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Brief paper

Second-order consensus in multi-agent dynamical systems with sampled position data[☆]

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ABSTRACT

This paper studies second-order consensus in multi-agent dynamical systems with sampled position data. A distributed linear consensus protocol with second-order dynamics is designed, where both the current and some sampled past position data are utilized. It is found that second-order consensus in such a multi-agent system cannot be reached without any sampled position data under the given protocol while it can be achieved by appropriately choosing the sampling period. A necessary and sufficient condition for reaching consensus of the system in this setting is established, based on which consensus regions are then characterized. It is shown that if all the eigenvalues of the Laplacian matrix are real, then second-order consensus in the multi-agent system can be reached for any sampling period except at some critical points depending on the spectrum of the Laplacian matrix. However, if there exists at least one eigenvalue of the Laplacian matrix with a nonzero imaginary part, second-order consensus cannot be reached for sufficiently small or sufficiently large sampling periods. In such cases, one nevertheless may be able to find some disconnected stable consensus regions determined by choosing appropriate sampling periods. Finally, simulation examples are given to verify and illustrate the theoretical analysis.

1. Introduction

Collective behaviors in networked systems with a group of autonomous mobile agents, e.g., synchronization (Arenas, Diaz-Guilera, Kurths, Moreno, & Zhou, 2008; Lü & Chen, 2005; Pecora & Carroll, 1990; Wang & Chen, 2002; Yu, Cao, & Lü, 2008; Yu, Chen, & Lü, 2009; Zhou, Lu, & Lü, 2006), consensus (Cao, Morse, & Anderson, 2008; Cao, Ren, & Chen, 2008; Hong, Chen, & Bushnell, 2008; Hong, Hu, & Gao, 2006; Jadbabaie, Lin, & Morse, 2003; Moreau, 2005; Olfati-Saber, 2004; Ren, 2008; Ren & Atkins, 2007; Ren & Beard, 2005; Vicsek, Cziok, Jacob, Cohen, & Shochet, 1995; Yu, Chen, & Cao, 2010; Yu, Chen, & Ren, 2010; Yu, Chen, Cao, & Kurths, 2010; Yu, Chen, Ren, Kurths, & Zheng, in press), swarming (Gazi & Passino, 2003), and flocking (Olfati-Saber, 2006), have received increasing interest recently due to broad applications in biological systems, sensor networks (Yu, Chen, Wang, & Yang, 2009), Unmanned Air Vehicle (UAV) formations, robotic teams, underwater vehicles, etc. The main idea is that each agent shares information only with its nearest neighbors while the whole network of agents can coordinate so as to achieve a certain global criterion of common interest, which by nature is a local distributed protocol. As one of the most typical collective behaviors, consensus refers to reaching an agreement among a group of autonomous agents, which serves as a foundation for the study of swarming and flocking behaviors of multi-agent systems. The consensus problem has a long history in the computer science especially for the field of distributed computing. The idea for consensus originated from statistical consensus theory by DeGroot (1974).

In the 1980s, various models of distributed asynchronous iterations were proposed and theoretically analyzed in Bertsekas and Tsitsiklis (1989). Recently, there are extensive studies of the conditions for reaching consensus among a group of autonomous agents in a dynamical network. In Vicsek et al. (1995), Vicsek et al. proposed a simple discrete-time model to study a group of autonomous agents moving in the plane with the same speed but different headings under noise perturbation, which in essence is

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the velocity consensus problem based on one of the heuristic rules proposed earlier by Reynolds (1987). By using algebraic graph theory (Fiedler, 1973) and similarly as some results in Bertsekas and Tsitsiklis (1989), the linear Vicsek's model was theoretically studied in Jadbabaie et al. (2003) and it was found that consensus in a network with a switching topology can be reached if the network is jointly connected frequently enough as the network evolves with time. Lately, the study of the consensus problem was further extended to the case of directed networks (Cao, Morse et al., 2008; Moreau, 2005; Olfati-Saber, 2004; Ren & Beard, 2005; Yu, Chen, and Cao, 2010).

In the literature, most existing studies address the simple case where agents are governed by first-order dynamics (Cao, Morse et al., 2008; Cao, Ren et al., 2008; Jadbabaie et al., 2003; Olfati-Saber, 2004; Ren & Beard, 2005; Vicsek et al., 1995). However, second-order dynamics (Hong et al., 2008, 2006; Ren, 2008; Ren & Atkins, 2007; Yu, Chen, and Cao, 2010; Yu, Chen, Cao, and Kurths, 2010) have recently received increasing attention due to many real-world applications where mobile agents are governed by both position and velocity states. In Yu, Chen, and Cao (2010), some necessary and sufficient conditions for second-order consensus in linear multi-agent dynamical systems with directed topologies and time delays were established. It was found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix associated with the corresponding network topology play key roles in reaching second-consensus. However, as shown in Hong et al. (2006), Hong et al. (2008) and Ren (2008), the velocity states of agents are often unavailable, therefore, some observers were designed with additional variables involved, which led to the investigation of higher-order dynamical systems.

In Cao, Ren et al. (2008) and Yu, Chen and Ren (2010), consensus in first-order and second-order multi-agent systems was studied, where information of both the current and delayed position states was utilized. In Cao, Ren et al. (2008), it was shown that the delay-involved algorithm converges faster than the standard consensus protocol without time delays. It was surprisingly found in Yu, Chen and Ren (2010) that consensus in a multi-agent system cannot be reached without delayed position information under the given protocol but it can be achieved with an even relatively small time delay by appropriately choosing the coupling strengths, which implies that delay can induce second-order consensus in some multi-agent dynamical systems. In the existing delay-involved consensus algorithms (Cao, Ren et al., 2008; Yu, Chen and Ren, 2010), however, all the position states in a certain time interval have to be kept in memory, which requires more information and induces higher energy cost.

On the other hand, hybrid systems are complex systems with both continuous-time and discrete-time event dynamics, which have been widely investigated recently in the literature. For example, continuous-time systems with impulsive responses, sampled data, quantization, to name just a few. Some real-world applications can be modeled by continuous-time systems together with some discrete-time events. For example, an A/C unit containing some discrete modes with on or off states, changes the temperature continuously over time. In this paper, sampled position data will be used instead, which is memoryless since only position information at some particular time instants is needed. Therefore, in order to utilize less information and to save energy, it is desirable to use only sampled data instead of the whole spectrum of delayed information. By using only sampled position data in this paper and without requiring the velocity information of agents in second-order dynamics as in Hong et al. (2006), Hong et al. (2008) and Ren (2008), it is found in this paper that second-order consensus in multi-agent system can be reached by appropriately choosing the sampling period. At this point, it should be noted that the current paper relying on sampled position data allows for studying a general directed network topology, while (Hong et al., 2008, 2006; Ren, 2008) relying on observers design typically deals with an undirected network topology.

The main contributions of this paper include some necessary and sufficient conditions derived for reaching second-order consensus in multi-agent systems using sampled position data, specifically showing that second-order consensus in a multi-agent system using both current and sampled past position data can be reached if and only if the sampling period is chosen from some particular time intervals depending on the coupling strengths and the spectrum of the Laplacian matrix of the network.

The rest of the paper is organized as follows. In Section 2, some preliminaries on graph theory and model formulation are given. The main results about second-order consensus in multi-agent dynamical systems with sampled position data are presented in Section 3. In Section 4, numerical examples are given to illustrate the theoretical analysis. Conclusions are finally drawn in Section 5.

2. Preliminaries

In this section, some basic concepts and results about algebraic graph theory (Godsil & Royle, 2001) are first introduced.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, G)$ be a weighted directed graph of order N, with the set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $G = (G_{ij})_{N\times N}$. A directed edge e_{ij} in network $\mathcal{G}$ is denoted by the ordered pair of nodes (v_i, v_j), where v_i and v_j are called the parent and child nodes, respectively, meaning that node v_j can receive information from node v_i. In this paper, only positively weighted directed graphs are considered, thus, $G_{ij} > 0$ if and only if there is a directed edge (v_j, v_i) in $\mathcal{G}$; otherwise, $G_{ij} = 0$.

A directed path from node v_i to node v_j in $\mathcal{G}$ is a sequence of edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_l}, v_j)$ in the directed network with distinct nodes v_{i_k}, $k = 1, 2, \ldots, l$ (Godsil & Royle, 2001; Horn & Johnson, 1985). *A root* r is a node such that for each node v different from r, there is a directed path from r to v. *A directed tree* is a directed graph, in which there is exactly one root and every node except for this root itself has exactly one parent. *A directed spanning tree* is a directed tree consisting of all the nodes and some edges in $\mathcal{G}$. A directed graph contains a directed spanning tree if one of its subgraphs is a directed spanning tree.

The second-order consensus protocol in multi-agent dynamical systems is described by Ren (2008), Ren and Atkins (2007) and Yu, Chen, and Cao (2010)

$$\begin{aligned}\dot{x}_i(t) &= v_i,\\ \dot{v}_i(t) &= \widetilde{\alpha}\sum_{j=1,j\neq i}^{N} G_{ij}(x_j(t)-x_i(t))\\ &\quad+\widetilde{\beta}\sum_{j=1,j\neq i}^{N} G_{ij}(v_j(t)-v_i(t)),\quad i=1,2,\ldots,N,\end{aligned} \tag{1}$$

where $x_i \in R^n$ and $v_i \in R^n$ are the position and velocity states of the ith agent (node), respectively, $\widetilde{\alpha} > 0$ and $\widetilde{\beta} > 0$ are the coupling strengths, and $G = (G_{ij})_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network and thus is the weighted adjacency matrix of the network. The Laplacian matrix $L = (L_{ij})_{N\times N}$ is defined by

$$L_{ii} = -\sum_{j=1,j\neq i}^{N} L_{ij},\qquad L_{ij} = -G_{ij},\quad i\neq j, \tag{2}$$

which ensures the diffusion property that $\sum_{j=1}^{N} L_{ij} = 0$. For notational simplicity, $n = 1$ is considered throughout the paper, but all the results obtained can be easily generalized to the case with $n > 1$ by using the Kronecker product operations (Horn & Johnson, 1991).

In many cases, it is literally difficult to measure the relative velocity difference between two neighboring agents. In Hong et al. (2006), Hong et al. (2008) and Ren (2008), distributed observers were designed for second-order multi-agent systems, where the velocity states were assumed to be unavailable, i.e., $\widetilde{\beta} = 0$, and some slack variables were introduced and a higher-order controller was designed. In this paper, only sampled position data is used. It will be shown that second-order consensus can be reached in the multi-agent dynamical systems under some conditions even if the velocity states are unavailable. To do so, the following consensus protocol with both current and sampled position data is considered

$$\dot{x}_i(t) = v_i,$$
$$\dot{v}_i(t) = \alpha \sum_{j=1, j\neq i}^{N} G_{ij}(x_j(t) - x_i(t)) - \beta \sum_{j=1, j\neq i}^{N} G_{ij} \times (x_j(t_k) - x_i(t_k)), \quad t \in [t_k, t_{k+1}),\ i = 1, \dots, N, \tag{3}$$

where t_k are the sampling instants satisfying $0 = t_0 < t_1 < \cdots < t_k < \cdots$, and α and β are the coupling strengths. For simplicity, assume that $t_{k+1} - t_k = T$, where $T > 0$ is the sampling period.

In Cao, Ren et al. (2008) and Yu, Chen and Ren (2010), consensus in multi-agent systems was studied, where information of both the current and delayed position states data was used. In the existing delay-involved consensus algorithms (Cao, Ren et al., 2008; Yu, Chen and Ren, 2010), all the position states in the time interval $[t-\tau, t]$ have to be kept in memory, where τ is the time delay constant. In order to utilize less information and save energy, it is desirable to use only sampled data instead of delayed information in a spectrum of a time interval.

Definition 1. Second-order consensus in multi-agent system (3) is said to be achieved if, for any initial conditions,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \qquad \lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \quad \forall i, j = 1, 2, \dots, N.$$

Because of (2), system (3) can be equivalently rewritten as follows:

$$\dot{x}_i(t) = v_i,$$
$$\dot{v}_i(t) = -\alpha \sum_{j=1}^{N} L_{ij} x_j(t) + \beta \sum_{j=1}^{N} L_{ij} x_j(t_k), \quad t \in [t_k, t_{k+1}),\ i = 1, 2, \dots, N. \tag{4}$$

The following notations will be used throughout the paper. Let $\mathcal{R}(u)$ and $\mathcal{I}(u)$ be the real and imaginary parts of a complex number u, $0 = \mu_1 \le \mathcal{R}(\mu_2) \le \cdots \le \mathcal{R}(\mu_N)$ be the N eigenvalues of the Laplacian matrix L, $I_m \in R^{m\times m}$ ($O_N \in R^{m\times m}$) be the m-dimensional identity (zero) matrix, $1_m \in R^m$ ($0_N \in R^m$) be the vector with all entries being 1 (0), and $\|a_1 + \mathbf{i}a_2\| = \sqrt{a_1^2 + a_2^2}$ be the norm of a complex number $a_1 + \mathbf{i}a_2$ where $\mathbf{i} = \sqrt{-1}$.

Lemma 1 (*Ren & Beard, 2005*). *The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree.*

Lemma 2 (*Horn & Johnson, 1985*). *The Kronecker product $\otimes$ has the following properties: For matrices A, B, C and D of appropriate dimensions,*

(1) $(A + B) \otimes C = A \otimes C + B \otimes C$;
(2) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Lemma 3 (*Parks & Hahn, 1993*). *Given a complex coefficient polynomial of order two as follows:*

$$g(s) = s^2 + (\xi_1 + \mathbf{i}\gamma_1)s + \xi_0 + \mathbf{i}\gamma_0,$$

where ξ_1, γ_1, ξ_0, and γ_0 are real constants. Then, $g(s)$ is stable if and only if $\xi_1 > 0$ and $\xi_1\gamma_1\gamma_0 + \xi_1^2\xi_0 - \gamma_0^2 > 0$.

3. Second-order consensus in multi-agent dynamical systems with sampled position data

Let $\eta_i = (x_i, v_i)^T$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then, system (4) can be rewritten as

$$\dot{\eta}_i(t) = A\eta_i(t) - \alpha \sum_{j=1}^{N} L_{ij} B \eta_j(t) + \beta \sum_{j=1}^{N} L_{ij} B \eta_j(t_k), \quad t \in [t_k, t_{k+1}),\ i = 1, 2, \dots, N. \tag{5}$$

Note that a solution of an isolated node of system (5) satisfies

$$\dot{s}(t) = As(t), \quad t \in [t_k, t_{k+1}), \tag{6}$$

where $s(t) = (s_1, s_2)^T$ is the state vector. Let $\eta = (\eta_1^T, \dots, \eta_N^T)^T$ and rewrite system (5) in a matrix form:

$$\dot{\eta}(t) = [(I_N \otimes A) - \alpha(L \otimes B)]\eta(t) + \beta(L \otimes B)\eta(t_k), \quad t \in [t_k, t_{k+1}), \tag{7}$$

where $\otimes$ is the Kronecker product (Horn & Johnson, 1991). Let J be the Jordan form associated with the Laplacian matrix L, i.e., $L = PJP^{-1}$, where P is a nonsingular matrix. By Lemma 2, one has

$$\begin{aligned} \dot{y}(t) &= (P^{-1} \otimes I_2)[(I_N \otimes A) - \alpha(L \otimes B)]\eta(t) + \beta(P^{-1} \otimes I_2)(L \otimes B)\eta(t_k) \\ &= [(P^{-1} \otimes A) - \alpha(JP^{-1} \otimes B)]\eta(t) + \beta(JP^{-1} \otimes B)\eta(t_k) \\ &= [(I_N \otimes A) - \alpha(J \otimes B)]y(t) + \beta(J \otimes B)y(t_k), \end{aligned} \quad t \in [t_k, t_{k+1}), \tag{8}$$

where $y(t) = (P^{-1} \otimes I_2)\eta(t)$. If the graph $\mathcal{G}$ is undirected, then L is symmetric and J is a diagonal matrix with real eigenvalues. However, when $\mathcal{G}$ is directed, some eigenvalues of L may be complex, and $J = \mathrm{diag}(J_1, J_2, \dots, J_r)$, where

$$J_l = \begin{pmatrix} \mu_l & 0 & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \mu_l \end{pmatrix}_{N_l \times N_l}, \tag{9}$$

in which μ_l are the eigenvalues of the Laplacian matrix L, with multiplicity N_l, $l = 1, 2, \dots, r$, $N_1 + N_2 + \cdots + N_r = N$.

Let $P = (p_1, \dots, p_N)$, $P^{-1} = (q_1, \dots, q_N)^T$, $y(t) = (P^{-1} \otimes I_2)\eta(t) = (y_1^T, \dots, y_N^T)^T$, and $y_i = (y_{i1}, y_{i2})^T$. Note that if the network $\mathcal{G}$ contains a directed spanning tree, then by Lemma 1, 0 is a simple eigenvalue of the Laplacian matrix L, so

$$\dot{y}_1(t) = Ay_1(t), \quad t \in [t_k, t_{k+1}). \tag{10}$$

Theorem 1. *Suppose that the network $\mathcal{G}$ contains a directed spanning tree. Then, second-order consensus in system (3) can be reached if and only if, in (8),*

$$\lim_{t\to\infty} \|y_i\| \to 0, \quad i = 2, \dots, N. \tag{11}$$

Proof. (Sufficiency). Since the network $\mathcal{G}$ contains a directed spanning tree, $p_1 = 1_N/\sqrt{N}$ is the unit right eigenvector of the Laplacian matrix L associated with the simple zero eigenvalue

$\mu_1 = 0$, where $LP = PJ$ and $P = (p_1, \ldots, p_N)$. From $\lim_{t\to\infty} \|y_i\| \to 0$ for $i = 2, \ldots, N$, one has

$$\lim_{t\to\infty} \left\| \eta(t) - \frac{1}{\sqrt{N}}(y_1(t)^T, \ldots, y_1(t)^T)^T \right\|$$
$$= \lim_{t\to\infty} \left\| (P \otimes I_2)y(t) - \frac{1}{\sqrt{N}}(y_1(t)^T, \ldots, y_1(t)^T)^T \right\| = 0,$$

where $y_1(t)$ satisfies (10). Therefore, second-order consensus in system (3) is reached.

(Necessity). If second-order consensus in system (3) can be reached, then there exists a vector $\eta^*(t) \in R^2$ such that $\lim_{t\to\infty} \|\eta(t) - 1_N \otimes \eta^*(t)\| = 0$. Then, one has $0_N = P^{-1}L1_N = JP^{-1}1_N = J(q_1^T 1_N, \ldots, q_N^T 1_N)^T$. From the Jordan form (9) and $0 = \mu_1 < \mathcal{R}(\mu_2) \le \cdots \le \mathcal{R}(\mu_N)$, one obtains $q_i^T 1_N = 0$ for $i = 2, \ldots, N$. Therefore, $\|y_i(t)\| = \|(q_i^T \otimes I_2)\eta(t)\| \to \|(q_i^T 1_N) \otimes \eta^*(t)\| = 0$, as $t \to \infty$, for all $i = 2, \ldots, N$. □

Corollary 1. *Suppose that the network $\mathcal{G}$ contains a directed spanning tree. Then, second-order consensus in system (3) can be reached if and only if the following $N-1$ systems are asymptotically stable:*

$$\dot{z}_i(t) = (A - \alpha\mu_i B)z_i(t) + \beta\mu_i B z_i(t_k),$$
$$t \in [t_k, t_{k+1}),\ i = 2, \ldots, N. \tag{12}$$

Proof. (Necessity). If $\lim_{t\to\infty} \|y_i\| \to 0$ for $i = 2, \ldots, N$, then the $N-1$ systems (12) are asymptotically stable since the variables in (12) are the first term of each Jordan block in system (8).

(Sufficiency). It suffices to prove that if the $N-1$ systems (12) are asymptotically stable, then $\lim_{t\to\infty} \|y_i\| \to 0$ for $i = 2, \ldots, N$. From the properties of the Jordan form (9), the asymptotical behavior in system (8) is dominated by the diagonal terms, therefore the conclusion follows. □

Until now, it is still very hard to check the conditions (11) and (12) in Theorem 1 and Corollary 1 which do not reveal how network structure affects the consensus behavior. Next, a theorem is derived to ensure consensus depending on the control gains, spectra of the Laplacian matrix, and the sampling period.

Theorem 2. *Suppose that the network $\mathcal{G}$ contains a directed spanning tree. Then, second-order consensus in system (3) can be reached if and only if*

$$0 < \frac{\beta}{\alpha} < 1, \tag{13}$$

and

$$f(\alpha, \beta, \mu_i, T) = \frac{(\beta/\alpha)^2}{1 - (\beta/\alpha)}\left(\sin^2(d_i T) - \sinh^2(c_i T)\right) \times \left(\cosh(c_i T) - \cos(d_i T)\right)^2 - 4\sin^2(d_i T)\sinh^2(c_i T) > 0, \quad i = 2, \ldots, N, \tag{14}$$

where $c_i = \sqrt{\frac{|\alpha|(\|\mu_i\| - \mathrm{sign}(\alpha)\mathcal{R}(\mu_i))}{2}}$ and $d_i = \sqrt{\frac{|\alpha|(\|\mu_i\| + \mathrm{sign}(\alpha)\mathcal{R}(\mu_i))}{2}}$.

Proof. It suffices to prove that system (12) is asymptotically stable if and only if the conditions (13) and (14) are satisfied.

From (12), it follows that

$$\left(e^{-(A-\alpha\mu_i B)t} z_i(t)\right)' = e^{-(A-\alpha\mu_i B)t}\beta\mu_i B z_i(t_k), \quad t \in [t_k, t_{k+1}). \tag{15}$$

Let $z_i = (z_{i1} z_{i2})^T$ in (12). If $\alpha = 0$, then system (12) is $\ddot{z}_{i1} = \beta\mu_i z_{i1}(t_k)$ for $t \in [t_k, t_{k+1})$, which is unstable. Thus, $\alpha \neq 0$.

Since $\mu_i \neq 0$ for $i = 2, \ldots, N$, $(A - \alpha\mu_i B) = \begin{pmatrix} 0 & 1 \\ -\alpha\mu_i & 0 \end{pmatrix}$ is nonsingular and $(A - \alpha\mu_i B)^{-1} = \begin{pmatrix} 0 & -1/\alpha\mu_i \\ 1 & 0 \end{pmatrix}$. Integrating both sides of (15) from t_k to t, one has

$$z_i(t) = e^{(A-\alpha\mu_i B)(t-t_k)}\left(I_2 + (A - \alpha\mu_i B)^{-1}\beta\mu_i B\right)z_i(t_k) - (A - \alpha\mu_i B)^{-1}\beta\mu_i B z_i(t_k)$$
$$= e^{(A-\alpha\mu_i B)(t-t_k)}\begin{pmatrix} 1-\beta/\alpha & 0 \\ 0 & 1 \end{pmatrix} z_i(t_k) + \begin{pmatrix} \beta/\alpha & 0 \\ 0 & 0 \end{pmatrix} z_i(t_k), \quad t \in [t_k, t_{k+1}). \tag{16}$$

If the sampled data is missing, i.e., $\beta = 0$, it is easy to see that $A - \alpha\mu_i B$ has at least one eigenvalue with a nonnegative real part and it thus follows from (16) that $z_i(t) = e^{(A-\alpha\mu_i B)(t-t_k)}z_i(t_k)$, which is unstable.

Next, it is to show that by introducing sampled position data, the state in (16) is asymptotically stable.

Let $a_j + \mathbf{i}b_j$ be the eigenvalues of $(A - \alpha\mu_i B)$, $j = 1, 2$. Then, one has

$$a_j^2 - b_j^2 = -\alpha\mathcal{R}(\mu_i),$$
$$2a_j b_j = -\alpha\mathcal{I}(\mu_i).$$

The solutions of $a_j + \mathbf{i}b_j$ can be classified according to two cases, i.e., $\alpha > 0$ and $\alpha < 0$.

Case I ($\alpha > 0$). By simple calculation, one obtains

$$a_1 + \mathbf{i}b_1 = c_i - \mathbf{i}d_i, \qquad a_2 + \mathbf{i}b_2 = -c_i + \mathbf{i}d_i, \quad \text{if } \mathcal{I}(\mu_i) \ge 0,$$
$$a_1 + \mathbf{i}b_1 = c_i + \mathbf{i}d_i, \qquad a_2 + \mathbf{i}b_2 = -c_i - \mathbf{i}d_i, \quad \text{if } \mathcal{I}(\mu_i) < 0, \tag{17}$$

where $c_i = \sqrt{\frac{\alpha(\|\mu_i\| - \mathcal{R}(\mu_i))}{2}}$ and $d_i = \sqrt{\frac{\alpha(\|\mu_i\| + \mathcal{R}(\mu_i))}{2}}$. Without loss of generality, assume that $\mathcal{I}(\mu_i) \ge 0$. For the case of $\mathcal{I}(\mu_i) < 0$, the derived conditions in (13) and (14) of the asymptotical stability of system (16) for both $\mathcal{I}(\mu_i) \ge 0$ and $\mathcal{I}(\mu_i) < 0$ are the same, since $f(\alpha, \beta, \mu_i, T)$ is an even function on c_i and d_i.

Let $\varrho_1(t) = \cosh((c_i - \mathbf{i}d_i)(t))$ and $\varrho_2(t) = \sinh((c_i - \mathbf{i}d_i)(t))$. By calculation, one obtains

$$e^{(A-\alpha\mu_i B)(t-t_k)} = \begin{pmatrix} \varrho_1(t-t_k) & \varrho_2(t-t_k)/(c_i - \mathbf{i}d_i) \\ (c_i - \mathbf{i}d_i)\varrho_2(t-t_k) & \varrho_1(t-t_k) \end{pmatrix}. \tag{18}$$

Substituting (18) into (16), one has

$$z_i(t) = \begin{pmatrix} \varrho_1(t-t_k)(1-\beta/\alpha) + \beta/\alpha & \varrho_2(t-t_k)/(c_i - \mathbf{i}d_i) \\ (c_i - \mathbf{i}d_i)\varrho_2(t-t_k)(1-\beta/\alpha) & \varrho_1(t-t_k) \end{pmatrix} \times z_i(t_k),\ t \in [t_k, t_{k+1}). \tag{19}$$

Let $C(t) = \begin{pmatrix} \varrho_1(t)(1-\beta/\alpha) + \beta/\alpha & \varrho_2(t)/(c_i - \mathbf{i}d_i) \\ (c_i - \mathbf{i}d_i)\varrho_2(t)(1-\beta/\alpha) & \varrho_1(t) \end{pmatrix}$. It is easy to see that $C(t)$ is bounded on $[0, T]$. So, for $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and $t_{k+1} - t_k = T$, one has

$$z_i(t) = C(t - t_k)C^k(T)z_i(t_0), \quad t \in [t_k, t_{k+1}). \tag{20}$$

Since $C(t - t_k)$ is bounded when $t \in [t_k, t_{k+1})$, $z_i(t) \to 0$ if and only if all eigenvalues λ of $C(T)$ satisfy $\|\lambda\| < 1$.

Let $|\lambda I_2 - C(T)| = 0$, one has

$$\lambda^2 - \left(2\cosh((c_i - \mathbf{i}d_i)T) + \frac{\beta}{\alpha}\left(1 - \cosh((c_i - \mathbf{i}d)_i T)\right)\right)\lambda + \left(1 - \frac{\beta}{\alpha}\right) + \frac{\beta}{\alpha}\cosh((c_i - \mathbf{i}d_i)T) = 0. \tag{21}$$

Let $\lambda = \frac{s+1}{s-1}$. Then, (21) can be transformed to

$$0 = (1-\cosh((c_i-\mathbf{i}d_i)T))\left(1-\frac{\beta}{\alpha}\right)s^2 + \frac{\beta}{\alpha}(1 - \cosh((c_i-\mathbf{i}d_i)T))s + (1+\cosh((c_i-\mathbf{i}d_i)T)). \quad (22)$$

It is well known that $\|\lambda\| < 1$ in (21) if and only if $\mathcal{R}(s) < 0$ in (22). Therefore, $z_i(t) \to 0$ if and only if all the roots in (22) have negative real parts.

If $\cosh((c_i-\mathbf{i}d_i)T) = 1$ or $\frac{\beta}{\alpha} = 1$, then $C(T)$ has an eigenvalue $\lambda = 1$. Therefore, $(1-\cosh((c_i-\mathbf{i}d_i)T))\left(1-\frac{\beta}{\alpha}\right) \neq 0$ and (22) can be simplified to

$$s^2 + \frac{\beta/\alpha}{1-\beta/\alpha}s + \frac{(1+\cosh((c_i-\mathbf{i}d_i)T))}{(1-\beta/\alpha)(1-\cosh((c_i-\mathbf{i}d_i)T))} = 0. \quad (23)$$

Let $D_i = \frac{1+\cosh((c_i-\mathbf{i}d_i)T)}{1-\cosh((c_i-\mathbf{i}d_i)T)}$. By Lemma 3, (23) is stable if and only if

$$\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}-1\right) < 0 \quad (24)$$

and

$$\frac{(\beta/\alpha)^2}{1-(\beta/\alpha)}\mathcal{R}(D_i) > \mathcal{I}^2(D_i). \quad (25)$$

By solving (25), one obtains the condition (14). Therefore, $z_i(t) \to 0$ if and only if both (13) and (14) are satisfied. By Corollary 1, second-order consensus in system (3) is reached if and only if both (13) and (14) are satisfied.

Case II ($\alpha < 0$). By simple calculation, one obtains

$$\begin{aligned} &a_1 + \mathbf{i}b_1 = c_i - \mathbf{i}d_i, \qquad a_2 + \mathbf{i}b_2 = -c_i + \mathbf{i}d_i, \quad \text{if } \mathcal{I}(\mu_i) \geq 0, \\ &a_1 + \mathbf{i}b_1 = c_i + \mathbf{i}d_i, \qquad a_2 + \mathbf{i}b_2 = -c_i - \mathbf{i}d_i, \quad \text{if } \mathcal{I}(\mu_i) < 0, \end{aligned} \quad (26)$$

where $c_i = \sqrt{\frac{-\alpha(\|\mu_i\|+\mathcal{R}(\mu_i))}{2}}$ and $d_i = \sqrt{\frac{-\alpha(\|\mu_i\|-\mathcal{R}(\mu_i))}{2}}$. The rest of the proof is similar to that of Case I. □

Remark 1. In Theorem 2, a necessary and sufficient condition for second-order consensus in the multi-agent dynamical system (3) is established. For a given network, one can design appropriate α, β, and T such that the conditions (13) and (14) in Theorem 2 are satisfied. It is interesting to see that f increases as the parameter β/α increases. Thus, one can choose a large value of β/α such that (14) holds. Since the condition (14) holds for all $i = 2, \ldots, N$, one can find a stable consensus region as follows: $\mathcal{S} = \{c + \mathbf{i}d \mid f(\alpha, \beta, c + \mathbf{i}d, T) > 0\}$, where c and d are real. Then, the problem is transformed to finding if all the nonzero eigenvalues of the Laplacian matrix lie in the stable consensus region $\mathcal{S}$, i.e., $\mu_i \in \mathcal{S}$ for all $i = 2, \ldots, N$. In Duan, Chen, and Huang (2009) and Liu, Duan, Chen, and Huang (2007), disconnected synchronization regions of complex networks were discussed. It was shown that there indeed exist some disconnected synchronization regions for several particular complex networks when the synchronous state is an equilibrium point. In this paper, by introducing sampled position data in the consensus algorithm, it will be shown in the simulation that there exist some disconnected regions for choosing appropriate sampling periods.

Corollary 2. *Suppose that the network $\mathcal{G}$ contains a directed spanning tree and all the eigenvalues of its Laplacian matrix are real. Then, second-order consensus in system* (3) *can be reached if and only if*

$$0 < \beta < \alpha \quad (27)$$

and

$$\sqrt{\alpha\mu_i}T \neq k\pi, \quad i = 2, \ldots, N, \ k = 0, 1, \ldots. \quad (28)$$

Proof. If $\alpha > 0$, then $c_i = 0$ and $d_i = \sqrt{\alpha\mu_i}$. Therefore, $\sinh(c_iT) = 0$ and $\cosh(c_iT) = 1$. The condition in (14) can be simplified to

$$\frac{(\beta/\alpha)^2}{1-(\beta/\alpha)}\sin^2(d_iT)\left(1-\cos(d_iT)\right)^2 > 0, \quad i = 2, \ldots, N, \quad (29)$$

which is equivalent to the conditions (27) and (28).

If $\alpha < 0$, then $c_i = \sqrt{-\alpha\mu_i}$ and $d_i = 0$. The condition in (14) is

$$\frac{(\beta/\alpha)^2}{1-(\beta/\alpha)}\left(-\sinh^2(c_iT)\right)(\cosh(c_iT)-1)^2 > 0, \quad i = 2, \ldots, N, \quad (30)$$

which cannot be satisfied since $\beta/\alpha < 1$ according to the condition (13). □

Remark 2. If all the eigenvalues of the Laplacian matrix are real, which includes the undirected network as a special case, then the condition (28), i.e., $T \neq \frac{k\pi}{\sqrt{\alpha\mu_i}}$, is very easy to be verified and applied. It is quite interesting to see that second-order consensus in the multi-agent system (3) can be reached if and only if $0 < \beta < \alpha$ and the sampling period T is not of some particular value.

Usually, the convergence rate around the critical points $T = \frac{k\pi}{\sqrt{\alpha\mu_i}}$ is very slow. Therefore, it is hard to achieve better performance for a large T in a very large-scale network. A corollary is given below to simplify the theoretical analysis.

Corollary 3. *Suppose that the network $\mathcal{G}$ contains a directed spanning tree and all the eigenvalues of its Laplacian matrix are real. Then, second-order consensus in system* (3) *can be reached if* (27) *is satisfied and*

$$0 < T < \frac{\pi}{\sqrt{\alpha\mu_N}}. \quad (31)$$

Corollary 3 implies that if the network $\mathcal{G}$ contains a directed spanning tree and all the eigenvalues of the Laplacian matrix are real, then second-order consensus in system (3) can be reached provided that the sampling period is less than the critical value $\frac{\pi}{\sqrt{\alpha\mu_N}}$ depending on the largest eigenvalue of the Laplacian matrix. However, to our surprise, second-order consensus in system (3) cannot be reached in a general directed network with complex Laplacian eigenvalues for a sufficiently small or a sufficiently large sampling period T.

Corollary 4. *Suppose that the network $\mathcal{G}$ contains a directed spanning tree and there is at least one eigenvalue of its Laplacian matrix with a nonzero imaginary part. Then, second-order consensus in the system* (3) *cannot be reached for a sufficiently small or a sufficiently large sampling period T.*

Proof. Without loss of generality, suppose that $\mathcal{I}(\mu_k) \neq 0$, $2 \leq k \leq N$. Then, from the condition (14), one has $c_k \neq 0$ and $d_k \neq 0$. Consider the following Taylor Series for a sufficiently small T:

$$\sinh(c_kT) = c_kT + \frac{(c_kT)^3}{3!} + o(T^3)$$

$$\cosh(c_kT) = 1 + \frac{(c_kT)^2}{2!} + o(T^3)$$

$$\sin(d_kT) = d_kT - \frac{(d_kT)^3}{3!} + o(T^3)$$
$$\cos(d_kT) = 1 - \frac{(d_kT)^2}{2!} + o(T^3). \tag{32}$$

Substituting (32) into (14), one has

$$\begin{aligned} f(\alpha,\beta,\mu_k,T) &= \frac{(\beta/\alpha)^2}{1-(\beta/\alpha)}\Big(\sin^2(d_kT) - \sinh^2(c_kT)\Big) \\ &\quad \times \Big(\cosh(c_kT) - \cos(d_kT)\Big)^2 \\ &\quad - 4\sin^2(d_kT)\sinh^2(c_kT) \\ &= \frac{(\beta/\alpha)^2}{4(1-(\beta/\alpha))}(c_k^2 - d_k^2)^3T^6 + o(T^6) \\ &\quad - 4c_k^2d_k^2T^2 - o(T^2) > 0 \end{aligned} \tag{33}$$

which cannot be satisfied for a sufficiently small sampling period T since $c_k \neq 0$ and $d_k \neq 0$.

If the sampling period T is sufficiently large, then $\sin^2(d_kT) - \sinh^2(c_kT) < 0$ for $c_k \neq 0$ and $d_k \neq 0$, and thus the condition (14) is not satisfied. □

Remark 3. If the network $\mathcal{G}$ contains a directed spanning tree and all the eigenvalues of the Laplacian matrix are real, then second-order consensus in system (3) can be reached for a sufficiently small sampling period T as stated in Corollary 2. However, if there is at least one eigenvalue of the Laplacian matrix having a nonzero imaginary part, then second-order consensus cannot be reached for a sufficiently small sampling period T as shown in Corollary 4, which is inconsistent with the common intuition that the consensus protocol (3) should be better if the sampled information is more accurate for a small sampling period. Interestingly, the nonzero imaginary part of the eigenvalue of the Laplacian matrix leads to possible instability of consensus.

4. Simulation examples

4.1. Second-order consensus in a multi-agent system with an undirected topology

Consider the multi-agent system (3) with an undirected topology, where $L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$, $\alpha = 1$, and $\beta = 0.8$. By simple calculation, one has $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 3$, and $\mu_4 = 4$. From Corollary 2, the system can reach second-order consensus if and only if $T \neq \frac{k\pi}{\sqrt{\alpha\mu_i}}$, for $i = 2, 3, 4$ and $k = 0, 1, \ldots$. It is easy to obtain $\frac{k\pi}{\sqrt{\alpha\mu_4}} \approx 1.5708$, $\frac{k\pi}{\sqrt{\alpha\mu_3}} \approx 1.8138$, and $\frac{k\pi}{\sqrt{\alpha\mu_2}} \approx 3.1416$. Consider the sampling period T as a variable of $f(\alpha,\beta,\mu_i,\cdot)$. The states of T vs. f are shown in Fig. 1. It is easy to see that second-order consensus in the system can be reached if $T = 1.0$, $T = 1.57$, or $T = 2.5$, while the convergence is not good when $T = 1.57$ around the critical point $\frac{k\pi}{\sqrt{\alpha\mu_4}}$. The position and velocity states of all the agents are shown in Fig. 2.

4.2. Second-order consensus in a multi-agent system with a directed topology

Consider the multi-agent system (3) with a directed topology, where $L = \begin{pmatrix} 2 & 0 & -2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$, $\alpha = 1$, and $\beta = 0.8$. By simple

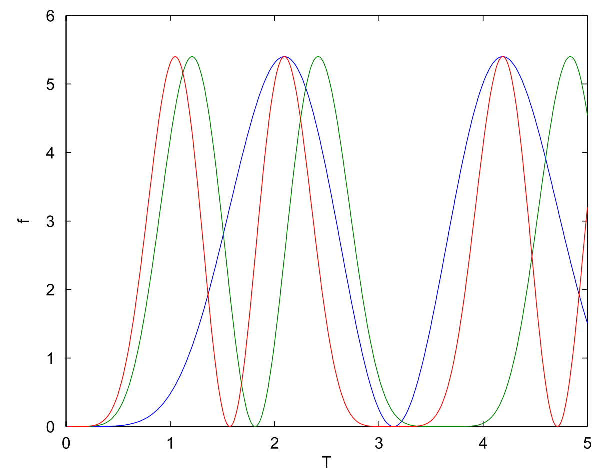

Fig. 1. States of $f(\alpha,\beta,\mu_i,\cdot)$ vs. the sampling period T, $i = 2, 3, 4$.

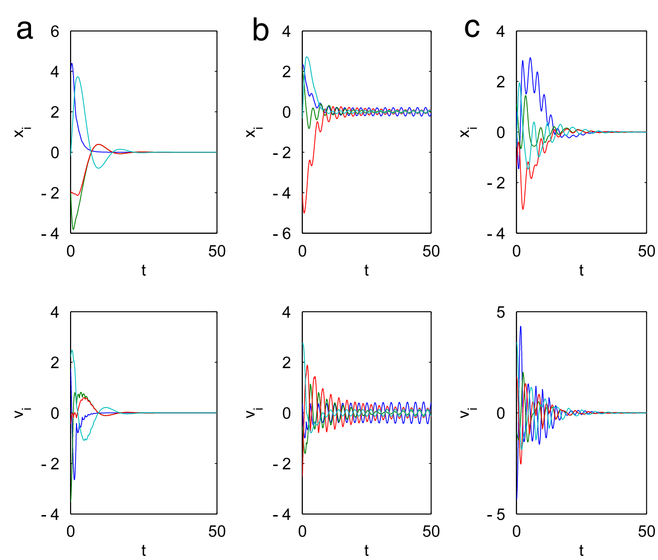

Fig. 2. Position and velocity states of agents, where $T = 1.0$ (a), $T = 1.57$ (b), and $T = 2.5$ (c).

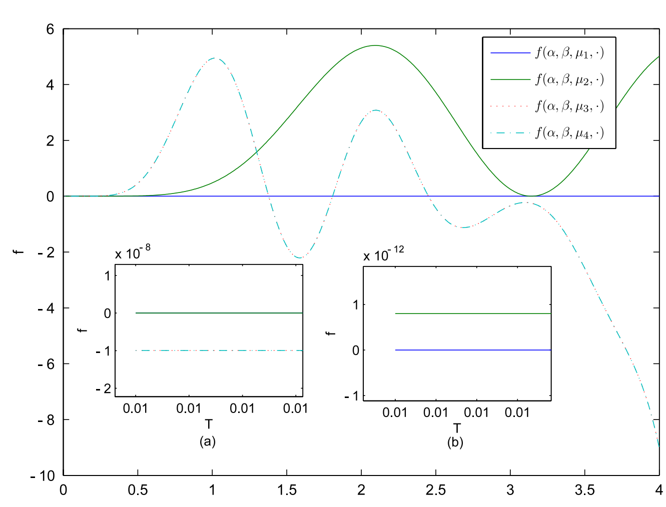

Fig. 3. States of $f(\alpha,\beta,\mu_i,\cdot)$ vs. the sampling period T, $i = 1, 2, 3, 4$.

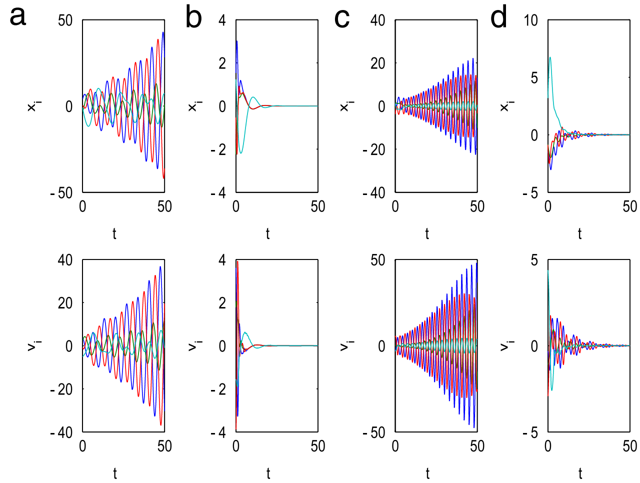

Fig. 4. Position and velocity states of agents, where $T = 0.1$ (a), $T = 1.0$ (b), $T = 1.5$ (c), and $T = 2.0$ (d).

calculation, one has $\mu_1 = 0, \mu_2 = 1, \mu_3 = 4 + \mathbf{i}$, and $\mu_4 = 4 - \mathbf{i}$. From Theorem 2, the multi-agent system can reach second-order consensus if and only if $f(\alpha, \beta, \mu_i, T) > 0$, for $i = 2, 3, 4$. Consider the sampling period T as a variable of $f(\alpha, \beta, \mu_i, \cdot)$. The states of T vs. f are shown in Fig. 3. It is easy to see that second-order consensus in the system can be reached if $T = 1.0$, or $T = 2.0$, while it cannot be reached for a sufficiently small $T = 0.1$ according to Corollary 4 or $T = 1.5$ where $f(\alpha, \beta, \mu_3, T) < 0$. Moreover, when Fig. 3 is amplified around the origin as in the inner fig (a), there are two same lines with complex eigenvalues, i.e., $f(\alpha, \beta, \mu_3, \cdot)$ and $f(\alpha, \beta, \mu_4, \cdot)$ under the zero. If Fig. 3 is further amplified around the origin as in the inner fig (b), there is one line with a real eigenvalue, i.e., $f(\alpha, \beta, \mu_2, \cdot)$, above zero. The position and velocity states of all the agents are shown in Fig. 4.

From Figs. 1 and 3, one can see that if the eigenvalue μ_i of the Laplacian matrix is real, then $f(\alpha, \beta, \mu_i, T) > 0$ except at some critical points $T = \frac{k\pi}{\sqrt{\alpha\mu_i}}$ where $f = 0$. However, if the eigenvalue μ_i of the Laplacian matrix is complex with a nonzero imaginary part, then $f(\alpha, \beta, \mu_i, T) \le 0$ for a sufficiently small T or a sufficiently large T. In this case, still, there may be some disconnected stable consensus regions by choosing an appropriate sampling period T as shown in Fig. 4. Therefore, the design of an appropriate sampling period T plays a key role in reaching consensus. In addition, one can design appropriate coupling strengths α and β such that second-order consensus in the multi-agent system (3) can be reached, as guaranteed by theory. Details are omitted due to space limitations.

5. Conclusions

In this paper, second-order consensus in multi-agent dynamical systems with sampled position data is investigated. A distributed linear consensus protocol in the second-order dynamics is designed based on both current and sampled position data. It is found that second-order consensus in a multi-agent system cannot be reached without sampled position data under the given protocol but it can be achieved by appropriately choosing the sampling period. A necessary and sufficient condition for reaching consensus in multi-agent dynamical systems is established and demonstrated.

There are still a number of related interesting problems deserving further investigation. For example, it is desirable to study multi-agent systems with nonuniform sampling intervals, nonlinear dynamics with time-varying velocities (Yu, Chen, Cao, and Kurths, 2010), more general consensus protocols, and so on, some of which will be investigated in the near future.

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