{q}_i(t) \to 0$ *as* $t \to \infty$ *if graph* $\mathcal{G}_A$ *is undirected connected and graph* $\mathcal{G}_B$ *is undirected.*

Proof: Let $q = [q_1^T, \ldots, q_n^T]^T$ and $\dot{q} = [\dot{q}_1^T, \ldots, \dot{q}_n^T]^T$. Let $\mathcal{L}_A$ and $\mathcal{L}_B$ be, respectively, the Laplacian matrix associated with $\mathcal{A}$ and $\mathcal{B}$. Note that both $\mathcal{L}_A$ and $\mathcal{L}_B$ are symmetric positive semidefinite because both $\mathcal{G}_A$ and $\mathcal{G}_B$ are undirected. Given square matrices A_1–A_m, let $\mathrm{diag}(A_1, \ldots, A_m)$ denotes a block diagonal matrix with diagonal blocks A_1–A_m. Let $M(q) = \mathrm{diag}[M_1(q_1), \ldots, M_n(q_n)]$, $C(q, \dot{q}) = \mathrm{diag}[C_1(q_1, \dot{q}_1), \ldots, C_n(q_n, \dot{q}_n)]$ and $K = \mathrm{diag}(K_1, \ldots, K_n)$. Using (3), (1) can be written in vector form as

$$M(q)\ddot{q} = -C(q, \dot{q})\dot{q} - (\mathcal{L}_A \otimes I_p)q - (\mathcal{L}_B \otimes I_p)\dot{q} - K\dot{q}. \qquad (4)$$

Using (3), (1) can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(q_i - q_j) = \dot{q}_i - \dot{q}_j$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_i = -M_i^{-1}(q_i)\Bigg[C_i(q_i, \dot{q}_i)\dot{q}_i + \sum_{j=1}^{n} a_{ij}(q_i - q_j) + \sum_{j=1}^{n} b_{ij}(\dot{q}_i - \dot{q}_j) + K_i\dot{q}_i\Bigg]. \qquad (5)$$

Let $\tilde{q}$ be a column stack vector of all $q_i - q_j$, where $i < j$ and $a_{ij} \neq 0$ (i.e. $(i,j) \in \mathcal{E}$). Consider the Lyapunov function candidate for (5) as

$$V = \frac{1}{2}q^T(\mathcal{L}_A \otimes I_p)q + \frac{1}{2}\dot{q}^T M(q)\dot{q}.$$

Because graph $\mathcal{G}_A$ is undirected, it follows that $q^T(\mathcal{L}_A \otimes I_p)q = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\|q_i - q_j\|^2$ as shown in Section 2. It thus follows that V is positive definite and decrescent with respect to $\tilde{q}$ and $\dot{q}$. Note that system (5) with states $q_i - q_j$ and $\dot{q}_i$ is non-autonomous due to the dependence of M_i and C_i on q_i. As a result, LaSalle's invariance principle is no longer applicable for (5). Instead, we apply Theorem 2.1 to prove the theorem. Note that Condition 1 in Theorem 2.1 is satisfied.

The derivative of V is given by

$$\begin{aligned}\dot{V} &= \dot{q}^T(\mathcal{L}_A \otimes I_p)q + \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{1}{2}\dot{q}^T M(q)\ddot{q}\\ &= \dot{q}^T(\mathcal{L}_A \otimes I_p)q + \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q},\end{aligned}$$

where we have used the fact that $M(q)$ is symmetric. Note that $\dot{M}(q) - 2C(q, \dot{q})$ is skew symmetric. By applying (4), the derivative of V can be written as

$$\dot{V} = -\dot{q}^T(\mathcal{L}_B \otimes I_p)\dot{q} - \dot{q}^T K\dot{q} \le 0, \qquad (6)$$

where we have used the fact that $-\dot{q}^T(\mathcal{L}_B \otimes I_p)\dot{q} \le 0$ because graph $\mathcal{G}_B$ is undirected. Therefore, Condition 2 in Theorem 2.1 is satisfied.

Let $W = \dot{q}^T M(q)(\mathcal{L}_A \otimes I_p)q$. It follows that $|W| \le \|\dot{q}\|\|M(q)(\mathcal{L}_A \otimes I_p)q\| \le \|\dot{q}\|\|M(q)\|\|(\mathcal{L}_A \otimes I_p)q\|$. Note that $\|M(q)\|$ is bounded. Also note that (6) implies that $V(t) \le V(0)$, $\forall t \ge 0$, which in turn implies that $\|\tilde{q}\|$ and $\|\dot{q}\|$ are bounded. Noting that $(\mathcal{L}_A \otimes I_p)q$ is a column stack vector of all $\sum_{j=1}^{n} a_{ij}(q_i - q_j)$, $i = 1, \ldots, n$, it follows that $\|(\mathcal{L}_A \otimes I_p)q\|$ is also bounded. It thus follows that $|W|$ is bounded along the solution trajectory, implying that Condition 3 in Theorem 2.1 is satisfied.

The derivative of W along the solution trajectory of (4) is

$$\begin{aligned}\dot{W} &= \ddot{q}^T M(q)(\mathcal{L}_A \otimes I_p)q + \dot{q}^T\dot{M}(q)(\mathcal{L}_A \otimes I_p)q\\ &\quad + \dot{q}^T M(q)(\mathcal{L}_A \otimes I_p)\dot{q}\\ &= -\dot{q}^T C^T(q, \dot{q})(\mathcal{L}_A \otimes I_p)q\\ &\quad - q^T(\mathcal{L}_A^2 \otimes I_p)q - \dot{q}^T(\mathcal{L}_B\mathcal{L}_A \otimes I_p)q\\ &\quad + \dot{q}^T\dot{M}(q)(\mathcal{L}_A \otimes I_p)q + \dot{q}^T M(q)(\mathcal{L}_A \otimes I_p)\dot{q},\end{aligned}$$

where we have used Lemma 2.3. Note that $\dot{V} = 0$, implies that $\dot{q} = 0$. On set $\{(\tilde{q}, \dot{q}) | \dot{V} = 0\}$, $\dot{W}$, becomes

$$\dot{W} = -q^T(\mathcal{L}_A^2 \otimes I_p)q \le 0.$$

Note that $|\dot{W}| = \|(\mathcal{L}_A \otimes I_p)q\|^2$ is positive definite with respect to $\tilde{q}$. It follows from Khalil (1996, Lemma 3.5) that there exists a class $\mathcal{K}$ function, α, such that $|\dot{W}| \ge \alpha(\|\tilde{q}\|)$. Also note that $|\dot{W}|$ does not explicitly depend on t. It follows from Lemma 2.2 that Condition 4 in Theorem 2.1 is satisfied. We conclude from Theorem 2.1 that the equilibrium of system (5) (i.e. $\tilde{q} = 0$ and $\dot{q} = 0$) is uniformly asymptotically stable, which implies that $q_i(t) \to q_j(t)$ and $\dot{q}_i(t) \to 0$ as $t \to \infty$ because graph $\mathcal{G}_A$ is undirected connected. □

3.2 *Algorithm accounting for actuator saturation*

In this section, we consider a consensus algorithm that explicitly accounts for actuator saturation as

$$\tau_i = -\sum_{j=1}^{n} a_{ij} \tanh[K_q(q_i - q_j)] - \sum_{j=1}^{n} b_{ij} \tanh[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)] - K_i \tanh(K_{di}\dot{q}_i), \quad (7)$$

where $i = 1, \ldots, n$, a_{ij} and b_{ij} are defined as in (3), K_q, $K_{\dot{q}}$, K_i and K_{di} are $p \times p$ positive-definite diagonal matrices, and $\tanh(\cdot)$ is defined component-wise for a vector.

Remark 1: In contrast to (3), bounded functions are introduced in (7) to explicitly account for actuator saturation. Using (7), it follows that $\|\tau_i(t)\|_\infty \le \tau_{\max}$ for all t, where $\tau_{\max} \triangleq \sum_{j=1}^{n}(a_{ij} + b_{ij}) + \|K_i\|_\infty$. Note that $\tau_{\max}$ is independent of the initial conditions of q_i and $\dot{q}_i$. In contrast, using (3), $\|\tau_i(t)\|_\infty$ is dependent on the initial conditions of q_i and $\dot{q}_i$.

Theorem 3.2: *Using* (7) *for system* (1), $q_i(t) \to q_j(t)$ *and* $\dot{q}_i(t) \to 0$ *as* $t \to \infty$ *if graph* $\mathcal{G}_A$ *is undirected connected and graph* $\mathcal{G}_B$ *is undirected.*

Proof: Similar to the proof of Theorem 3.1, using (7), (1) can be written as a non-autonomous system with states $q_i - q_j$ and $\dot{q}_i$. We apply Theorem 2.1 to prove the theorem. Let $\tilde{q}$ and $\dot{q}$ be defined as in the proof of Theorem 3.1. Consider the Lyapunov function candidate

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \mathbf{1}_p^T K_q^{-1} \log\{\cosh[K_q(q_i - q_j)]\} + \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^T M_i(q_i)\dot{q}_i,$$

where $\log(\cdot)$ and $\cosh(\cdot)$ are defined component-wise for a vector. Note that V is positive definite with respect to $\tilde{q}$ and $\dot{q}$. Therefore, Condition 1 in Theorem 2.1 is satisfied.

The derivative of V is given by

$$\dot{V} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\dot{q}_i - \dot{q}_j)^T \tanh[K_q(q_i - q_j)] + \frac{1}{2}\sum_{i=1}^{n}[\ddot{q}_i^T M_i(q_i)\dot{q}_i + \dot{q}_i^T \dot{M}_i(q_i)\dot{q}_i + \dot{q}_i^T M_i(q_i)\ddot{q}_i].$$

Using (7), (1) can be written as

$$M_i(q_i)\ddot{q}_i = -C_i(q_i, \dot{q}_i)\dot{q}_i - \sum_{j=1}^{n} a_{ij} \tanh[K_q(q_i - q_j)] - \sum_{j=1}^{n} b_{ij} \tanh[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)] - K_i \tanh(K_{di}\dot{q}_i). \quad (8)$$

Because graph $\mathcal{G}_A$ is undirected, it follows from Lemma 2.4 that $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\dot{q}_i - \dot{q}_j)^T \tanh[K_q(q_i - q_j)] = \sum_{i=1}^{n} \dot{q}_i^T\{\sum_{j=1}^{n} a_{ij} \tanh[K_q(q_i - q_j)]\}$. Also note that $M_i(q_i)$ is symmetric and that $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric. By applying (8), it follows that

$$\dot{V} = -\sum_{i=1}^{n} \dot{q}_i^T \left\{\sum_{j=1}^{n} b_{ij} \tanh[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)] + K_i \tanh(K_{di}\dot{q}_i)\right\}.$$

By noting that graph $\mathcal{G}_B$ is undirected and applying Lemma 2.4 again, it follows that the derivative of V becomes

$$\dot{V} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(\dot{q}_i - \dot{q}_j)^T \tanh[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)] - \sum_{i=1}^{n} \dot{q}_i^T K_i \tanh(K_{di}\dot{q}_i).$$

Given a vector x and positive-definite diagonal matrices K_1 and K_2, x and $K_1 \tanh(K_2 x)$ have the same signs for each component. Therefore, it follows that $\dot{V} \le 0$, which implies that Condition 2 in Theorem 2.1 is satisfied.

Let $W = \sum_{i=1}^{n} \dot{q}_i^T M_i(q_i)\chi_i$, where

$$\chi_i \triangleq \sum_{j=1}^{n} a_{ij} \tanh[K_q(q_i - q_j)].$$

Note that $\dot{V} = 0$ implies that $V(t) \le V(0)$, $\forall t \ge 0$, which in turn implies that $\tilde{q}$ and $\dot{q} = 0$ are bounded. It thus follows that $\|\chi_i\|$ is also bounded. Similar to the proof of Theorem 3.1, it follows that $|W|$ is bounded along the solution trajectory, implying that Condition 3 in Theorem 2.1 is satisfied.

The derivative of W along the solution trajectory of (8) is

$$\dot{W} = -\sum_{i=1}^{n} \dot{q}_i^T C_i^T(q_i, \dot{q}_i)\chi_i - \sum_{i=1}^{n} \chi_i^T \chi_i - \sum_{i=1}^{n}\left\{\sum_{j=1}^{n} b_{ij} \tanh[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)]\right\}^T \chi_i - \sum_{i=1}^{n}[K_i \tanh(K_{di}\dot{q}_i)]^T \chi_i + \sum_{i=1}^{n} \dot{q}_i^T \dot{M}_i(q_i)\chi_i + \sum_{i=1}^{n} \dot{q}_i^T M_i(q_i)\dot{\chi}_i.$$

Note that $\dot{V} = 0$ implies that $\dot{q} = 0$. On set $\{(\tilde{q}, \dot{q}) | \dot{V} = 0\}$, $\dot{W}$ becomes

$$\dot{W} = -\sum_{i=1}^{n} \chi_i^T \chi_i \le 0.$$

If $|\dot{W}| = \sum_{i=1}^{n} \chi_i^T \chi_i$ is positive definite with respect to $\tilde{q}$, then a similar argument to that in Theorem 3.1 implies that Condition 4 in Theorem 2.1 is satisfied. Because $|\dot{W}| = 0$, equivalently we only need to show that $\dot{W} = 0$ implies that $\tilde{q} = 0$. Suppose that $|\dot{W}| = 0$, which implies that $\chi_i = \sum_{j=1}^n a_{ij} \tanh K_q(q_i - q_j) = 0$. It thus follows that $\sum_{i=1}^n q_i^T \{\sum_{j=1}^n a_{ij} \tanh[K_q(q_i - q_j)]\} = 0$, which implies from Lemma 2.4 that $\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}(q_i - q_j)^T \tanh[K_q(q_i - q_j)] = 0$. Note that graph $\mathcal{G}_A$ is undirected and $q_i - q_j$ and tanh $[K_q(q_i - q_j)]$ have the same signs for each component. It follows that $q_i - q_j = 0$ for all $a_{ij} \neq 0$ (i.e. $\tilde{q} = 0$) when $\dot{W} = 0$. Combining the above-mentioned arguments, we conclude from Theorem 2.1 that the equilibrium $\tilde{q} = \dot{q} = 0$ is uniformly asymptotically stable, which implies that $q_i(t) \to q_j(t)$ and $\dot{q}_i(t) \to 0$ as $t \to \infty$ because graph $\mathcal{G}_A$ is undirected connected. □

3.3 *Algorithm accounting for unavailability of measurements of generalised coordinate derivatives*

Note that (3) and (7) require measurements of $\dot{q}_i$ and $\dot{q}_i - \dot{q}_j$, where $b_{ij} \neq 0$. In this section, we consider a consensus algorithm that removes the requirement for the measurements of $\dot{q}_i$ and $\dot{q}_i - \dot{q}_j$ as

$$\dot{\hat{x}}_i = \Gamma \hat{x}_i + \sum_{j=1}^n b_{ij}(q_i - q_j) + \kappa q_i \tag{9a}$$

$$y_i = P\left[\Gamma \hat{x}_i + \sum_{j=1}^n b_{ij}(q_i - q_j) + \kappa q_i\right] \tag{9b}$$

$$\tau_i = -\sum_{j=1}^n a_{ij} \tanh[K_q(q_i - q_j)] - y_i \tag{9c}$$

where $i = 1, \ldots, n$, $\Gamma \in \mathbb{R}^{p\times p}$ is Hurwitz, κ is a positive scalar, a_{ij} is the (i,j) entry of weighted adjacency matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}_A$ for q_i in (9c), b_{ij} is the (i,j) entry of weighted adjacency matrix $\mathcal{B} \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}_B$ for q_i in (9a) and $P = P^T \in \mathbb{R}^{p\times p}$ is the positive-definite solution to the Lyapunov equation $\Gamma^T P + P\Gamma = -Q$ with $Q = Q^T \in \mathbb{R}^{p\times p}$ being positive definite.

Theorem 3.3: *Using* (9) *for system* (1), $q_i(t) \to q_j(t)$ *and* $\dot{q}_i(t) \to 0$ *as* $t \to \infty$ *if graph* $\mathcal{G}_A$ *is undirected connected and graph* $\mathcal{G}_B$ *is undirected.*

Proof: Similar to the proofs of Theorems 3.1 and 3.2, we apply Theorem 2.1 to prove the theorem. Consider the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij} \mathbf{1}_p^T K_q^{-1} \log(\cosh[K_q(q_i - q_j)]) + \frac{1}{2}\sum_{i=1}^n \dot{q}_i^T M_i(q_i)\dot{q}_i + \frac{1}{2}\dot{\hat{x}}^T(\Theta \otimes I_p)^{-1}(I_n \otimes P)\dot{\hat{x}},$$

where $\hat{x} = [\hat{x}_1^T, \ldots, \hat{x}_n^T]^T$, $\Theta = \mathcal{L}_B + \kappa I_n$. Note that $\mathcal{L}_B$ is symmetric positive semidefinite because graph $\mathcal{G}_B$ is undirected. It thus follows that Θ is symmetric positive definite, so is Θ^{-1}. From Lemma 2.3, note that $(\Theta \otimes I_p)^{-1} = (\Theta^{-1} \otimes I_p)$. Also from Lemma 2.3 note that $(\Theta^{-1} \otimes I_p)(I_n \otimes P) = \Theta^{-1} I_n \otimes I_p P = I_n\ \Theta^{-1} \otimes P I_p = (I_n \otimes P)(\Theta^{-1} \otimes I_p)$. That is, $(\Theta \otimes I_p)^{-1}$ and $I_n \otimes P$ commute. Similarly, it is straightforward to show that $(\Theta \otimes I_p)^{-1}$ and $I_n \otimes \Gamma^T$ also commute. Note that $\Theta^{-1} I_n \otimes I_p P$ is symmetric positive definite, so is $(\Theta^{-1} \otimes I_p)(I_n \otimes P)$. Let $\tilde{q}$ and $\dot{q}$ be defined as in the proof of Theorem 3.2. It follows that V is positive definite with respect to $\tilde{q}$, $\dot{q}$, and $\dot{\hat{x}}$. Therefore, Condition 1 in Theorem 2.1 is satisfied.

Following the proof of Theorem 3.2, we derive the derivative of V as

$$\begin{aligned}
\dot{V} &= -\sum_{i=1}^n \dot{q}_i^T y_i + \frac{1}{2}\dot{\hat{x}}^T(I_n \otimes \Gamma^T)(\Theta \otimes I_p)^{-1}(I_n \otimes P)\dot{\hat{x}} \\
&\quad + \frac{1}{2}\dot{q}^T(\Theta \otimes I_p)^T(\Theta \otimes I_p)^{-1}(I_n \otimes P)\dot{\hat{x}} \\
&\quad + \frac{1}{2}\dot{\hat{x}}^T(\Theta \otimes I_p)^{-1}(I_n \otimes P)(I_n \otimes \Gamma)\dot{\hat{x}} \\
&\quad + \frac{1}{2}\dot{\hat{x}}^T(\Theta \otimes I_p)^{-1}(I_n \otimes P)(\Theta \otimes I_p)\dot{q} \\
&= -\sum_{i=1}^n \dot{q}_i^T y_i + \frac{1}{2}\dot{\hat{x}}^T(\Theta \otimes I_p)^{-1}[I_n \otimes (\Gamma^T P + P\Gamma)]\dot{\hat{x}} \\
&\quad + \dot{q}^T(I_n \otimes P)\dot{\hat{x}} \\
&= -\frac{1}{2}\dot{\hat{x}}^T(\Theta \otimes I_p)^{-1}(I_n \otimes Q)\dot{\hat{x}} \le 0,
\end{aligned}$$

where we have used the fact that

$$\ddot{\hat{x}} = (I_n \otimes \Gamma)\dot{\hat{x}} + (\Theta \otimes I_p)\dot{q}, \tag{10}$$

$(\Theta \otimes I_p)^{-1}$ and $I_n \otimes \Gamma^T$ commute, $(\Theta \otimes I_p)^{-1}$ and $I_n \otimes P$ commute, $\Theta \otimes I_p = (\Theta \otimes I_p)^T$, $y = (I_n \otimes P)\dot{\hat{x}}$ with $y = [y_1^T, \ldots, y_n^T]^T$ and $(\Theta \otimes I_p)^{-1}$ $(I_n \otimes Q) = \Theta^{-1} I_n \otimes Q I_p$ is symmetric positive definite. Therefore, Condition 2 in Theorem 2.1 is satisfied.

Let W and χ_i be defined as in the proof of Theorem 3.2. Similar to the proof of Theorem 3.2, it follows that $|W|$ is bounded along the solution trajectory, implying that Condition 3 in Theorem 2.1 is satisfied.

The derivative of W along the solution trajectory of closed-loop system (1) using (9) is

$$\dot{W} = -\sum_{i=1}^n \dot{q}_i^T C_i^T(q_i, \dot{q}_i)\chi_i - \sum_{i=1}^n \chi_i^T \chi_i - \sum_{i=1}^n y_i^T \chi_i + \dot{q}_i^T \dot{M}_i(q_i)\chi_i + \dot{q}_i^T M_i(q_i)\dot{\chi}_i.$$

Note that $\dot{V} = 0$ implies that $\dot{\hat{x}} = 0$, which in turn implies that $(\Theta \otimes I_p)\dot{q} = 0$ according to (10) and $y_i = 0$ by noting that $y_i = P\dot{\hat{x}}_i$ according to (9b). Because $\Theta \otimes I_p$ is symmetric positive definite, it follows that

$\dot{q}=0$. On set $\{(\tilde{q},\dot{q},\dot{\hat{x}})\,|\,\dot{V}=0\}$, $\dot{W}$ becomes

$$\dot{W}=-\sum_{i=1}^{n}\chi_i^T\chi_i\le 0.$$

Therefore, similar to the proof of Theorem 3.2, we conclude that $q_i(t)\to q_j(t)$ and $\dot{q}_i(t)\to 0$ as $t\to\infty$. □

Remark 2: Note that without the terms $-\sum_{j=1}^{n} b_{ij}(\dot{q}_i-\dot{q}_j)$ in (3) $-\sum_{j=1}^{n} b_{ij}\tanh K_{\dot{q}}(\dot{q}_i-\dot{q}_j)$ in (7) and $\sum_{j=1}^{n} b_{ij}(q_i-q_j)$ in (9a) or equivalently $b_{ij}\equiv 0$, Theorems 3.1–3.3 are still valid as long as graph $\mathcal{G}_A$ is undirected connected. However, these terms introduce relative damping between neighbouring systems.

4. Simulation results

In this section, we simulate a scenario where six two-link revolute joint arms are synchronised through local communication using, respectively, algorithms (3), (7) and (9). For simplicity, we assume that each arm is identical. The Euler–Lagrange equation of each two-link revolute joint arm is given in Spong and Vidyasagar (2006, pp. 259–262). In particular, we assume that the masses of links 1 and 2 are, respectively, 0.5 and 0.4 kg, the lengths of links 1 and 2 are, respectively, 0.4 and 0.3 m, the distances from the previous joint to the centre of mass of links 1 and 2 are, respectively, 0.2 and 0.15 m and the moments of inertia of links 1 and 2 are, respectively, 0.1 and 0.05 kg m^2.

For simplicity, we assume that graphs $\mathcal{G}_A$ and $\mathcal{G}_B$ are identical. Figure 1 shows graph $\mathcal{G}_A$ (equivalently, $\mathcal{G}_B$) for the six two-link revolute joint arms. Table 1 shows the control parameters for each algorithm. In simulation, we let $q_i(0)=[\frac{\pi}{7}i,\frac{\pi}{8}i]^T$ rad and $\dot{q}(0)=[0.1i-0.4,-0.1i+0.5]^T$ rad/s, where $i=1,\ldots,6$. In the following, we use a superscript (j) to denote the j-th component of a vector.

Figures 2, 3 and 4 show, respectively, the joint angles, their derivatives and the control torques of arms 1, 3, and 5 using (3). Note that the joint angles of all arms reach consensus while their derivatives converge to zero.

Figures 5, 6 and 7 show, respectively, the joint angles, their derivatives and the control torques of arms 1, 3, and 5 using (7). Note that the joint angles of all arms reach consensus while their derivatives converge to zero. By comparing Figure 7 with Figure 4, we can see that the maximum control torque using (7) is much smaller than that using (3) with the introduction of bounded functions in (7). As a tradeoff, joint angles and their derivatives reach consensus more slowly using (7) than using (3). In addition, the upper bound for the control torque using (7) is independent of the initial joint angles and their derivatives.

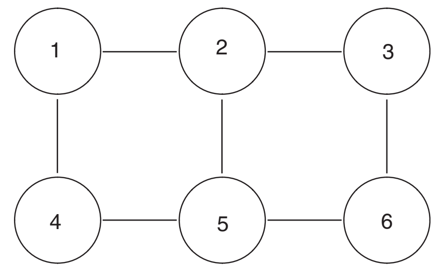

Figure 1. Graph $\mathcal{G}_A$ (equivalently, $\mathcal{G}_B$) for six two-link revolute joint arms. The graph is connected.

Table 1. Control parameters for each algorithm.

Algorithm (3):
$K_i=0.1I_2$, $a_{ij}=b_{ij}=0.1$ if $(j,i)\in\mathcal{E}$
Algorithm (7):
$K_q=K_{\dot{q}}=K_i=K_{di}=0.1I_2$
$a_{ij}=b_{ij}=0.1$ if $(j,i)\in\mathcal{E}$
Algorithm (9):
$\Gamma=-I_2$, $\kappa=0.2$, $P=0.5I_2$, $K_q=0.2I_2$
$a_{ij}=b_{ij}=0.2$ if $(j,i)\in\mathcal{E}$

Figures 8, 9 and 10 show, respectively, the joint angles, their derivatives, and the control torques of arms 1, 3, and 5 using (9). The initial conditions $\hat{x}_i(0)$ are chosen randomly. Note that the joint angles of all arms reach consensus while their derivatives converge to zero even without measurements of absolute and relative joint angle derivatives.

We have shown simulation results where six-networked Euler–Lagrange systems are synchronized perfectly with zero final consensus errors. However, in actual applications, there will exist measurement noise when the systems measure the generalised coordinates and/or their derivatives and time delay when the systems communicate with their neighbours. It is expected that the measurement noise and time delay will cause imperfect final consensus. That is, the systems might achieve only ϵ-consensus rather than consensus, where there exist non-zero small final consensus errors.

5. Conclusion and future work

We have proposed and analysed distributed, leaderless, model-independent consensus algorithms for systems modelled by Euler–Lagrange equations. In particular, we have studied a fundamental algorithm, an algorithm accounting for actuator saturation and an

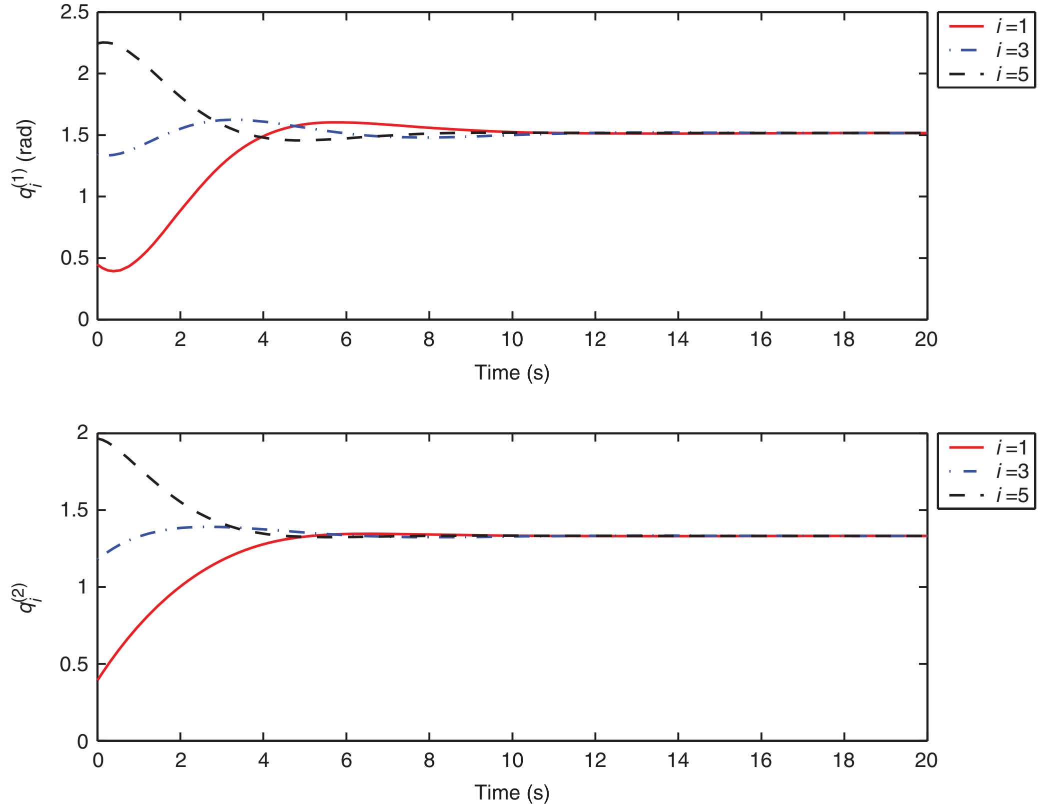

Figure 2. Joint angles of arms 1, 3 and 5 using (3). $q_i^{(j)}$ denotes the j-th joint angle of arm i.

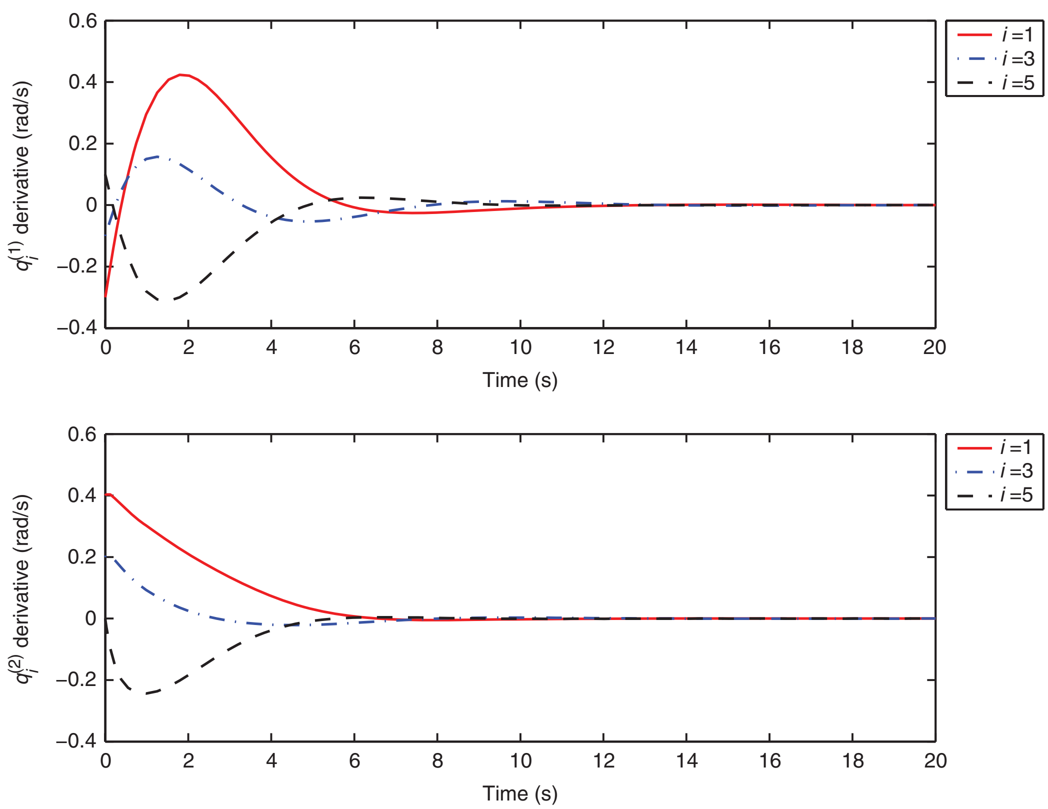

Figure 3. Joint angle derivatives of arms 1, 3 and 5 using (3). $\dot{q}_i^{(j)}$ denotes the j-th joint angle derivative of arm i.

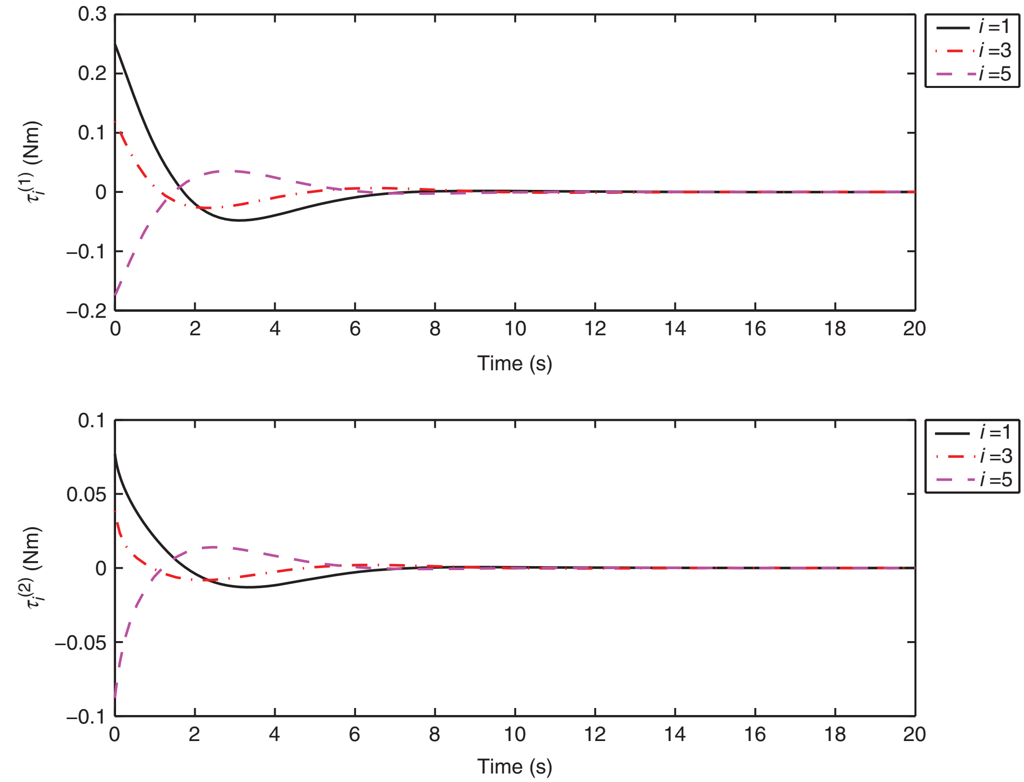

Figure 4. Control torques of arms 1, 3 and 5 using (3). $\tau_i^{(j)}$ denotes the j-th joint torque of arm i.

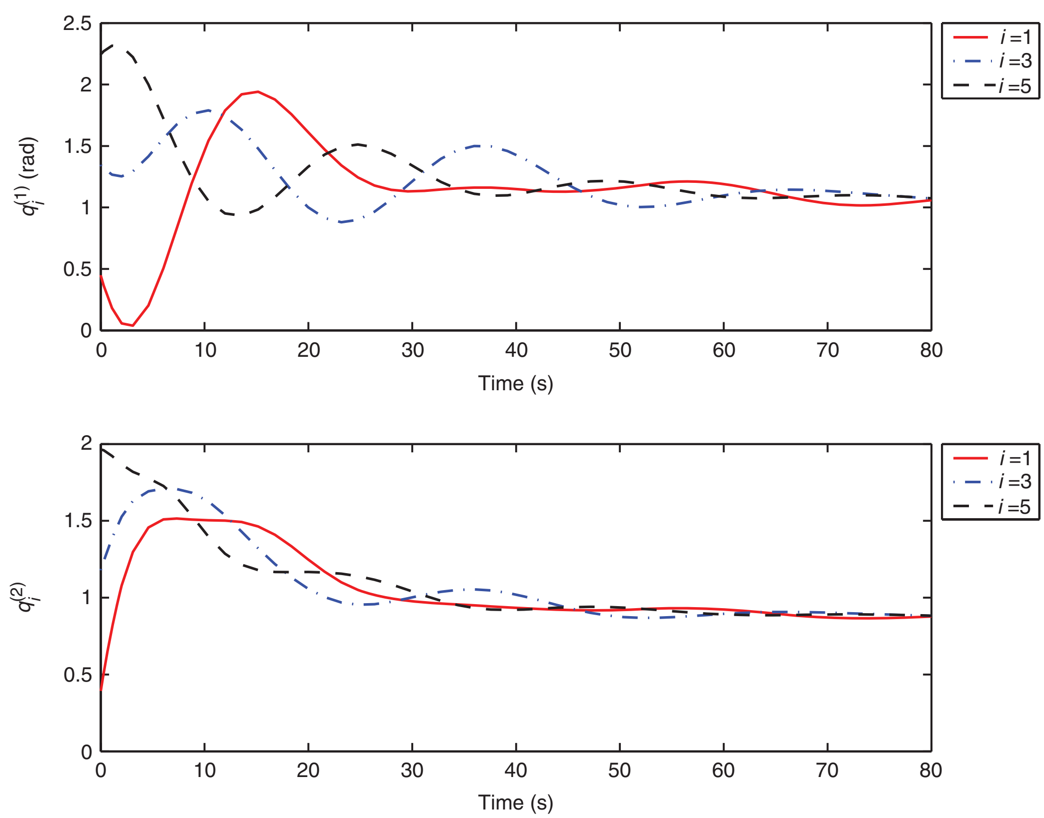

Figure 5. Joint angles of arms 1, 3 and 5 using (7).

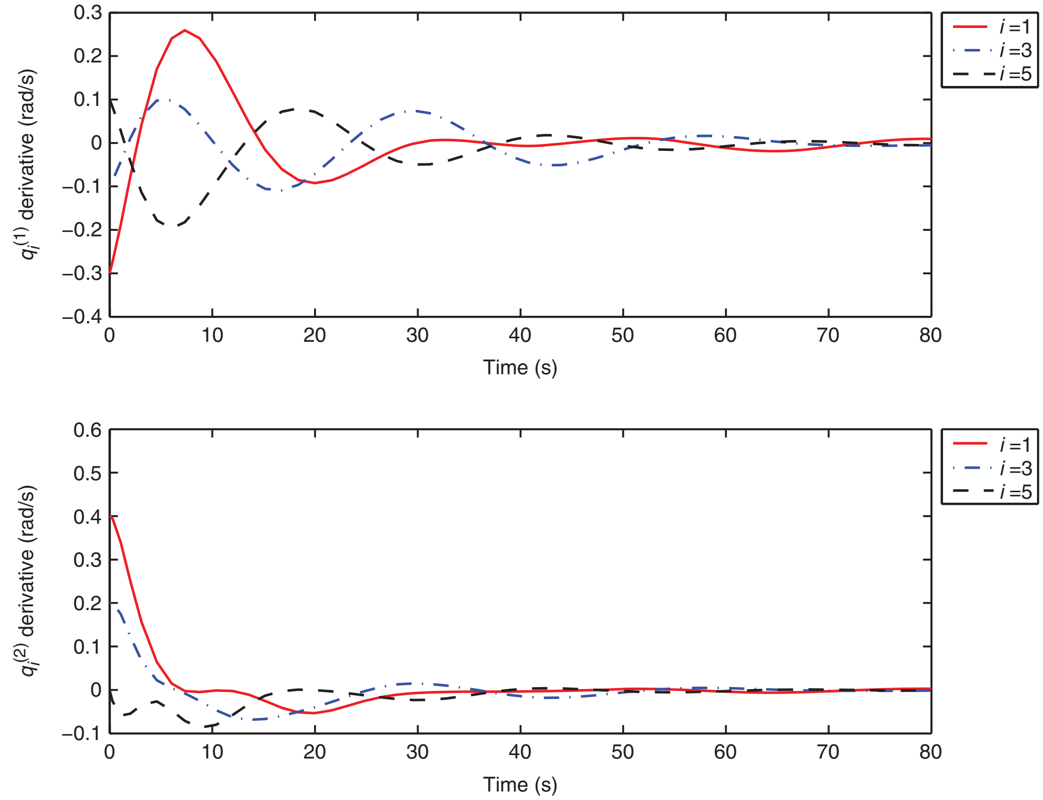

Figure 6. Joint angle derivatives of arms 1, 3 and 5 using (7).

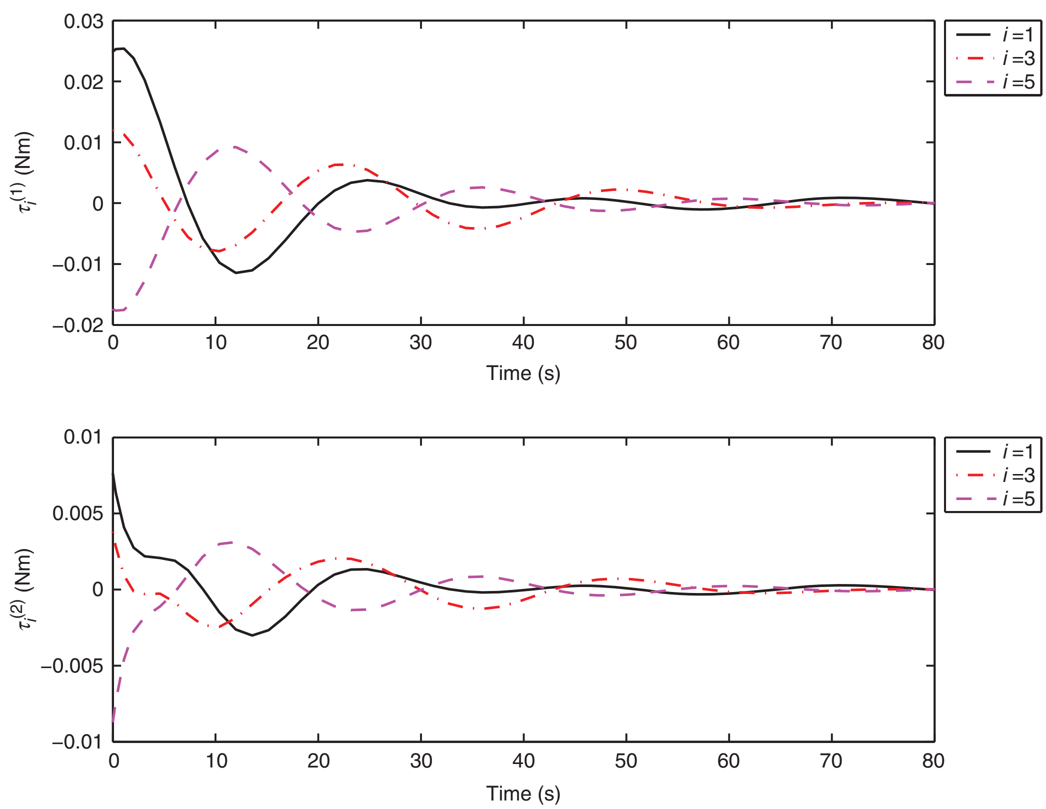

Figure 7. Control torques of arms 1, 3 and 5 using (7).

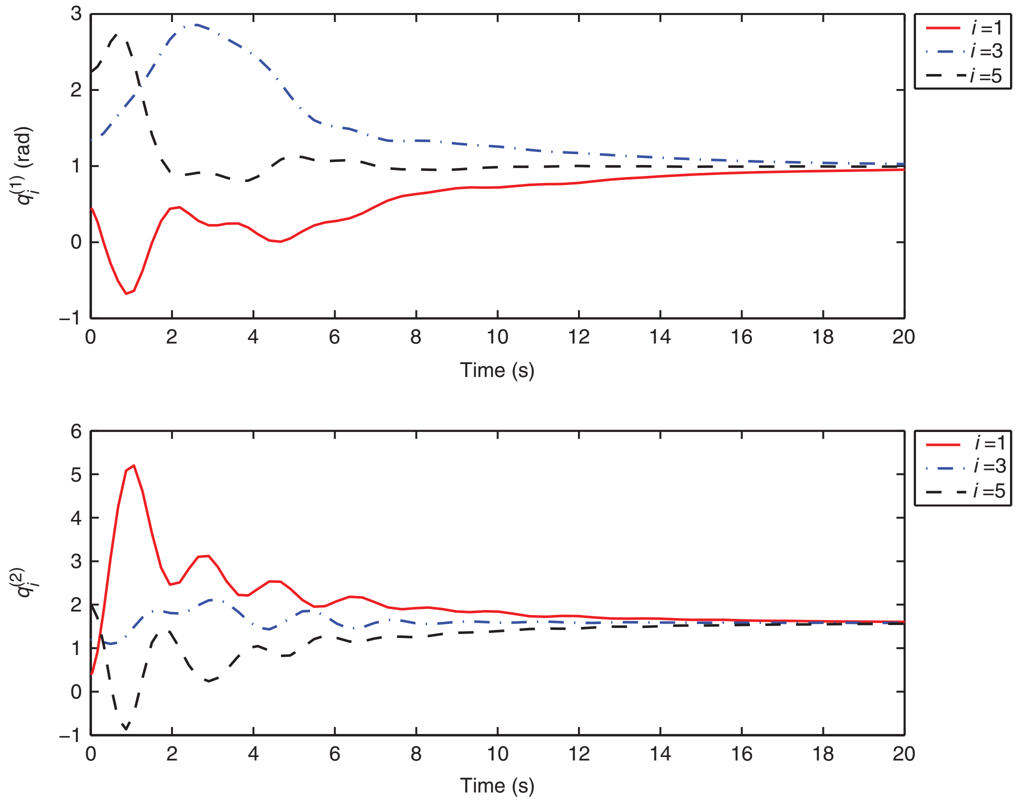

Figure 8. Joint angles of arms 1, 3 and 5 using (9).

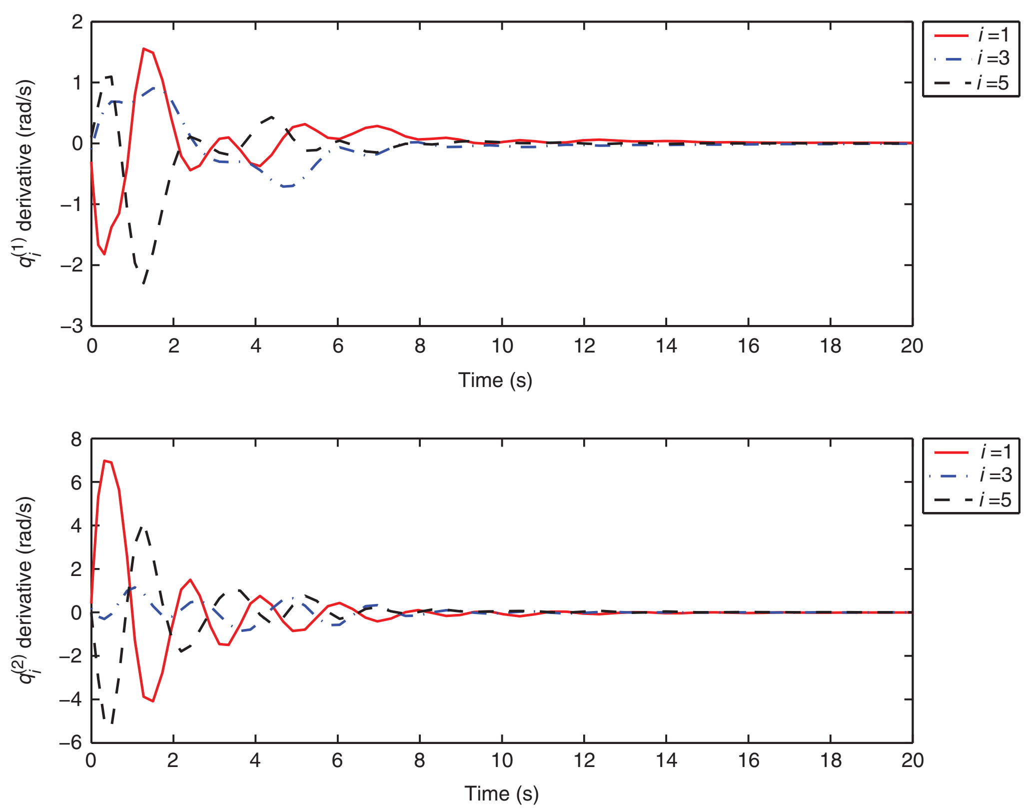

Figure 9. Joint angle derivatives of arms 1, 3 and 5 using (9).

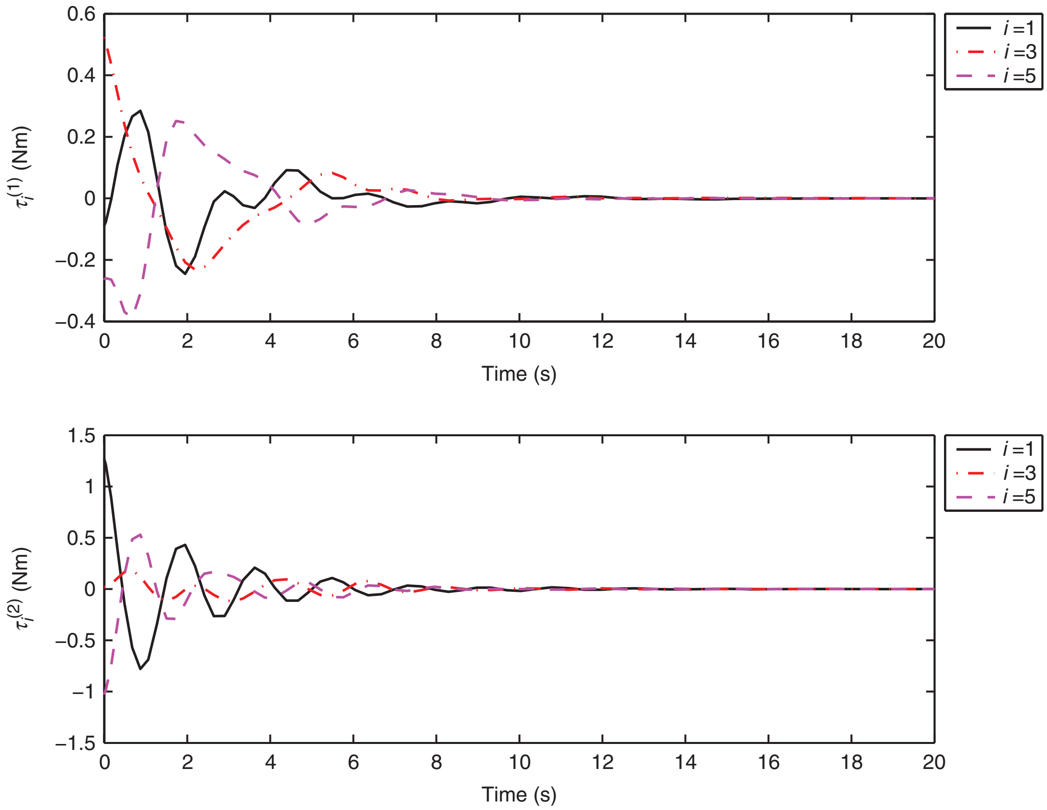

Figure 10. Control torques of arms 1, 3 and 5 using (9).

algorithm accounting for unavailability of measurements of generalised coordinate derivatives. Matrosov's theorem has been used to show that all algorithms converge as long as the undirected communication topology is connected. The algorithms have been used to synchronise six-networked two-link revolute joint arms. While the current article complements some existing results in the literature, there are some issues that need to addressed in future work. For example, in the current article, we have shown that consensus is reached for the networked Euler–Lagrange systems without explicitly deriving the final consensus equilibrium. It seems that the final consensus equilibrium is dependent on three factors, namely, the communication topology, the control gains and the initial conditions of the systems. In future work, it will be interesting to formally characterise the relationship between the final consensus equilibrium and the three factors. In addition, in the current article, we did not consider the effects of time delay and time-varying communication topologies. These effects play an important role in real-world applications. In future work, it will be interesting to derive convergence conditions under which consensus can still be reached in the presence of time delay and/or time-varying communication topologies. Furthermore, in future work, it will be interesting to experimentally implement and validate the algorithms in this article on hardware platforms.

Acknowledgment

This work was supported by a National Science Foundation CAREER Award (ECCS-0748287).

References

Chopra, N., and Spong, M.W. (2006), 'Passivity-based Control of Multi-agent Systems', in *Advances in Robot Control: From Everyday Physics to Human-Like Movements*, eds. S. Kawamura and M. Svinin, Berlin: Springer-Verlag, pp. 107–134.

Chung, S.J., and Slotine, J.J.E. (2007). 'Cooperative Robot Control and Synchronization of Lagrangian Systems', in *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, LA, December 2007, pp. 2504–2509.

Graham, A. (1981), *Kronecker Products and Matrix Calculus with Applications*, New York: Halsted Press.

Khalil, H.K. (1996), *Nonlinear Systems* (2nd ed.), Upper Saddle River, NJ: Prentice Hall.

Krogstad, T.R., and Gravdahl, J.T. (2006), '6-DOF Mutual Synchronisation of Formation Flying Spacecraft', in *Proceedings of the IEEE Conference on Decision and Control*, December, San Diego, pp. 5706–5711.

Lafferriere, G., Williams, A., Caughman, J., and Veerman, J.J.P. (2005), 'Decentralised Control of Vehicle Formations', *Systems and Control Letters*, 54, 899–910.

Lee, D., and Spong, M.W. (2007), 'Stable Flocking of Multiple Inertial Agents on Balanced Graphs', *IEEE Transactions on Automatic Control*, 52, 1469–1475.

Merris, R. (1994), 'Laplacian Matrices of Graphs: A Survey', *Linear Algebra and its Applications*, 197–198, 143–176.

Olfati-Saber, R. (2006), 'Flocking for Multi-Agent Dynamic Systems: Algorithms and Theory', *IEEE Transactions on Automatic Control*, 51, 401–420.

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007), 'Consensus and Cooperation in Networked Multi-Agent Systems', *Proceedings of the IEEE*, 95, 215–233.

Paden, B., and Panja, R. (1988), 'Globally Asymptotically Stable 'PD+' Controller for Robot Manipulators', *International Journal of Control*, 47, 1697–1712.

Ren, W. (2007), 'Distributed Attitude Alignment in Spacecraft Formation Flying', *International Journal of Adaptive Control and Signal Processing*, 21, 95–113.

Ren, W. (2008), 'On Consensus Algorithms for Double-integrator Dynamics', *IEEE Transactions on Automatic Control*, 53, 1503–1509.

Ren, W., and Atkins, E.M. (2007), 'Distributed Multi-vehicle Coordinated Control via Local Information Exchange', *International Journal of Robust and Nonlinear Control*, 17, 1002–1033.

Ren, W., Beard, R.W., and Atkins, E.M. (2007), 'Information Consensus in Multivehicle Cooperative Control', *IEEE Control Systems Magazine*, 27, 71–82.

Spong, M.W., and Vidyasagar, M. (2006), *Robot Modeling and Control*, New York: John Wiley & Sons.

Su, Y., Sun, D., Ren, L., and Mills, J.K. (2006), 'Integration of Saturated PI Synchronous Control and PD Feedback for Control of Parallel Manipulators', *IEEE Transactions on Robotics*, 22, 202–207.

Sun, D., Shao, X., and Feng, G. (2007), 'A Model-Free Cross-coupled Control for Position Synchronization of Multi-aXis Motions: Theory and Experiments', *IEEE Transactions on Control Systems Technology*, 15, 306–314.

Tanner, H.G., Jadbabaie, A., and Pappas, G.J. (2007), 'Flocking in Fixed and Switching Networks', *IEEE Transactions on Automatic Control*, 52, 863–868.

Veerman, J.J.P., Lafferriere, G., Caughman, J.S., and Williams, A. (2005), 'Flocks and Formations', *Journal of Statistical Physics*, 121, 901–936.

Xie, G., and Wang, L. (2007), 'Consensus Control for a Class of Networks of Dynamic Agents', *International Journal of Robust and Nonlinear Control*, 17, 941–959.